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SOME GENERALIZED DIFFERENCE SEQUENCE SPACES DEFINED BY A SEQUENCE OF ORLICZ FUNCTIONS

Abstract. In this paper, we define and examine some new difference sequence spaces combining with de la Vallee-Poussin mean and a sequence of Orlicz functions which completes the gap of the literature. We also introduce the concept of $S_\lambda^{u\Delta^m}$-statistical convergent sequences and give some inclusion relations between these defined spaces with the space of $S_\lambda^{u\Delta^m}$-statistical convergent sequences.

Key words: difference sequence, Orlicz function, dela Vallee-Poussin means, statistical convergence.

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1. Introduction

Let w be the set of all sequences of real or complex numbers and l_∞, c and c_0 be, respectively, the Banach spaces of bounded, convergent and null sequences $x = (x_k)$ with the usual norm $\|x\| = \sup_k |x_k|$.

Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to ∞ and $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$.

The generalized de la Vallee-Poussin mean is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k$$

where $I_n = [n - \lambda_n + 1, n]$.

A sequence $x = (x_k)$ is said to be (V, λ)-summable to a number l [1] if $t_n(x) \to l$ as $n \to \infty$. If $\lambda_n = n$, then (V, λ)-summability and strongly (V, λ)-summability reduce to $(C, 1)$-summability and $[C, 1]$-summability, respectively.

An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If the convexity of an Orlicz function M is replaced by

$$M(x + y) \leq M(x) + M(y)$$

then this function is called modulus function, defined and discussed by Ruckle [2] and Maddox [3].

Lindenstrauss and Tzafriri [4] used the idea of Orlicz function to construct the sequence space

$$l_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}.$$

The space l_M becomes a Banach space with the norm

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}$$

which is called an Orlicz sequence space. The space l_M is closely related to the space l_p which is an Orlicz sequence space with $M(x) = x^p$ for $1 \leq p < \infty$. Orlicz sequence spaces were introduced and studied by Parashar and Choudhary [5], Nuray and Gulcu [6], Bhardwaj and Singh [7] and many others.

It is well known that if M is a convex function and $M(0) = 0$, then $M(tx) \leq tM(x)$ for all t with $0 < t < 1$.

An Orlicz funtion M is said to satisfy Δ_2-condition for all values of u, if there exists constant $K > 0$, such that $M(2u) \leq KM(u)$ $(u \geq 0)$. The Δ_2-condition is equivalent to the inequality $M(Lu) \leq K.L.M(u)$ for all values of u and for $L > 1$ being satisfied [8].

The difference sequence space $X(\Delta)$ was introduced by Kızmaz [9] as follows:

$$X(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in X\}$$

for $X = l_\infty$, c and c_0 ; where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$.

The notion of difference sequence spaces was further generalized by Et and Colak [10] as follows:

$$X(\Delta^m) = \{x = (x_k) \in w : (\Delta^m x_k) \in X\}$$

for $X = l_\infty$, c and c_0, where $\Delta^m x_k = \Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}$ and $\Delta^0 x_k = x_k$ for all $k \in$. Taking $X = l_\infty(p)$, $c(p)$ and $c_0(p)$, these sequence spaces has been generalized by Et and Basarır [11].

The generalized difference has the following binomial representation:

$$\Delta^m x_k = \sum_{v=0}^{m} (-1)^v \binom{m}{v} x_{k+v}$$

for all $k \in \mathbb{N}$.

Subsequently, difference sequence spaces have been discussed by several authors [12], [13] and [14].

The following inequality will be used throughout this paper. Let $p = (p_k)$ be a sequence of positive real numbers with $0 < p_k \leq \sup p_k = G$, and let $D = \max\left(1, 2^{G-1}\right)$. Then for a_k, $b_k \in \mathbb{C}$, the set of complex numbers for all $k \in \mathbb{N}$, we have [15]

$$|a_k + b_k|^{p_k} \ \leq \ D\left\{|a_k|^{p_k} + |b_k|^{p_k}\right\}. \tag{1}$$

Now we introduce the following sequence spaces.

Let $M = (M_i)$ be a sequence of Orlicz functions, m be a positive integer and $u = (u_i)$ be any sequence such that $u_i \neq 0$ for all i, then we define:

$$w_0(\lambda, M_i, p, u, s)_{\Delta^m} = \{x \in w : \lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{i \in I_n} i^{-s} M_i\Big(\frac{|u_i \Delta^m x_i|}{\rho}\Big)^{p_i} = 0, \text{ for some } \ \rho > 0, \ \ s \geq 0\}$$

$$w(\lambda, M_i, p, u, s)_{\Delta^m} = \{x \in w : \lim_{n\to\infty} \frac{1}{\lambda_n} \sup_{i \in I_n} i^{-s} M_i\Big(\frac{|u_i \Delta^m x_i - le|}{\rho}\Big)^{p_i} = 0, \text{ for some } \ l, \rho > 0, \ \ s \geq 0\}$$

and

$$w_\infty(\lambda, M_i, p, u, s)_{\Delta^m} = \{x \in w : \sup_n \frac{1}{\lambda_n} \sup_{i \in I_n} i^{-s} M_i\Big(\frac{|u_i \Delta^m x_i|}{\rho}\Big)^{p_i} < \infty, \text{ for some } \ \rho > 0, \ \ s \geq 0\}$$

where $u_i \Delta^m x_i = (u_i \Delta^{m-1} x_i - u_{i+1} \Delta^{m-1} x_{i+1})$ such that $u_i \Delta^m x_i = \sum_{n=0}^{m} (-1)^n \times mn u_{i+n} x_{i+n}$, $u_i \Delta^0 x_i = (u_i x_i)$ and $u_i \Delta x_i = (u_i x_i - u_{i+1} x_{i+1})$.

Here for convenience, we put $M_i\big(\frac{|u_i \Delta^m x_i - le|}{\rho}\big)^{p_i}$ instead of $\big[M_i\big(\frac{|u_i \Delta^m x_i - le|}{\rho}\big)\big]^{p_i}$.

We get the following sequence spaces from the above sequence spaces on giving particular values to s, m, $M = (M_i)$, $p = (p_i)$, $u = (u_i)$ for all i.

(i) If $M_i(x) = x$ for all i, then we write $w_0(\lambda, p, u, s)_{\Delta^m}$, $w(\lambda, p, u, s)_{\Delta^m}$ and $w_\infty(\lambda, p, u, s)_{\Delta^m}$, respectively.

(ii) If $M_i(x) = x$ and $p = (p_i) = 1$ for all i, $u = e$, $s = 0$ and $m = 1$ then we write $w_0(\lambda)_\Delta$, $w_0(\lambda)_\Delta$ and $w_\infty(\lambda)_\Delta$, respectively.

2. Main results

In this section, we examine some topological properties of $w_0(\lambda, M_i, p, u, s)_{\Delta^m}$, $w(\lambda, M_i, p, u, s)_{\Delta^m}$ and $w_\infty(\lambda, M_i, p, u, s)_{\Delta^m}$ spaces and investigate some inclusion relations between these spaces.

Theorem 1. *Let (M_i) be a sequence of Orlicz functions, m be a positive integer and $p = (p_i)$ be a bounded sequence of strictly positive real numbers. Then $w_0(\lambda, M_i, p, u, s)_{\Delta^m}$, $w(\lambda, M_i, p, u, s)_{\Delta^m}$ and $w_\infty(\lambda, M_i, p, u, s)_{\Delta^m}$ are linear spaces over the set of complex numbers.*

Proof. It is easy to prove this theorem using (1). ■

Theorem 2. *Let (M_i) be a sequence of Orlicz functions and m be a positive integer. If $\sup_i (M_i(x))^{p_i} < \infty$ for all fixed $x > 0$ then*

$$w(\lambda, M_i, p, u, s)_{\Delta^m} \subset w_\infty(\lambda, M_i, p, u, s)_{\Delta^m}.$$

Proof. Let $x \in w(\lambda, M_i, p, u, s)_{\Delta^m}$. Then there exists some positive ρ_1 such that

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{i\in I_n} i^{-s} M_i \left(\frac{|u_i \Delta^m x_i - le|}{\rho_1} \right)^{p_i} = 0.$$

Define $\rho = 2\rho_1$. Since M_i is non-decreasing and convex, for each i, by using (1), we have

$$\begin{aligned} \sup_n \frac{1}{\lambda_n} \sum_{i\in I_n} i^{-s} M_i \left(\frac{|u_i \Delta^m x_i|}{\rho} \right)^{p_i} & \\ &= \sup_n \frac{1}{\lambda_n} \sum_{i\in I_n} i^{-s} M_i \left(\frac{|u_i \Delta^m x_i| - le + le}{\rho} \right)^{p_i} \\ &\le D \left\{ \sup_n \frac{1}{\lambda_n} \sum_{i\in I_n} i^{-s} \frac{1}{2^{p_i}} M_i \left(\frac{|u_i \Delta^m x_i| - le}{\rho_1} \right)^{p_i} \right. \\ &\quad \left. + \sup_n \frac{1}{\lambda_n} \sum_{i\in I_n} i^{-s} \frac{1}{2^{p_i}} M_i \left(\frac{|le|}{\rho_1} \right)^{p_i} \right\} < \infty. \end{aligned}$$

Hence $x \in w_\infty(\lambda, M_i, p, u, s)_{\Delta^m}$. This completes the proof. ■

Theorem 3. *Let (M_i) be a sequence of Orlicz functions, m be a positive integer and $0 < h = \inf p_i$. Then $w_\infty(\lambda, M_i, p, u, s)_{\Delta^m} \subset w_0(\lambda, p, u, s)_{\Delta^m}$ if and only if*

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{i\in I_n} i^{-s} M_i(t)^{p_i} = \infty \tag{2}$$

for some $t > 0$.

Proof. Let $x \in w_\infty(\lambda, M_i, p, u, s)_{\Delta^m} \subset w_0(\lambda, p, u, s)_{\Delta^m}$. Suppose that (2) does not hold. Therefore there is a subinterval $I_{n(k)}$ of the set of interval I_n and a number $t_0 > 0$, where $t_0 = \frac{|u_i\Delta^m x_i|}{\rho}$ for all i such that

$$\text{(3)} \qquad \frac{1}{\lambda_{n(k)}} \sum_{i\in I_{n(k)}} i^{-s} M_i(t_0)^{p_i} \ \le\ K\ <\ \infty, \qquad k = 1, 2, 3, \ldots .$$

Let us define $x = (x_i)$ as following

$$u_i\Delta^m x_i = \begin{cases} \rho t_0, & i \in I_{n(k)}\ , \\ 0, & i \notin I_{n(k)}\ . \end{cases}$$

Thus by (3), $x \in w_\infty(\lambda, M_i, p, u, s)_{\Delta^m}$. But $x \notin w_0(\lambda, p, u)_\Delta$. Hence (2) must hold.

Conversely, suppose that (2) holds and that $x \in w_\infty(\lambda, M_i, p, u, s)_{\Delta^m}$. Then for each n

$$\text{(4)} \qquad \frac{1}{\lambda_n} \sum_{i\in I_n} i^{-s} M_i \left(\frac{|u_i\Delta^m x_i|}{\rho}\right)^{p_i} \ \le\ K\ <\ \infty.$$

Suppose that $x \notin w_0(\lambda, p, u, s)_{\Delta^m}$. Then for some number $0 < \varepsilon < 1$, there is a number i_0 such that for a subinterval I_{n_1} of the set of interval I_n, $\frac{|u_i\Delta^m x_i|}{\rho} > \varepsilon$ for $i \ge i_0$. From properties of the Orlicz function, we can write $M_i(\frac{|u_i\Delta^m x_i|}{\rho})^{p_i} \ge M_i(\varepsilon)^{p_i}$ which contradicts (2), by using (4). Hence we get $x \in w_\infty(\lambda, M_i, p, u, s)_{\Delta^m} \subset w_0(\lambda, p, u, s)_{\Delta^m}$. This completes the proof. ∎

Theorem 4. *Let $0 < h = \inf p_i \le p_i \le \sup p_i = H < \infty$. Then for a sequence of Orlicz functions (M_i) which satisfies the Δ_2-condition, we have $w_0(\lambda, p, u, s)_{\Delta^m} \subset w_0(\lambda, M_i, p, u, s)_{\Delta^m}$, $w(\lambda, p, u, s)_{\Delta^m} \subset w(\lambda, M_i, p, u, s)_{\Delta^m}$ and $w_\infty(\lambda, p, u, s)_{\Delta^m} \subset w_\infty(\lambda, M_i, p, u, s)_{\Delta^m}$.*

Proof. Let $x \in w(\lambda, p, u, s)_{\Delta^m}$. Then we have

$$\frac{1}{\lambda_n} \sum_{i\in I_n} i^{-s} M_i \left(\frac{|u_i\Delta^m x_i - le|}{\rho}\right)^{p_i} \to 0 \text{ as } n \to \infty, \text{ for some } l.$$

Let $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M_i(t) < \varepsilon$ for $0 \le t \le \delta$. Then we can write

$$\frac{1}{\lambda_n} \sum_{i\in I_n} i^{-s} M_i \left(\frac{|u_i\Delta^m x_i - le|}{\rho}\right)^{p_i} = \frac{1}{\lambda_n} \sum_{\substack{i\in I_n \\ \frac{|u_i\Delta^m x_i - le|}{\rho} \le \delta}} i^{-s} M_i \left(\frac{|u_i\Delta^m x_i - le|}{\rho}\right)^{p_i}$$

$$+ \frac{1}{\lambda_n} \sum_{\substack{i\in I_n \\ \frac{|u_i\Delta^m x_i - le|}{\rho} > \delta}} i^{-s} M_i \left(\frac{|u_i\Delta^m x_i - le|}{\rho}\right)^{p_i} = \sum\nolimits_1 + \sum\nolimits_2 .$$

For the first summation above, we immediately write

$$\sum\nolimits_1 = \frac{1}{\lambda_n} \sum_{\substack{i \in I_n \\ \frac{|u_i \Delta^m x_i - le|}{\rho} \leq \delta}} i^{-s} M_i \left(\frac{|u_i \Delta^m x_i - le|}{\rho} \right)^{p_i} < \max(\varepsilon, \varepsilon^H)$$

by using continuity of M_i. For the second summation, we will make the following procedure. We have

$$\left(\frac{|u_i \Delta^m x_i - le|}{\rho} \right) < 1 + \frac{\left(\frac{|u_i \Delta^m x_i - le|}{\rho} \right)}{\delta}.$$

Since M_i is non-decreasing and convex, it follows that

$$\begin{aligned} M_i \left(\frac{|u_i \Delta^m x_i - le|}{\rho} \right) &< M_i \left\{ 1 + \frac{\left(\frac{|u_i \Delta^m x_i - le|}{\rho} \right)}{\delta} \right\} \\ &\leq \frac{1}{2} M_i(2) + \frac{1}{2} M_i \left\{ 2 \frac{\left(\frac{|u_i \Delta^m x_i - le|}{\rho} \right)}{\delta} \right\}. \end{aligned}$$

Since M_i satisfies the Δ_2-condition, we can write

$$\begin{aligned} M_i \left(\frac{|u_i \Delta^m x_i - le|}{\rho} \right) &\leq \frac{1}{2} L \left\{ \frac{\left(\frac{|u_i \Delta^m x_i - le|}{\rho} \right)}{\delta} \right\} M_i(2) \\ + \frac{1}{2} L \left\{ \frac{\left(\frac{|u_i \Delta^m x_i - le|}{\rho} \right)}{\delta} \right\} M_i(2) &= L \left\{ \frac{\left(\frac{|u_i \Delta^m x_i - le|}{\rho} \right)}{\delta} \right\} M_i(2). \end{aligned}$$

In this way, we write

$$\sum\nolimits_2 = \max \left\{ 1, \left[\frac{L M_i(2)}{\delta} \right]^H \right\} \frac{1}{\lambda_n} \sum_{i \in I_n} i^{-s} \left(\frac{|u_i \Delta^m x_i - le|}{\rho} \right)^{p_i}.$$

Taking the limit as $\varepsilon \to 0$ and $n \to \infty$, it follows that $x \in w(\lambda, M_i, p, u, s)_{\Delta^m}$. Following similar arguments, we can prove that $w_0(\lambda, p, u, s)_{\Delta^m} \subset w_0(\lambda, M_i, p, u, s)_{\Delta^m}$ and $w_\infty(\lambda, p, u, s)_{\Delta^m} \subset w_\infty(\lambda, M_i, p, u, s)_{\Delta^m}$. ■

Theorem 5. *Let (M_i) be a sequence of Orlicz functions. Then the following statements are equivalent:*

(i) $w_\infty(\lambda, p, u, s)_{\Delta^m} \subset w_\infty(\lambda, M_i, p, u, s)_{\Delta^m}$,
(ii) $w_0(\lambda, p, u, s)_{\Delta^m} \subset w_\infty(\lambda, M_i, p, u, s)_{\Delta^m}$,
(iii) $\sup_n \frac{1}{\lambda_n} \sum_{i \in I_n} i^{-s} M_i(t)^{p_i} < \infty$ *for all* $t > 0$.

Proof. $(i) \Rightarrow (ii)$: Let (i) holds. To verify (ii), it is enough to prove $w_0(\lambda, p, u, s)_{\Delta^m} \subset w_\infty(\lambda, p, u, s)_{\Delta^m}$. Let $x \in w_0(\lambda, p, u, s)_{\Delta^m}$. Then there exists $n \geq n_0$, for $\varepsilon > 0$, such that

$$\frac{1}{\lambda_n} \sum_{i \in I_n} i^{-s} \left(\frac{|u_i \Delta^m x_i|}{\rho} \right)^{p_i} < \varepsilon.$$

Hence there exists $K > 0$ such that

$$\sup_n \frac{1}{\lambda_n} \sum_{i \in I_n} i^{-s} \left(\frac{|u_i \Delta^m x_i|}{\rho} \right)^{p_i} < K.$$

So we get $x \in w_\infty(\lambda, p, u, s)_{\Delta^m}$.

$(ii) \Rightarrow (iii)$: Let (ii) holds. Suppose (iii) does not hold. Then for some $t > 0$

$$\sup_n \frac{1}{\lambda_n} \sum_{i \in I_n} i^{-s} M_i(t)^{p_i} = \infty$$

and we can find a subinterval $I_{n(k)}$ of the set of interval I_n such that

$$\frac{1}{\lambda_{n(k)}} \sum_{i \in I_{n(k)}} i^{-s} M_i \left(\frac{1}{r} \right)^{p_i} > r, \qquad r = 1, 2, 3, \ldots . \tag{5}$$

Let us define $x = (x_i)$ as following

$$u_i \Delta^m x_i = \begin{cases} \frac{\rho}{r}, & i \in I_{n(k)}, \\ 0, & i \notin I_{n(k)}. \end{cases}$$

Then $x \in w_0(\lambda, p, u, s)_{\Delta^m}$, but by (5) $x \notin w_\infty(\lambda, M_i, p, u, s)_{\Delta^m}$, which contradicts (ii). Hence (iii) must holds.

$(iii) \Rightarrow (i)$: Let (iii) holds and $x \in w_\infty(\lambda, p, u, s)_{\Delta^m}$. Suppose that $x \notin w_\infty(\lambda, M_i, p, u, s)_{\Delta^m}$. Then for $x \in w_\infty(\lambda, p, u, s)_{\Delta^m}$

$$\sup_n \frac{1}{\lambda_n} \sum_{i \in I_n} i^{-s} M_i \left(\frac{|u_i \Delta^m x_i|}{\rho} \right)^{p_i} = \infty. \tag{6}$$

Let $t = \frac{|u_i \Delta^m x_i|}{\rho}$ for each i, then by (6)

$$\sup_n \frac{1}{\lambda_n} \sum_{i \in I_n} i^{-s} M_i(t)^{p_i} = \infty$$

which contradicts (iii). Hence (i) must holds. ■

Theorem 6. *Let (M_i) be a sequence of Orlicz functions. Then the following statements are equivalent:*

(i) $w_0(\lambda, M_i, p, u, s)_{\Delta^m} \subset w_0(\lambda, p, u, s)_{\Delta^m}$,

(ii) $w_0(\lambda, M_i, p, u, s)_{\Delta^m} \subset w_\infty(\lambda, p, u, s)_{\Delta^m}$,

(iii) $\inf_n \frac{1}{\lambda_n} \sum_{i\in I_n} i^{-s} M_i(t)^{p_i} < \infty$ *for all* $t > 0$.

Proof. *(i)* $\Rightarrow$ *(ii)*: It is obvious.

(ii) $\Rightarrow$ *(iii)*: Let *(ii)* holds. Suppose *(iii)* does not hold. Then

$$\inf_n \frac{1}{\lambda_n} \sum_{i\in I_n} i^{-s} M_i(t)^{p_i} = 0 \quad \text{for some } t > 0$$

and we can find a subinterval $I_{n(k)}$ of the set of interval I_n such that

$$\frac{1}{\lambda_{n(k)}} \sum_{i\in I_{n(k)}} i^{-s} M_i(r)^{p_i} < \frac{1}{r}, \qquad r = 1, 2, 3, \ldots . \tag{7}$$

Let us define $x = (x_i)$ as following

$$u_i \Delta^m x_i = \begin{cases} \rho r, & i \in I_{n(k)}, \\ 0, & i \notin I_{n(k)}. \end{cases}$$

Thus by (7), $x \in w_0(\lambda, p, u, s)_{\Delta^m}$, but $x \notin w_\infty(\lambda, p, u, s)_{\Delta^m}$ which contradicts *(ii)*. Hence *(iii)* must holds.

(iii) $\Rightarrow$ *(i)*: Let *(iii)* holds. Suppose that $x \in w_0(\lambda, M_i, p, u, s)_{\Delta^m}$. Then

$$\frac{1}{\lambda_n} \sum_{i\in I_n} i^{-s} M_i \left(\frac{|u_i \Delta^m x_i|}{\rho} \right)^{p_i} \to 0 \text{ as } n \to \infty. \tag{8}$$

Again suppose that $x \notin w_0(\lambda, p, u, s)_{\Delta^m}$. Then for some number $\varepsilon > 0$ and a subinterval $I_{n(k)}$ of the set of interval I_n, we have $\frac{|u_i \Delta^m x_i|}{\rho} > \varepsilon$ for all i. From properties of the Orlicz function, we can write $M_i(\frac{|u_i \Delta^m x_i|}{\rho})^{p_i} \geq M_i(\varepsilon)^{p_i}$.

Consequently by (8), we have

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{i\in I_n} i^{-s} M_i(\varepsilon)^{p_i} = 0$$

which contradicts *(iii)*. Hence *(i)* must holds. ■

Theorem 7. *Let (p_i) be any bounded sequences of strictly positive real numbers. Then*

(i) If $0 < \inf p_i \leq p_i \leq 1$ *for all* i*, then*

$$w(\lambda, M_i, u, s)_{\Delta^m} \subseteq w(\lambda, M_i, p, u, s)_{\Delta^m},$$

(ii) If $1 \leq p_i \leq \sup p_i = H < \infty$ *then*

$$w(\lambda, M_i, p, u, s)_{\Delta^m} \subseteq w(\lambda, M_i, u, s)_{\Delta^m}.$$

Proof. (i) Let $x \in w(\lambda, M_i, u, s)$. Then since $0 < \inf p_i \le p_i \le 1$, we get

$$\frac{1}{\lambda_n} \sum_{i \in I_n} i^{-s} M_i \left(\frac{|u_i \Delta^m x_i|}{\rho} \right)^{p_i} \le \frac{1}{\lambda_n} \sum_{i \in I_n} i^{-s} M_i \left(\frac{|u_i \Delta^m x_i|}{\rho} \right)$$

and hence $x \in w(\lambda, M_i, p, u, s)_{\Delta^m}$.

(ii) Let $1 \le p_i \le \sup p_i = H < \infty$ and $x \in w(\lambda, M_i, p, u, s)_{\Delta^m}$. Then for each $0 < \varepsilon < 1$, there exists a positive integer n_0 such that

$$\frac{1}{\lambda_n} \sum_{i \in I_n} i^{-s} M_i \left(\frac{|u_i \Delta^m x_i|}{\rho} \right) \le \varepsilon < 1$$

for all $n \ge n_0$. This implies that

$$\frac{1}{\lambda_n} \sum_{i \in I_n} i^{-s} M_i \left(\frac{|u_i \Delta^m x_i|}{\rho} \right) \le \frac{1}{\lambda_n} \sum_{i \in I_n} i^{-s} M_i \left(\frac{|u_i \Delta^m x_i|}{\rho} \right)^{p_i}.$$

Therefore $x \in w(\lambda, M_i, u, s)$. ■

Theorem 8. *Let* $0 < p_i \le q_i$ *for all* i *and let* $(\frac{p_i}{q_i})$ *be bounded. Then* $w(\lambda, M_i, q, u, s)_{\Delta^m} \subseteq w(\lambda, M_i, p, u, s)_{\Delta^m}$.

Proof. Let $x \in w(\lambda, M_i, q, u, s)_{\Delta^m}$. Write $t_i = \left[M_i (\frac{|u_i \Delta^m x_i - le|}{\rho}) \right]^{q_i}$ and $\mu_i = \frac{p_i}{q_i}$ for all $i \in \mathbb{N}$. Then $0 < \mu_i \le 1$ for all $i \in \mathbb{N}$. Take $0 < \mu < \mu_i$ for all $i \in \mathbb{N}$. Define the sequences (u_i) and (v_i) as follows:

For $t_i \ge 1$, let $u_i = t_i$ and $v_i = 0$ and for $t_i < 1$, let $u_i = 0$ and $v_i = t_i$.

Then clearly for all $i \in \mathbb{N}$ we have $t_i = u_i + v_i$, $t_i^{\mu_i} = u_i^{\mu_i} + v_i^{\mu_i}$.Now it follows that $u_i^{\mu_i} \le u_i \le t_i$ and $v_i^{\mu_i} \le v_i^{\mu}$.

Therefore

$$\frac{1}{\lambda_n} \sum_{i \in I_n} t_i^{\mu_i} = \frac{1}{\lambda_n} \sum_{i \in I_n} (u_i^{\mu_i} + v_i^{\mu_i}) \le \frac{1}{\lambda_n} \sum_{i \in I_n} t_i + \frac{1}{\lambda_n} \sum_{i \in I_n} v_i^{\mu}.$$

Now for each i,

$$\begin{aligned} \frac{1}{\lambda_n} \sum_{i \in I_n} v_i^{\mu} &= \sum_{i \in I_n} \left(\frac{1}{\lambda_n} v_i \right)^{\mu} \left(\frac{1}{\lambda_n} \right)^{1-\mu} \\ &\le \left(\sum_{i \in I_n} \left[\left(\frac{1}{\lambda_n} v_i \right)^{\mu} \right]^{\frac{1}{\mu}} \right)^{\mu} \left(\sum_{i \in I_n} \left[\left(\frac{1}{\lambda_n} \right)^{1-\mu} \right]^{\frac{1}{1-\mu}} \right)^{1-\mu} \\ &= \left(\frac{1}{\lambda_n} \sum_{i \in I_n} v_i \right)^{\mu} \end{aligned}$$

and so

$$\frac{1}{\lambda_n}\sum_{i\in I_n} t_i^{\mu_i} \;\le\; \frac{1}{\lambda_n}\sum_{i\in I_n} t_i + \left(\frac{1}{\lambda_n}\sum_{i\in I_n} v_i\right)^{\mu}.$$

Hence $x \in w(\lambda, M_i, p, u, s)_{\Delta^m}$. ■

3. $S_\lambda^{u\Delta^m}$-statistical convergence

In this section, we introduce the concept of $S_\lambda^{u\Delta^m}$-statistical convergence and give some inclusion relations related to these sequence spaces.

The notion of statistical convergence was introduced by Fast [16] and was studied by [17], [18], [19] and [20].

Definition 1. *A sequence is said to be $S_\lambda^{u\Delta^m}$-statistically convergent to l, if for any $\varepsilon > 0$*

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \left|\{i \in I_n : |u_i\Delta^m x_i - le| \ge \varepsilon\}\right| \;=\; 0.$$

In this case, we write $S_\lambda^{u\Delta^m} - \lim x$ or $x_i \to L(S_\lambda^{u\Delta^m})$ and $S_\lambda^{u\Delta^m} = \left\{x = (x_i) : S_\lambda^{u\Delta^m} - \lim x = l, \text{for some } l\right\}$.

If $\lambda_n = n$, we shall write $S^{u\Delta^m}$ instead of $S_\lambda^{u\Delta^m}$.

Let Λ denote the set of all non-decreasing sequences $\lambda = (\lambda_n)$ of positive numbers tending to ∞ such that $\lambda_{n+1} \le \lambda_n + 1$ and $\lambda_1 = 1$.

The proof of Theorem 9 and Theorem 10 are obtained by using the same techniques of Mursaleen [19].

Theorem 9. *Let $\lambda \in \Lambda$ and $u = (u_i) \in l_\infty$, then*

(i) $x_i \to L(w(\lambda, u)_{\Delta^m}) \Rightarrow x_i \to L(S_\lambda^{u\Delta^m})$ and the inclusion is strict,

(ii) If $x \in l_\infty(\Delta^m)$ and $x_i \to L(S_\lambda^{u\Delta^m})$ then $x_i \to L(w(\lambda, u)_{\Delta^m})$ and hence $x_i \to L(w(u)_{\Delta^m})$ provided $x = (x_i)$ is not eventually constant,

(iii) $\left\{S_\lambda^{u\Delta^m} \cap l_\infty(\Delta^m)\right\} = \{w(\lambda, u)_{\Delta^m} \cap l_\infty(\Delta^m)\}$

where $l_\infty(\Delta^m) = \{x \in w : (\Delta^m x_i) \in l_\infty,\ m \in \mathbb{N}\}$.

Proof. *(i)* Let $\varepsilon > 0$ be given and $(x_i) \in w(\lambda, u)_{\Delta^m}$. Then we can write

$$\begin{aligned}\frac{1}{\lambda_n}\sum_{i\in I_n} |u_i\Delta^m x_i - le| \;&\ge\; \frac{1}{\lambda_n}\sum_{\substack{i\in I_n \\ |u_i\Delta^m x_i - le|\ge\varepsilon}} |u_i\Delta^m x_i - le| \\ &\ge\; \frac{1}{\lambda_n}\left|\{i \in I_n : |u_i\Delta^m x_i - le| \ge \varepsilon\}\right| \varepsilon.\end{aligned}$$

Hence $x_i \to L(S_\lambda^{u\Delta^m})$.

(ii) Let $x \in l_\infty(\Delta^m)$, $u \in l_\infty$, $x_i \to L(S_\lambda^{u\Delta^m})$ and say $|u_i\Delta^m x_i - le| \le M$ for all i. Given $\varepsilon > 0$, we have

$$\begin{aligned}\frac{1}{\lambda_n}\sum_{i\in I_n}|u_i\Delta^m x_i - le| &= \frac{1}{\lambda_n}\sum_{\substack{i\in I_n\\ |u_i\Delta^m x_i - le|\ge\varepsilon}}|u_i\Delta^m x_i - le| \\ &\quad + \frac{1}{\lambda_n}\sum_{\substack{i\in I_n\\ |u_i\Delta^m x_i - le|<\varepsilon}}|u_i\Delta^m x_i - le| \\ &\le \frac{M}{\lambda_n}\left|\{i\in I_n : |u_i\Delta^m x_i - le|\ge\varepsilon\}\right| + \varepsilon\end{aligned}$$

which implies that $x_i \to L(w(\lambda,u)_{\Delta^m})$.

Further, we have

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n}|u_i\Delta^m x_i - le| &\le \frac{1}{n}\sum_{i=1}^{n-\lambda_n}|u_i\Delta^m x_i - le| + \frac{1}{n}\sum_{i\in I_n}|u_i\Delta^m x_i - le| \\ &\le \frac{1}{\lambda_n}\sum_{i=1}^{n-\lambda_n}|u_i\Delta^m x_i - le| + \frac{1}{\lambda_n}\sum_{i\in I_n}|u_i\Delta^m x_i - le| \\ &\le \frac{2}{\lambda_n}\sum_{i\in I_n}|u_i\Delta^m x_i - le|\,.\end{aligned}$$

Hence $x_i \to L(w(u)_{\Delta^m})$, since $x_i \to L(w(\lambda,u)_{\Delta^m})$.

(iii) This immediately follows from (i) and (ii). ∎

It is easily seen that $S_\lambda^{u\Delta^m} \subset S^{u\Delta^m}$ for all λ, since $\frac{\lambda_n}{n}$ is bounded.
Now, we prove the following relation.

Theorem 10. *If* $\liminf_n \frac{\lambda_n}{n} > 0$ *then* $S^{u\Delta^m} \subset S_\lambda^{u\Delta^m}$.

Proof. For given $\varepsilon > 0$ we have

$$\{i \le n : |u_i\Delta^m x_i - le| \ge \varepsilon\} \supset \{i \in I_n : |u_i\Delta^m x_i - le| \ge \varepsilon\}\,.$$

Therefore

$$\begin{aligned}\frac{1}{n}\left|\{i \le n : |u_i\Delta^m x_i - le| \ge \varepsilon\}\right| &\ge \frac{1}{n}\left|\{i \in I_n : |u_i\Delta^m x_i - le| \ge \varepsilon\}\right| \\ &= \frac{\lambda_n}{n}\frac{1}{\lambda_n}\left|\{i \in I_n : |u_i\Delta^m x_i - le| \ge \varepsilon\}\right|\,.\end{aligned}$$

Taking the limit as $n \to \infty$ and using $\liminf_n \frac{\lambda_n}{n} > 0$, we get

$$x_i \to L\left(S^{u\Delta^m}\right) \Rightarrow x_i \to L\left(S_\lambda^{u\Delta^m}\right).$$

∎

Theorem 11. *Let (M_i) be a sequence of Orlicz functions, $0 < h = \inf p_i \le p_i \le \sup p_i = H < \infty$ and $u = (u_i) \in l_\infty$. Then $w(\lambda, M_i, p, u)_{\Delta^m} \subset S_\lambda^{u\Delta^m}$.*

Proof. Let $x \in w(\lambda, M_i, , p, u)_{\Delta^m}$. Then there exists $\rho > 0$ such that

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{i\in I_n} M_i \left(\frac{|u_i \Delta^m x_i - le|}{\rho} \right)^{p_i} = 0.$$

Then

$$\begin{aligned}
\frac{1}{\lambda_n}\sum_{i\in I_n} M_i \left(\frac{|u_i \Delta^m x_i - le|}{\rho} \right)^{p_i} &= \frac{1}{\lambda_n} \sum_{\substack{i\in I_n \\ |u_i\Delta^m x_i - le| \ge \varepsilon}} M_i \left(\frac{|u_i \Delta^m x_i - le|}{\rho} \right)^{p_i} \\
&+ \frac{1}{\lambda_n} \sum_{\substack{i\in I_n \\ |u_i\Delta^m x_i - le| < \varepsilon}} M_i \left(\frac{|u_i \Delta^m x_i - le|}{\rho} \right)^{p_i} \\
&\ge \frac{1}{\lambda_n} \sum_{\substack{i\in I_n \\ |u_i\Delta^m x_i - le| \ge \varepsilon}} M_i \left(\frac{|u_i \Delta^m x_i - le|}{\rho} \right)^{p_i} \\
&\ge \frac{1}{\lambda_n} \sum_{\substack{i\in I_n \\ |u_i\Delta^m x_i - le| \ge \varepsilon}} M_i(\varepsilon_1)^{p_i} \quad (\text{where} \frac{\varepsilon}{\rho} = \varepsilon_1) \\
&\ge \frac{1}{\lambda_n} \sum_{\substack{i\in I_n \\ |u_i\Delta^m x_i - le| \ge \varepsilon}} \min \left\{ M_i(\varepsilon_1)^{\inf p_i}, M_i(\varepsilon_1)^H \right\} \\
&\ge \frac{1}{\lambda_n} \left| \{ i \in I_n : |u_i \Delta^m x_i - le| \ge \varepsilon \} \right| \min \left\{ M_i(\varepsilon_1)^{\inf p_i}, M_i(\varepsilon_1)^H \right\}.
\end{aligned}$$

Hence $x \in S_\lambda^{u\Delta^m}$. ■

Theorem 12. *Let $0 < h = \inf p_i \le p_i \le \sup p_i = H < \infty$ and $u = (u_i) \in l_\infty$. Then*

$$\left\{ S_\lambda^{u\Delta^m} \cap l_\infty(\Delta^m) \right\} = \left\{ w(\lambda, M_i, p, u)_{\Delta^m} \cap l_\infty(\Delta^m) \right\}.$$

Proof. By Theorem 11, we need only show that

$$\left\{ S_\lambda^{u\Delta^m} \cap l_\infty(\Delta^m) \right\} \subset \left\{ w(\lambda, M_i, p, u)_{\Delta^m} \cap l_\infty(\Delta^m) \right\}.$$

Let $y_i = |u_i \Delta^m x_i - le| \to \theta(S_\lambda)$.

Since $x \in l_\infty(\Delta^m)$ and $u \in l_\infty$, so there exists $K > 0$ such that

$$M_i\left(\frac{y_i}{\rho}\right) \leq K.$$

Then for a given $\varepsilon > 0$ and for each $n \in \mathbb{N}$, we have

$$\begin{aligned}\frac{1}{\lambda_n}\sum_{i\in I_n} M_i\left(\frac{|u_i\Delta^m x_i - le|}{\rho}\right)^{p_i} &= \frac{1}{\lambda_n}\sum_{\substack{i\in I_n\\ |u_i\Delta^m x_i - le|\geq\varepsilon}} M_i\left(\frac{|u_i\Delta^m x_i - le|}{\rho}\right)^{p_i}\\ &\quad + \frac{1}{\lambda_n}\sum_{\substack{i\in I_n\\ |u_i\Delta^m x_i - le|<\varepsilon}} M_i\left(\frac{|u_i\Delta^m x_i - le|}{\rho}\right)^{p_i}\\ &\leq \max\left(K^h K^H\right)\frac{1}{\lambda_n}\left|\{i \in I_n : |u_i\Delta^m x_i - le| \geq \varepsilon\}\right|\\ &\quad + \max\left(\left[M_i\left(\frac{\varepsilon}{\rho}\right)\right]^h, \left[M_i\left(\frac{\varepsilon}{\rho}\right)\right]^H\right).\end{aligned}$$

Hence $x \in \{w(\lambda, M_i, p, u)_{\Delta^m} \cap l_\infty(\Delta^m)\}$. ■

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