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Gülen Başcanbaz-Tunca and Yalcin Tuncer

SOME PROPERTIES OF MULTIVARIATE BETA OPERATOR

Abstract. This work relates to multivariate beta operator which is expressed shortly as B_n. We show that the operator preserves Lipschitz constant of a Lipschitz continuous function and semi-additivity of the relevant operand. Furthermore, we provide an r-th order generalisation, $B_n^{[r]}$, of B_n.

Key words: multivariate beta operator, Dirichlet distribution, Lipschitz constant, modulus of continuity function.

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1. Introduction

The univariate linear positive beta operator is given by

$$B_n^*(f;x) = \int_0^1 \frac{t^{nx-1}(1-t)^{n(1-x)-1}}{B(nx, n(1-x))} f(t)\,dt, \tag{1}$$

whose kernel is the well-known beta probability density function with the support $(0,1)$ such that t denotes a value of the generic (random) variable T, where $n \in \mathbb{N}$, $0 < x < 1$ and f is any real measurable, Lebesgue integrable function defined on $[0,1]$. Corresponding to $x = 0$ and $x = 1$, we set $B_n^*(f;x) = f(x)$ for all n. Khan [8] considers the beta operator $B_n^*(f;x)$ for $f \in C[0,1]$, $C[0,1] := \{f : f \text{is real valued, continuous and defined on } [0,1]\}$. Various results related to beta operator can be found in [2], [10] and references therein. Another form of beta operator which is slightly different from (1) can be found in [13], and yet a different type of beta operator is due to Upreti [15]. The kernel of the latter consists of the probability density function of the well-known F distribution of Probability Theory.

We can interpret and express (1) alternately in probabilistic terms. In fact, the operator (1) corresponds to mathematical expectation operator usually denoted as $E[f]$ for $f \in C[0,1]$ in Probability Theory (see, Feller [7, p.107]). Provided it exists, the operator $E[f]$ applies actually to all probability distributions irrespective of their types and dimensions. We shall not however follow this line of approach in this work any further.

We give first some notation which will be used throughout the paper. Let $\mathbf{x}=(x_1,...,x_k)$, $\mathbf{t}=(t_1,...,t_k)\in\mathbb{R}^k$, $k\in\mathbb{N}$, $\mathbf{m}=(m_1,...,m_k)\in\mathbb{N}_0^k$ and $n\in\mathbb{N}$. For $\boldsymbol{\alpha}=(\alpha_1,...,\alpha_k)<\boldsymbol{\beta}=(\beta_1,...,\beta_k)\in\mathbb{R}^k$,

$$\int_{\boldsymbol{\alpha}}^{\boldsymbol{\beta}}\mathbf{d}\boldsymbol{\xi} = \int_{\alpha_1}^{\beta_1}...\int_{\alpha_k}^{\beta_k}d\xi_k\,...\,d\xi_1,$$

stands for the k-fold integration, where, for $\boldsymbol{\alpha}=(\alpha_1,...,\alpha_k)$, $\boldsymbol{\beta}=(\beta_1,...,\beta_k)$ $\in\mathbb{R}^k$, $\boldsymbol{\alpha}\le\boldsymbol{\beta}$ means $\alpha_i\le\beta_i$, $i=1,...,k$. Furthermore, we set $\mathbf{x}^{\mathbf{m}}=\prod_{i=1}^{k}x_i^{m_i}$ and $|\mathbf{x}|=\sum_{i=1}^{k}x_i$.

Let $S_k\subset\mathbb{R}^k$, $k\in\mathbb{N}$, be the standard simplex defined as

$$S_k = \left\{\boldsymbol{\xi}=(\xi_1,...,\xi_k)\in\mathbb{R}^k : \xi_i\ge 0,\ i=1,...,k,\ |\boldsymbol{\xi}|\le 1\right\}.$$

We also refer below to an open simplex which will be denoted by S_k^0 and defined as

$$S_k^0 := \left\{\boldsymbol{\xi}=(\xi_1,...,\xi_k)\in\mathbb{R}^k : \xi_i> 0,\ i=1,...,k,\ |\boldsymbol{\xi}|< 1\right\}.$$

We consider next a linear positive multivariate beta operator for $f\in C(S_k)$, i.e., the set of all rel valued, continuous functions on S_k. This operator is defined as

$$(2)\qquad B_n(f;\mathbf{x}) = \int_0^1\int_0^{1-t_1}...\int_0^{1-t_1-...-t_{k-1}}\Psi_{n,\mathbf{x}}(\mathbf{t})\,f(t_1,...,t_k)\,dt_k...dt_2dt_1,$$

where

$$(3)\qquad \Psi_{n,\mathbf{x}}(\mathbf{t}) = \frac{\Gamma(n)\,t_1^{nx_1-1}...t_k^{nx_k-1}(1-t_1-...-t_k)^{n(1-x_1-...-x_k)-1}}{\Gamma(nx_1)...\Gamma(nx_k)\,\Gamma(n(1-x_1-...-x_k))}$$

is the k-variate Dirichlet distribution with parameters.

$$(n,\ x_1,...,x_k,\ 1-|\mathbf{x}|),$$

$n\in\mathbb{N}$, $\mathbf{x}\in S_k^0$, and $\mathbf{t}\in S_k$. Similar to the univariate case, we shall let $B_n(f;\mathbf{x}) := f(\mathbf{x})$. Note that if $k=1$, then the Dirichlet distribution reduces to the beta distribution. It must be noted further at this point that (2) is not a natural tensor product generalisation of the univariate beta operator given by (1). With the notation given above, a compact form of (3) can be expressed as

$$\Psi_{n,\mathbf{x}}(\mathbf{t}) = \frac{\mathbf{t}^{n\mathbf{x}-\mathbf{1}}(1-|\mathbf{t}|)^{n(1-|\mathbf{x}|)-1}}{\mathbf{B}(n\mathbf{x}, n(1-|\mathbf{x}|))}, \tag{4}$$

where $\mathbf{B}$ is the multinomial beta function. Hence, for simplicity, the multivariate beta operator defined in (2) can be represented as

$$B_n(f;\mathbf{x}) = \int\limits_{S_k} \Psi_{n,\mathbf{x}}(\mathbf{t}) f(\mathbf{t})\, \mathbf{dt}. \tag{5}$$

On the other hand, it is hardly difficult to show that

$$B_n(1;\mathbf{x}) = \int\limits_{S_k} \Psi_{n,\mathbf{x}}(\mathbf{t})\, \mathbf{dt} = 1 \tag{6}$$

(see [17], p. 177-178). Moreover, it is likewise easy to show that

$$B_n(t_i;\mathbf{x}) = x_i, \; B_n\left(t_i^2;\mathbf{x}\right) = \frac{nx_i^2 + x_i}{n+1} \tag{7}$$

for $i = 1, ..., k$.

By virtue of the multivariate extension of the Bohman-Korovkin theorem (see [16]) it readily follows that

Theorem 1. *For $f \in C(S_k)$, $\|B_n - f\|_{C(S_k)} \to 0$ as $n \to \infty$.*

Recall that a continuous function f from $A \subseteq R^k$ into R is said to be Lipschitz continuous of order μ, $\mu \in (0,1]$, if there exists a constant $M > 0$ such that for every $\mathbf{x} = (x_1, ..., x_k)$, $\mathbf{y} = (y_1, ..., y_k) \in A$, f satisfies

$$|f(\mathbf{x}) - f(\mathbf{y})| \le M \sum_{i=1}^{k} |x_i - y_i|^{\mu}.$$

The set of Lipschitz continuous functions defined above is denoted by $Lip_M(\mu, A)$.

Also, a continuous non-negative function $\omega(\mathbf{u})$ defined in S_k is called the modulus of continuity type function, if it satisfies the following conditions ([14]):

1. $\omega(\mathbf{0}) = 0$, *here* $\mathbf{0} = (0, 0, ..., 0)$,
2. $\omega(\mathbf{u})$ *is non-decreasing in* $\mathbf{u}$, *i.e.;* $\omega(\mathbf{u}) \ge \omega(\mathbf{v})$ *for* $\mathbf{u} \ge \mathbf{v}$,
3. $\omega(\mathbf{u})$ *is semi-additive, i.e.;* $\omega(\mathbf{u} + \mathbf{v}) \le \omega(\mathbf{u}) + \omega(\mathbf{v})$.

In this work we show that the multivariate beta operator B_n preserves the properties of the modulus of continuity type function and Lipschitz condition for a given function. We design a generalisation $B_n^{[r]}$ that extends the idea of Kirov and Popova in [11] to the multivariate beta operator. We also investigate the approximation of $B_n^{[r]}$.

2. Preservation of some properties by B_n

As it is known, the bearing of the modulus of continuity of a continuous function shows the global smoothness properties of the function. Hence, the problem whether the approximation operator is able to preserve the Lipschitz constant of a given Lipschitz continuous function arises naturally. Briefly, the first elementary proof related to this problem was given by Brown, Elliott and Paget for the univariate Bernstein polynomials in their elegant work [4]. Their work is a continuation of the work of Bloom and Elliott [3]. The same problem was also investigated for some classes of univariate and multivariate operators by Khan and Peters in [9] by probabilistic point of view. For more information concerning this subject we refer to the book of Anastassiou and Gal [1]. Cal and Valle obtained best constants in global smoothness preservation inequalities for multivariate beta operator also some other multivariate operators by probabilistic approach [5-6].

We should note here that the preservation of the modulus of continuity function by Bernstein polynomials is proved by Li in [12]. Also preservation of Lipschitz constant for univariate beta operator is shown by Khan in [8].

To show that multivariate beta operator preserves similarly the modulus of continuity type function and Lipschitz condition, let D be a set defined by

$$D := \left\{(\boldsymbol{\alpha},\boldsymbol{\beta}) : \boldsymbol{\alpha} \geq 0,\ \boldsymbol{\beta} \geq 0,\ |\boldsymbol{\alpha}| + |\boldsymbol{\beta}| \leq 1,\ \boldsymbol{\alpha},\boldsymbol{\beta} \in \mathbb{R}^k\right\},$$

$\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_k)$, $\boldsymbol{\beta} = (\beta_1, ..., \beta_k)$ and $\mathbf{x} < \mathbf{y},\ \mathbf{x},\mathbf{y} \in S_k^0$. We then have an auxillary function

$$h_{n,\mathbf{x},\mathbf{y}}(\boldsymbol{\alpha},\boldsymbol{\beta}) := \frac{\Gamma(n)\,\alpha_1^{nx_1-1}...\alpha_k^{nx_k-1}\beta_1^{n(y_1-x_1)-1}....}{\Gamma(nx_1)...\Gamma(nx_k)\,\Gamma(ny_1-nx_1)...} \times \frac{\beta_k^{n(y_k-x_k)-1}(1-|\boldsymbol{\alpha}|-|\boldsymbol{\beta}|)^{n(1-|\mathbf{y}|)-1}}{\Gamma(ny_k-nx_k)\,\Gamma(n(1-|\mathbf{y}|))}, \tag{8}$$

if $(\boldsymbol{\alpha},\boldsymbol{\beta}) \in D$ and zero otherwise.

Related to $h_{n,\mathbf{x},\mathbf{y}}(\boldsymbol{\alpha},\boldsymbol{\beta})$, we need the following lemma which will be used in a subsequent theorem.

Lemma 1. *For any* $\mathbf{t} \in S_k^0$ *we have*

$$\text{(9) } a)\ \Psi_{n,\mathbf{y}}(\mathbf{t}) = \int_{\mathbf{0}}^{\mathbf{t}} h_{n,\mathbf{x},\mathbf{y}}(\mathbf{u}, \mathbf{t}-\mathbf{u})\,\mathbf{du}$$

$$b)\ \Psi_{n,\mathbf{x}}(\mathbf{t}) = \int_0^{1-|\mathbf{t}|}\int_0^{1-|\mathbf{t}|-v_1}\cdots\int_0^{1-|\mathbf{t}|-v_1-\ldots-v_{k-1}} h_{n,\mathbf{x},\mathbf{y}}(\mathbf{t},\mathbf{v})\,dv_k\ldots dv_2 dv_1,$$

$$c)\ \Psi_{n,\mathbf{y}-\mathbf{x}}(\mathbf{t}) = \int_0^{1-|\mathbf{t}|}\int_0^{1-|\mathbf{t}|-u_1}\cdots\int_0^{1-|\mathbf{t}|-u_1-\ldots-u_{k-1}} h_{n,\mathbf{x},\mathbf{y}}(\mathbf{u},\mathbf{t})\,du_k\ldots du_2 du_1,$$

where $\Psi_{n,\cdot}$ *is probability density function of the Dirichlet distribution in (3) and (4).*

Proof. *a*) By (8),

$$\int_{\mathbf{0}}^{\mathbf{t}} h_{n,\mathbf{x},\mathbf{y}}(\mathbf{u},\mathbf{t}-\mathbf{u})\,\mathbf{du}$$
$$= \frac{\Gamma(n)(1-|\mathbf{t}|)^{n(1-|\mathbf{y}|)-1}}{\Gamma(nx_1)\ldots\Gamma(nx_k)\Gamma(ny_1-nx_1)\ldots\Gamma(ny_k-nx_k)\Gamma(n(1-|\mathbf{y}|))}$$
$$\times \int_0^{t_1}\ldots\int_0^{t_k} u_1^{nx_1-1}\ldots u_k^{nx_k-1}$$
$$\times\ (t_1-u_1)^{n(y_1-x_1)-1}\ldots(t_k-u_k)^{n(y_k-x_1)-1}\,du_k\ldots du_1.$$

Straightforward computation for each iterated integral gives the result.

b) Again from (8), we have

$$\text{(10) } \int_0^{1-|\mathbf{t}|}\int_0^{1-|\mathbf{t}|-v_1}\cdots\int_0^{1-|\mathbf{t}|-v_1-\ldots-v_{k-1}} h_{n,\mathbf{x},\mathbf{y}}(\mathbf{t},\mathbf{v})\,dv_k\ldots dv_2 dv_1$$
$$= \frac{\Gamma(n)\,\mathbf{t}^{n\mathbf{x}-\mathbf{1}}(1-|\mathbf{t}|)^{n(1-|\mathbf{x}|)-1}}{\Gamma(nx_1)\ldots\Gamma(nx_k)\Gamma(ny_1-nx_1)\ldots\Gamma(ny_k-nx_k)\Gamma(n(1-|\mathbf{y}|))}$$
$$\times \int_0^{1-|\mathbf{t}|}\int_0^{1-|\mathbf{t}|-v_1}\cdots\int_0^{1-|\mathbf{t}|-v_1-\ldots-v_{k-1}} \left(\frac{\mathbf{v}}{1-|\mathbf{t}|}\right)^{n(\mathbf{y}-\mathbf{x})-\mathbf{1}}$$
$$\times \left(1-\frac{|\mathbf{v}|}{1-|\mathbf{t}|}\right)^{n(1-|\mathbf{y}|)-1}\frac{dv_k}{1-|\mathbf{t}|}\cdots\frac{dv_2}{1-|\mathbf{t}|}\frac{dv_1}{1-|\mathbf{t}|},$$

where

$$\left(\frac{\mathbf{v}}{1-|\mathbf{t}|}\right)^{n(\mathbf{y}-\mathbf{x})-\mathbf{1}} = \left(\frac{v_1}{1-|\mathbf{t}|}\right)^{n(y_1-x_1)-1}\left(\frac{v_2}{1-|\mathbf{t}|}\right)^{n(y_2-x_2)-1}\cdots \times\left(\frac{v_k}{1-|\mathbf{t}|}\right)^{n(y_k-x_k)-1}.$$

Let us make the following transformation in (10).

$$\begin{aligned}\frac{v_1}{1-|\mathbf{t}|} &= \varphi_1,\\ \frac{v_2}{1-|\mathbf{t}|} &= \varphi_2\left(1-\frac{v_1}{1-|\mathbf{t}|}\right),\ \ldots,\\ \frac{v_k}{1-|\mathbf{t}|} &= \varphi_k\left(1-\frac{v_1}{1-|\mathbf{t}|}-\ldots-\frac{v_{k-1}}{1-|\mathbf{t}|}\right),\end{aligned}$$

where $0\le\varphi_i\le 1$, $i=1,2,...,k$. Therefore, (10) yields

$$\int_0^{1-|\mathbf{t}|}\int_0^{1-|\mathbf{t}|-v_1}\cdots\int_0^{1-|\mathbf{t}|-v_1-\ldots-v_{k-1}} h_{n,\mathbf{x},\mathbf{y}}(\mathbf{t},\mathbf{v})\,dv_k\ldots dv_2dv_1 \tag{11}$$

$$\begin{aligned}&= \frac{\Gamma(n)\,\mathbf{t}^{n\mathbf{x}-\mathbf{1}}(1-|\mathbf{t}|)^{n(1-|\mathbf{x}|)-1}}{\Gamma(nx_1)\ldots\Gamma(nx_k)\,\Gamma(ny_1-nx_1)\ldots\Gamma(ny_k-nx_k)\,\Gamma(n(1-|\mathbf{y}|))}\\ &\times\int_0^1\varphi_1^{n(y_1-x_1)-1}(1-\varphi_1)^{n(y_2-x_2)+\ldots+n(y_k-x_k)+n(1-|\mathbf{y}|)-1}d\varphi_1\\ &\times\int_0^1\varphi_2^{n(y_2-x_2)-1}(1-\varphi_2)^{n(y_3-x_3)+\ldots+n(y_k-x_k)+n(1-|\mathbf{y}|)-1}d\varphi_2\\ &\times\cdots\\ &\times\int_0^1\varphi_k^{n(y_k-x_k)-1}(1-\varphi_k)^{n(1-|\mathbf{y}|)-1}d\varphi_k.\end{aligned}$$

Making use of the familiar beta function, the right hand side of (11) reduces to $\Psi_{n,\mathbf{x}}(\mathbf{t})$ given in (4). Finally, (c) can be proved similarly. ■

For modulus of continuity type function, we have:

Theorem 2. *Let f be a modulus of continuity type function. Then B_n, $n\in\mathbb{N}$, have the same property.*

Proof. Let $\mathbf{x}=(x_1,...,x_k)$, $\mathbf{y}=(y_1,...,y_k)\in S_k^0$, and $\mathbf{x}\le\mathbf{y}$. Using (a) and (b) of (9), we have

$$\begin{aligned}B_n(f;\mathbf{y})-B_n(f;\mathbf{x}) &= \int\limits_{S_k}\left[\Psi_{n,\mathbf{y}}(\mathbf{t})-\Psi_{n,\mathbf{x}}(\mathbf{t})\right]f(\mathbf{t})\,\mathbf{dt}\\ &= \int\limits_0^1\int\limits_0^{1-t_1}\cdots\int\limits_0^{1-t_1-...-t_{k-1}}\int\limits_0^{t_1}\int\limits_0^{t_2}\cdots\int\limits_0^{t_k}f(\mathbf{t})\\ &\qquad\times h_{n,\mathbf{x},\mathbf{y}}(\mathbf{u},\mathbf{t}-\mathbf{u})\,du_k...du_2du_1dt_k...dt_2dt_1\\ &\quad-\int\limits_0^1\int\limits_0^{1-t_1}\cdots\int\limits_0^{1-t_1-...-t_{k-1}}\int\limits_0^{1-|\mathbf{t}|}\int\limits_0^{1-|\mathbf{t}|-v_1}\cdots\int\limits_0^{1-|\mathbf{t}|-v_1-...-v_{k-1}}f(\mathbf{t})\\ &\qquad\times h_{n,\mathbf{x},\mathbf{y}}(\mathbf{t},\mathbf{v})\,dv_k...dv_2dv_1dt_k...dt_2dt_1.\end{aligned}$$

Interchanging the order of integration in the first $2k-$fold integration, then the above formula boils down to

$$\begin{aligned}B_n(f;\mathbf{y})-B_n(f;\mathbf{x}) &= \int\limits_0^1\int\limits_{u_1}^1\int\limits_0^{1-t_1}\int\limits_{u_2}^{1-t_1}\cdots\int\limits_0^{1-t_1-...-t_{k-1}}\int\limits_{u_k}^{1-t_1-...-t_{k-1}}f(\mathbf{t})\\ &\qquad\times h_{n,\mathbf{x},\mathbf{y}}(\mathbf{u},\mathbf{t}-\mathbf{u})\,dt_kdu_k...dt_2du_2dt_1du_1\\ &\quad-\int\limits_0^1\int\limits_0^{1-t_1}\cdots\int\limits_0^{1-t_1-...-t_{k-1}}\int\limits_0^{1-|\mathbf{t}|}\int\limits_0^{1-|\mathbf{t}|-v_1}\cdots\int\limits_0^{1-|\mathbf{t}|-v_1-...-v_{k-1}}f(\mathbf{t})\\ &\qquad\times h_{n,\mathbf{x},\mathbf{y}}(\mathbf{t},\mathbf{v})\,dv_k...dv_2dv_1dt_k...dt_2dt_1.\end{aligned}$$

Letting $t_i=u_i+v_i,\ i=1,2,...,k$ in the first expression and replace $t_i=u_i$, $i=1,2,...,k$ in the second, this difference becomes

$$\begin{aligned}&B_n(f;\mathbf{y})-B_n(f;\mathbf{x})\\ &= \int\limits_0^1\int\limits_0^{1-t_1}\cdots\int\limits_0^{1-t_1-...-t_{k-1}}\int\limits_0^{1-|\mathbf{u}|}\int\limits_0^{1-|\mathbf{u}|-v_1}\cdots\int\limits_0^{1-|\mathbf{u}|-v_1-...-v_{k-1}}f(\mathbf{u}+\mathbf{v})\\ &\qquad\times h_{n,\mathbf{x},\mathbf{y}}(\mathbf{u},\mathbf{v})\,dv_k...dv_2dv_1du_k...du_2du_1\\ &\quad-\int\limits_0^1\int\limits_0^{1-t_1}\cdots\int\limits_0^{1-t_1-...-t_{k-1}}\int\limits_0^{1-|\mathbf{u}|}\int\limits_0^{1-|\mathbf{u}|-v_1}\cdots\int\limits_0^{1-|\mathbf{u}|-v_1-...-v_{k-1}}f(\mathbf{u})\\ &\qquad\times h_{n,\mathbf{x},\mathbf{y}}(\mathbf{u},\mathbf{v})\,dv_k...dv_2dv_1du_k...du_2du_1,\end{aligned}$$

which gives that

$$(12)\qquad B_n(f;\mathbf{y})-B_n(f;\mathbf{x}) = \int\limits_{S_k}\int\limits_0^{1-|\mathbf{u}|}\int\limits_0^{1-|\mathbf{u}|-v_1}\cdots\int\limits_0^{1-|\mathbf{u}|-v_1-\ldots-v_{k-1}} h_{n,\mathbf{x},\mathbf{y}}(\mathbf{u},\mathbf{v}) \times [f(\mathbf{u}+\mathbf{v})-f(\mathbf{u})]\,dv_k\ldots dv_2dv_1du_k\ldots du_2du_1$$

Since f is semi-additive, then (12) reduces to

$$B_n(f;\mathbf{y})-B_n(f;\mathbf{x}) \le \int\limits_{S_k}\int\limits_0^{1-|\mathbf{u}|}\int\limits_0^{1-|\mathbf{u}|-v_1}\cdots\int\limits_0^{1-|\mathbf{u}|-v_1-\ldots v_{k-1}} h_{n,\mathbf{x},\mathbf{y}}(\mathbf{u},\mathbf{v}) \times f(\mathbf{v})\,dv_k\ldots dv_2dv_1du_k\ldots du_2du_1$$

or

$$B_n(f;\mathbf{y})-B_n(f;\mathbf{x}) \le \int\limits_{S_k}\int\limits_0^{1-|\mathbf{v}|}\int\limits_0^{1-|\mathbf{v}|-u_1}\cdots\int\limits_0^{1-|\mathbf{v}|-u_1-\ldots-u_{k-1}} h_{n,\mathbf{x},\mathbf{y}}(\mathbf{u},\mathbf{v}) \times f(\mathbf{v})\,du_k\ldots du_2du_1dv_k\ldots dv_2dv_1$$

$$= \int\limits_{S_k}\Psi_{n,\mathbf{y}-\mathbf{x}}(\mathbf{v})\,f(\mathbf{v})\,dv_k\ldots dv_2dv_1 = B_n(f;\mathbf{y}-\mathbf{x}),$$

by (c) of (9). Last inequality shows the semi-additivity of B_n. From (8) and (12) we get that $B_n(f;\mathbf{y})-B_n(f;\mathbf{x})\ge 0$ for $\mathbf{y}\ge\mathbf{x}$. Finally we have $B_n(f;\mathbf{0})=f(\mathbf{0})=0$ by the definition of the multivariate beta operator. The proof is complete. ■

Next result is about preservation of the Lipschitz condition for the multivariate beta operator.

Theorem 3. *If* $f\in Lip_M(\mu;S_k)$, $0<\mu\le 1$, *then* $B_n(f;\mathbf{x})\in Lip_M(\mu;S_k)$.

Proof. Assume that $\mathbf{x}=(x_1,\ldots,x_k)$, $\mathbf{y}=(y_1,\ldots,y_k)\in S_k^0$ and $\mathbf{x}\le\mathbf{y}$. From (12) we have

$$|B_n(f;\mathbf{y})-B_n(f;\mathbf{x})| \le \int\limits_{S_k}\int\limits_0^{1-|\mathbf{u}|}\int\limits_0^{1-|\mathbf{u}|-v_1}\cdots\int\limits_0^{1-|\mathbf{u}|-v_1-\ldots-v_{k-1}} h_{n,\mathbf{x},\mathbf{y}}(\mathbf{u},\mathbf{v}) \times |f(\mathbf{u}+\mathbf{v})-f(\mathbf{u})|\,dv_k\ldots dv_2dv_1du_k\ldots du_2du_1.$$

Using the fact that $f \in Lip_M(\mu; S_k)$, (c) of (9) and Jensen's inequality, then the last inequality yields

$$
\begin{aligned}
(13) \quad & |B_n(f;\mathbf{y}) - B_n(f;\mathbf{x})| \\
&\le M \int_{S_k} \int_0^{1-|\mathbf{u}|} \int_0^{1-|\mathbf{u}|-v_1} \cdots \int_0^{1-|\mathbf{u}|-v_1-\ldots-v_{k-1}} h_{n,\mathbf{x},\mathbf{y}}(\mathbf{u},\mathbf{v}) \\
&\qquad \times \left[v_1^\mu + v_2^\mu + \ldots + v_k^\mu\right] dv_k \ldots dv_2 dv_1 du_k \ldots du_2 du_1 \\
&= M \int_{S_k} \int_0^{1-|\mathbf{v}|} \int_0^{1-|\mathbf{v}|-u_1} \cdots \int_0^{1-|\mathbf{v}|-u_1-\ldots-u_{k-1}} h_{n,\mathbf{x},\mathbf{y}}(\mathbf{u},\mathbf{v}) \\
&\qquad \times \left[v_1^\mu + v_2^\mu + \ldots + v_k^\mu\right] du_k \ldots du_2 du_1 dv_k \ldots dv_2 dv_1 \\
&\le M \left\{ \left(\int_{S_k} v_1 \Psi_{n,\mathbf{y}-\mathbf{x}}(\mathbf{v})\, \mathbf{dv} \right)^\mu + \left(\int_{S_k} v_2 \Psi_{n,\mathbf{y}-\mathbf{x}}(\mathbf{v})\, \mathbf{dv} \right)^\mu \right. \\
&\qquad \left. + \ldots + \left(\int_{S_k} v_k \Psi_{n,\mathbf{y}-\mathbf{x}}(\mathbf{v})\, \mathbf{dv} \right)^\mu \right\}.
\end{aligned}
$$

With a view to (13) takes the following form.

$$
|B_n(f;\mathbf{y}) - B_n(f;\mathbf{x})| \le M \left\{ (y_1 - x_1)^\mu + (y_2 - x_2)^\mu \ldots + (y_k - x_k)^\mu \right\}
$$

which shows that $B_n \in Lip_M(\mu; S_k)$. In a similar way we can verify the case $\mathbf{x} \ge \mathbf{y}$. Finally if $x_1 \ge y_1, \ldots, x_{i-1} \ge y_{i-1}, x_{i+1} \ge y_{i+1}, \ldots, x_k \ge y_k$ and $x_i \le y_i$. Since $(y_1, \ldots, y_{i-1}, x_i, y_{i+1}, \ldots, y_k) \in S_k^0$, then we obtain from the above arguments that

$$
\begin{aligned}
& |B_n(f;\mathbf{y}) - B_n(f;\mathbf{x})| \\
&\le |B_n(f;(x_1, \ldots, x_k)) - B_n(f;(y_1, \ldots, y_{i-1}, x_i, y_{i+1}, \ldots, y_k))| \\
&\quad + |B_n(f;(y_1, \ldots, y_k)) - B_n(f;(y_1, \ldots, y_{i-1}, x_i, y_{i+1}, \ldots, y_k))| \\
&\le M \sum_{i=1}^{k} |x_i - y_i|^\mu .
\end{aligned}
$$

Clearly if the last case holds for more than one components, then the result can be reached similarly. ∎

3. A generalisation of order r of B_n

This section provides an r th order generalisation of the multivariate beta operator analogous to Kirov and Popova's construction [11]. For convenience of exposition, we simply take the case $k=2$ into consideration. The result will be similar for higher dimentions.

Let thus $S_2=\{\mathbf{x}=(x_1,x_2)\in\mathbb{R},\ x_1,x_2\geq 0,\ x_1+x_2\leq 1\}$, and $C^r(S_2)$, $r\in\mathbb{N}_0$, denote the space of all functions f defined on S_2 and having all continuous partial derivatives up to order r and consider the representation (5). By $B_n^{[r]}$, we denote the following generalisation of B_n. For $\mathbf{x},\mathbf{t}\in S_2$,

$$B_n^{[r]}(f;\mathbf{x})=B_n\left(P_{r,\mathbf{t}}(\mathbf{\Delta x},f);\mathbf{x}\right)=\int_{S_2}\Psi_{n,\mathbf{x}}(\mathbf{t})\,P_{r,\mathbf{t}}(\mathbf{\Delta x},f)\,\mathbf{dt},\tag{14}$$

where $\mathbf{\Delta x}:=\mathbf{x}-\mathbf{t}=(x_1-t_1,x_2-t_2)=(\Delta x_1,\Delta x_2)$, $\mathbf{\nabla}:=\left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2}\right)$,

$$(\mathbf{\Delta x}.\mathbf{\nabla})^r:=\sum_{i+j=r}\binom{r}{j}(\Delta x_1)^i(\Delta x_2)^j\frac{\partial^r}{\partial x_1^i\partial x_2^j},\tag{15}$$

$\binom{r}{j}$ are binomial coefficients, and

$$P_{r,\mathbf{t}}(\mathbf{\Delta x},f)=f(\mathbf{t})+(\mathbf{\Delta x}.\mathbf{\nabla})f(\mathbf{t})+\frac{(\mathbf{\Delta x}.\mathbf{\nabla})^2}{2!}f(\mathbf{t})+\ldots+\frac{(\mathbf{\Delta x}.\mathbf{\nabla})^r}{r!}f(\mathbf{t}),\tag{16}$$

the Taylor polynomial for $f(\mathbf{x})$ at $\mathbf{t}\in S_2$. Now, we have the following result for $k=2$.

Theorem 4. *Let $f\in C^r(S_2)$ and $\frac{\partial^r f}{\partial x_1^i\partial x_2^j}\in Lip_M(\mu;S_2)$, $i+j=r$, then*

$$\left|B_n^{[r]}(f;\mathbf{x})-f(\mathbf{x})\right|\leq\frac{\mathbf{M}}{(r-1)!}\frac{\mu}{\mu+r}B(\mu,r)\,B_n\left(|\mathbf{x}-\mathbf{t}|^{r+\mu};\mathbf{x}\right),$$

where B is the familiar beta function.

Proof. From (14) and (16) we have

$$\left|f(\mathbf{x})-B_n^{[r]}(f;\mathbf{x})\right|\leq\int_{S_2}\Psi_{n,\mathbf{x}}(\mathbf{t})\left|f(\mathbf{x})-\sum_{\nu=0}^{r}\frac{1}{\nu!}(\mathbf{\Delta x}.\mathbf{\nabla})^{\nu}f(\mathbf{t})\right|\mathbf{dt}.\tag{17}$$

The familiar formula for the remainder term in (17) can be given by

$$f(\mathbf{x}) - \sum_{\nu=0}^{r} \frac{1}{\nu!} (\mathbf{\Delta x}.\mathbf{\nabla})^{\nu} f(\mathbf{t}) = \frac{1}{(r-1)!} \int_0^1 (\mathbf{\Delta x}.\mathbf{\nabla})^{r} [f(\mathbf{t} + z\mathbf{\Delta x}) - f(\mathbf{t})] (1-z)^{r-1} dz. \tag{18}$$

Taking (15) into account, then (18) results in

$$f(\mathbf{x}) - \sum_{\nu=0}^{r} \frac{1}{\nu!} (\mathbf{\Delta x}.\mathbf{\nabla})^{\nu} f(\mathbf{t}) = \frac{1}{(r-1)!} \int_0^1 \sum_{i+j=r} \binom{r}{j} (\Delta x_1)^i (\Delta x_2)^j \times \frac{\partial^r}{\partial x_1^i \partial x_2^j} [f(\mathbf{t} + z\mathbf{\Delta x}) - f(\mathbf{t})] (1-z)^{r-1} dz. \tag{19}$$

Using (19) and the hyphothesis, we obtain from (18) that

$$\begin{aligned} \left| f(\mathbf{x}) - \sum_{\nu=0}^{r} \frac{1}{\nu!} (\mathbf{\Delta x}.\mathbf{\nabla})^{\nu} f(\mathbf{t}) \right| &\leq \frac{M}{(r-1)!} \sum_{i+j=r} \binom{r}{j} |\Delta x_1|^i |\Delta x_2|^j \\ &\quad \times \int_0^1 z^{\mu} [|\Delta x_1|^{\mu} + |\Delta x_2|^{\mu}] (1-z)^{r-1} dz \\ &\leq \frac{\mathbf{M}}{(r-1)!} [|\Delta x_1| + |\Delta x_2|]^{r+\mu} B(\mu+1, r) \\ &= \frac{\mathbf{M}}{(r-1)!} |\mathbf{x} - \mathbf{t}|^{r+\mu} \frac{\mu}{\mu + r} B(\mu, r). \end{aligned} \tag{20}$$

Inserting (20) into (17), we have

$$\left| f(\mathbf{x}) - B_n^{[r]}(f;\mathbf{x}) \right| \leq \frac{\mathbf{M}}{(r-1)!} \frac{\mu}{\mu + r} B(\mu, r) \int_{S_2} \Psi_{n,\mathbf{x}}(\mathbf{t}) |\mathbf{x} - \mathbf{t}|^{r+\mu} \mathbf{dt} \tag{21}$$

which is the desired result, where $\Psi_{n,\mathbf{x}}(\mathbf{t})$ is given by (4).

Now we take a function $g \in C(S_2)$ which is given by $g(\mathbf{t}) = |\mathbf{x} - \mathbf{t}|^{r+\mu}$. Clearly $g(\mathbf{x}) = 0$. In accordance with Theorem 1 that $\|B_n(g;\mathbf{x})\|_{C(S_2)} \to 0$ as $n \to \infty$. Hence, From (21) we reach to the following

$$\left\| B_n^{[r]}(f;\mathbf{x}) - f(x) \right\|_{C(S_2)} \to 0 \quad \text{as} \quad n \to \infty.$$

■

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Gülen Başcanbaz-Tunca
Ankara University
Faculty of Science
Department of Mathematics
06100, Tandoğan, Ankara, Turkey
e-mail: tunca@science.ankara.edu.tr

Yalcin Tuncer
Professor Emeritus, METU and Ankara University
e-mail: tunceryal@gmail.com

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