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ON THE SOLUTION OF THE RECURSIVE SEQUENCE $x_{n+1} = \max\left\{x_{n-2}, \frac{1}{x_{n-2}}\right\}$

Abstract. We obtain in this paper the solution of the following recursive sequence

$$x_{n+1} = \max\left\{x_{n-2}, \frac{1}{x_{n-2}}\right\}, \quad n = 0, 1, ...$$

where the initial conditions x_{-2}, x_{-1}, x_0 are arbitrary non zero real numbers.

Key words: difference equation, recursive sequence, periodic solution.

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1. Introduction

Our purpose in this paper is to obtain the solution of the following recursive sequence

$$(1) \qquad x_{n+1} = \max\left\{x_{n-2}, \frac{1}{x_{n-2}}\right\}, \qquad n = 0, 1, ...$$

where the initial conditions x_{-2}, x_{-1}, x_0 are arbitrary non zero real numbers.

The periodic nature of nonlinear difference equations of the max-type has been investigated by many authors. See for example [1-9]. Since Elabbasy et al. [4] investigated the boundedness and the periodicity character of the following recursive sequence

$$x_{n+1} = \max\left\{\frac{1}{x_n}, \frac{A_n}{x_{n-1}}\right\}.$$

Simsek et al. [8] investigated the solutions of the following difference equation

$$x_{n+1} = \max\left\{x_{n-1}, \frac{1}{x_{n-1}}\right\}.$$

Definition (Periodicity). *A sequence* $\{x_n\}_{n=-k}^{\infty}$ *is said to be periodic with period* p *if* $x_{n+p} = x_n$ *for all* $n \geq -k$.

2. Main result

In this section we give a specific form of the solutions of the difference equation (1) and in each cases we can deduce that every solution of this equation is periodic with period three.

Theorem 1. *Consider the difference equation (1) for* $x_{-2}, x_{-1}, x_0 > 0$.

a) If $0 < x_{-2}, x_{-1}, x_0 \leq 1$. *Then*

$$\{x_n\}_{n=1}^{\infty} = \left\{ \frac{1}{x_{-2}}, \frac{1}{x_{-1}}, \frac{1}{x_0}, \frac{1}{x_{-2}}, \frac{1}{x_{-1}}, \frac{1}{x_0}, \ldots \right\}.$$

b) If $0 < x_{-1}, x_0 \leq 1$, $x_{-2} \geq 1$. *Then*

$$\{x_n\}_{n=1}^{\infty} = \left\{ x_{-2}, \frac{1}{x_{-1}}, \frac{1}{x_0}, x_{-2}, \frac{1}{x_{-1}}, \frac{1}{x_0}, \ldots \right\}.$$

c) If $0 < x_{-2}, x_0 \leq 1$, $x_{-1} \geq 1$. *Then*

$$\{x_n\}_{n=1}^{\infty} = \left\{ \frac{1}{x_{-2}}, x_{-1}, \frac{1}{x_0}, \frac{1}{x_{-2}}, x_{-1}, \frac{1}{x_0}, \ldots \right\}.$$

d) If $0 < x_{-2}, x_{-1} \leq 1$, $x_0 \geq 1$. *Then*

$$\{x_n\}_{n=1}^{\infty} = \left\{ \frac{1}{x_{-2}}, \frac{1}{x_{-1}}, x_0, \frac{1}{x_{-2}}, \frac{1}{x_{-1}}, x_0, \ldots \right\}.$$

e) If $0 < x_0 \leq 1$, $x_{-2}, x_{-1} \geq 1$. *Then*

$$\{x_n\}_{n=1}^{\infty} = \left\{ x_{-2}, x_{-1}, \frac{1}{x_0}, x_{-2}, x_{-1}, \frac{1}{x_0}, \ldots \right\}.$$

f) If $0 < x_{-1} \leq 1$, $x_{-2}, x_0 \geq 1$. *Then*

$$\{x_n\}_{n=1}^{\infty} = \left\{ x_{-2}, \frac{1}{x_{-1}}, x_0, x_{-2}, \frac{1}{x_{-1}}, x_0, \ldots \right\}.$$

g) If $0 < x_{-2} \leq 1$, $x_{-1}, x_0 \geq 1$. *Then*

$$\{x_n\}_{n=1}^{\infty} = \left\{ \frac{1}{x_{-2}}, x_{-1}, x_0, \frac{1}{x_{-2}}, x_{-1}, x_0, \ldots \right\}.$$

h) If $x_{-2}, x_{-1}, x_0 \geq 1$. *Then*

$$\{x_n\}_{n=1}^{\infty} = \{ x_{-2}, x_{-1}, x_0, x_{-2}, x_{-1}, x_0, \ldots \}.$$

and so the solution is periodic with period three.

Proof. We will prove Case (a) (the other cases are similar and the proof will be omitted).

(a) Let $0 < x_{-2},\ x_{-1},\ x_0 \le 1$, then $x_1 = \max\left\{x_{-2}, \dfrac{1}{x_{-2}}\right\} = \dfrac{1}{x_{-2}}$ (since $x_{-2} \le 1$, then $x_{-2}^2 \le 1$).

Also, $x_2 = \max\left\{x_{-1}, \dfrac{1}{x_{-1}}\right\} = \dfrac{1}{x_{-1}}$ (since $x_{-1} \le 1$, then $x_{-1}^2 \le 1$), and $x_3 = \max\left\{x_0, \dfrac{1}{x_0}\right\} = \dfrac{1}{x_0}$ (since $x_0 \le 1$, then $x_0^2 \le 1$). By induction we obtain that $x_{3n+1} = \dfrac{1}{x_{-2}}$, $x_{3n+2} = \dfrac{1}{x_{-1}}$, $x_{3n+3} = \dfrac{1}{x_0}$ for $n \ge 0$, that is $\{x_n\}_{n=1}^{\infty} = \left\{\dfrac{1}{x_{-2}}, \dfrac{1}{x_{-1}}, \dfrac{1}{x_0}, \dfrac{1}{x_{-2}}, \dfrac{1}{x_{-1}}, \dfrac{1}{x_0}, ...\right\}$. Thus, the proof is completed. ■

The following theorems can be proved similarly.

Theorem 2. *Consider the difference equation (1) for* $x_0 < 0 < x_{-2}, x_{-1}$.

a) If $x_{-2},\ x_{-1} \le 1,\ x_0 \ge -1$. *Then*

$$\{x_n\}_{n=1}^{\infty} = \left\{\frac{1}{x_{-2}}, \frac{1}{x_{-1}}, x_0, \frac{1}{x_{-2}}, \frac{1}{x_{-1}}, x_0, ...\right\}.$$

b) If $x_{-2} \ge 1,\ x_{-1} \le 1,\ x_0 \ge -1$. *Then*

$$\{x_n\}_{n=1}^{\infty} = \left\{x_{-2}, \frac{1}{x_{-1}}, x_0, x_{-2}, \frac{1}{x_{-1}}, x_0, ...\right\}.$$

c) If $x_{-2} \le 1,\ x_{-1} \ge 1,\ x_0 \ge -1$. *Then*

$$\{x_n\}_{n=1}^{\infty} = \left\{\frac{1}{x_{-2}}, x_{-1}, x_0, \frac{1}{x_{-2}}, x_{-1}, x_0, ...\right\}.$$

d) If $x_{-2},\ x_{-1} \ge 1,\ x_0 \ge -1$. *Then*

$$\{x_n\}_{n=1}^{\infty} = \{x_{-2}, x_{-1}, x_0, x_{-2}, x_{-1}, x_0, ...\}.$$

e) If $x_{-2},\ x_{-1} \le 1,\ x_0 \le -1$. *Then*

$$\{x_n\}_{n=1}^{\infty} = \left\{\frac{1}{x_{-2}}, \frac{1}{x_{-1}}, \frac{1}{x_0}, \frac{1}{x_{-2}}, \frac{1}{x_{-1}}, \frac{1}{x_0}, ...\right\}.$$

f) If $x_{-2} \ge 1,\ x_{-1} \le 1,\ x_0 \le -1$. *Then*

$$\{x_n\}_{n=1}^{\infty} = \left\{x_{-2}, \frac{1}{x_{-1}}, \frac{1}{x_0}, x_{-2}, \frac{1}{x_{-1}}, \frac{1}{x_0}, ...\right\}.$$

g) If $x_{-2} \leq 1$, $x_{-1} \geq 1$, $x_0 \leq -1$. Then

$$\{x_n\}_{n=1}^{\infty} = \left\{ \frac{1}{x_{-2}}, x_{-1}, \frac{1}{x_0}, \frac{1}{x_{-2}}, x_{-1}, \frac{1}{x_0}, ... \right\}.$$

h) If x_{-2}, $x_{-1} \geq 1$, $x_0 \leq -1$. Then

$$\{x_n\}_{n=1}^{\infty} = \left\{ x_{-2}, x_{-1}, \frac{1}{x_0}, x_{-2}, x_{-1}, \frac{1}{x_0}, ... \right\}.$$

Theorem 3. *Consider the difference equation (1) for $x_{-1} < 0 < x_{-2}$, x_0.*

a) If x_{-2}, $x_0 \leq 1$, $x_{-1} \geq -1$. Then

$$\{x_n\}_{n=1}^{\infty} = \left\{ \frac{1}{x_{-2}}, x_{-1}, \frac{1}{x_0}, \frac{1}{x_{-2}}, x_{-1}, \frac{1}{x_0}, ... \right\}.$$

b) If $x_{-2} \geq 1$, $x_0 \leq 1$, $x_{-1} \geq -1$. Then

$$\{x_n\}_{n=1}^{\infty} = \left\{ x_{-2}, x_{-1}, \frac{1}{x_0}, x_{-2}, x_{-1}, \frac{1}{x_0}, ... \right\}.$$

c) If $x_{-2} \leq 1$, $x_0 \geq 1$, $x_{-1} \geq -1$. Then

$$\{x_n\}_{n=1}^{\infty} = \left\{ \frac{1}{x_{-2}}, x_{-1}, x_0, \frac{1}{x_{-2}}, x_{-1}, x_0, ... \right\}.$$

d) If x_{-2}, $x_0 \geq 1$, $x_{-1} \geq -1$. Then

$$\{x_n\}_{n=1}^{\infty} = \{x_{-2}, x_{-1}, x_0, x_{-2}, x_{-1}, x_0, ...\}.$$

e) If x_{-2}, $x_0 \leq 1$, $x_{-1} \leq -1$. Then

$$\{x_n\}_{n=1}^{\infty} = \left\{ \frac{1}{x_{-2}}, \frac{1}{x_{-1}}, \frac{1}{x_0}, \frac{1}{x_{-2}}, \frac{1}{x_{-1}}, \frac{1}{x_0}, ... \right\}.$$

f) If $x_{-2} \geq 1$, $x_0 \leq 1$, $x_{-1} \leq -1$. Then

$$\{x_n\}_{n=1}^{\infty} = \left\{ x_{-2}, \frac{1}{x_{-1}}, \frac{1}{x_0}, x_{-2}, \frac{1}{x_{-1}}, \frac{1}{x_0}, ... \right\}.$$

g) If $x_{-2} \leq 1$, $x_0 \geq 1$, $x_{-1} \leq -1$. Then

$$\{x_n\}_{n=1}^{\infty} = \left\{ \frac{1}{x_{-2}}, \frac{1}{x_{-1}}, x_0, \frac{1}{x_{-2}}, \frac{1}{x_{-1}}, x_0, ... \right\}.$$

h) If x_{-2}, $x_0 \geq 1$, $x_{-1} \leq -1$. Then

$$\{x_n\}_{n=1}^{\infty} = \left\{ x_{-2}, \frac{1}{x_{-1}}, x_0, x_{-2}, \frac{1}{x_{-1}}, x_0, ... \right\}.$$

Theorem 4. *Consider the difference equation (1) for* $x_{-2} < 0 < x_{-1}$, x_0.

a) If $x_{-1},\ x_0 \le 1,\ x_{-2} \ge -1$. *Then*

$$\{x_n\}_{n=1}^{\infty} = \left\{x_{-2}, \frac{1}{x_{-1}}, \frac{1}{x_0}, x_{-2}, \frac{1}{x_{-1}}, \frac{1}{x_0}, ...\right\}.$$

b) If $x_{-1} \ge 1,\ x_0 \le 1,\ x_{-2} \ge -1$. *Then*

$$\{x_n\}_{n=1}^{\infty} = \left\{x_{-2}, x_{-1}, \frac{1}{x_0}, x_{-2}, x_{-1}, \frac{1}{x_0}, ...\right\}.$$

c) If $x_{-1} \le 1,\ x_0 \ge 1,\ x_{-2} \ge -1$. *Then*

$$\{x_n\}_{n=1}^{\infty} = \left\{x_{-2}, \frac{1}{x_{-1}}, x_0, x_{-2}, \frac{1}{x_{-1}}, x_0, ...\right\}.$$

d) If $x_{-1},\ x_0 \ge 1,\ x_{-2} \ge -1$. *Then*

$$\{x_n\}_{n=1}^{\infty} = \{x_{-2}, x_{-1}, x_0, x_{-2}, x_{-1}, x_0, ...\}.$$

e) If $x_{-1},\ x_0 \le 1,\ x_{-2} \le -1$. *Then*

$$\{x_n\}_{n=1}^{\infty} = \left\{\frac{1}{x_{-2}}, \frac{1}{x_{-1}}, \frac{1}{x_0}, \frac{1}{x_{-2}}, \frac{1}{x_{-1}}, \frac{1}{x_0}, ...\right\}.$$

f) If $x_{-1} \ge 1,\ x_0 \le 1,\ x_{-2} \le -1$. *Then*

$$\{x_n\}_{n=1}^{\infty} = \left\{\frac{1}{x_{-2}}, x_{-1}, \frac{1}{x_0}, \frac{1}{x_{-2}}, x_{-1}, \frac{1}{x_0}, ...\right\}.$$

g) If $x_{-1} \le 1,\ x_0 \ge 1,\ x_{-2} \le -1$. *Then*

$$\{x_n\}_{n=1}^{\infty} = \left\{\frac{1}{x_{-2}}, \frac{1}{x_{-1}}, x_0, \frac{1}{x_{-2}}, \frac{1}{x_{-1}}, x_0, ...\right\}.$$

h) If $x_{-1},\ x_0 \ge 1,\ x_{-2} \le -1$. *Then*

$$\{x_n\}_{n=1}^{\infty} = \left\{\frac{1}{x_{-2}}, x_{-1}, x_0, \frac{1}{x_{-2}}, x_{-1}, x_0, ...\right\}.$$

Theorem 5. *Consider the difference equation (1) for* $x_{-1},\ x_0 < 0 < x_{-2}$.
a) If $x_{-1},\ x_0 \le -1,\ x_{-2} \ge 1$. *Then*

$$\{x_n\}_{n=1}^{\infty} = \left\{x_{-2}, \frac{1}{x_{-1}}, \frac{1}{x_0}, x_{-2}, \frac{1}{x_{-1}}, \frac{1}{x_0}, ...\right\}.$$

b) If $x_{-1} \geq -1$, $x_0 \leq -1$, $x_{-2} \geq 1$. *Then*

$$\{x_n\}_{n=1}^{\infty} = \left\{x_{-2}, x_{-1}, \frac{1}{x_0}, x_{-2}, x_{-1}, \frac{1}{x_0}, ...\right\}.$$

c) If $x_{-1} \leq -1$, $x_0 \geq -1$, $x_{-2} \geq 1$. *Then*

$$\{x_n\}_{n=1}^{\infty} = \left\{x_{-2}, \frac{1}{x_{-1}}, x_0, x_{-2}, \frac{1}{x_{-1}}, x_0, ...\right\}.$$

d) If x_{-1}, $x_0 \geq -1$, $x_{-2} \geq 1$. *Then*

$$\{x_n\}_{n=1}^{\infty} = \{x_{-2}, x_{-1}, x_0, x_{-2}, x_{-1}, x_0, ...\}.$$

e) If x_{-1}, $x_0 \leq -1$, $x_{-2} \leq 1$. *Then*

$$\{x_n\}_{n=1}^{\infty} = \left\{\frac{1}{x_{-2}}, \frac{1}{x_{-1}}, \frac{1}{x_0}, \frac{1}{x_{-2}}, \frac{1}{x_{-1}}, \frac{1}{x_0}, ...\right\}.$$

f) If $x_{-1} \geq -1$, $x_0 \leq -1$, $x_{-2} \leq 1$. *Then*

$$\{x_n\}_{n=1}^{\infty} = \left\{\frac{1}{x_{-2}}, x_{-1}, \frac{1}{x_0}, \frac{1}{x_{-2}}, x_{-1}, \frac{1}{x_0}, ...\right\}.$$

g) If $x_{-1} \leq -1$, $x_0 \geq -1$, $x_{-2} \leq 1$. *Then*

$$\{x_n\}_{n=1}^{\infty} = \left\{\frac{1}{x_{-2}}, \frac{1}{x_{-1}}, x_0, \frac{1}{x_{-2}}, \frac{1}{x_{-1}}, x_0, ...\right\}.$$

h) If x_{-1}, $x_0 \geq -1$, $x_{-2} \leq -1$. *Then*

$$\{x_n\}_{n=1}^{\infty} = \left\{\frac{1}{x_{-2}}, x_{-1}, x_0, \frac{1}{x_{-2}}, x_{-1}, x_0, ...\right\}.$$

Theorem 6. *Consider the difference equation (1) for* x_{-2}, $x_0 < 0 < x_{-1}$.
a) If x_{-2}, $x_0 \leq -1$, $x_{-1} \geq 1$. *Then*

$$\{x_n\}_{n=1}^{\infty} = \left\{\frac{1}{x_{-2}}, x_{-1}, \frac{1}{x_0}, \frac{1}{x_{-2}}, x_{-1}, \frac{1}{x_0}, ...\right\}.$$

b) If $x_{-2} \geq -1$, $x_0 \leq -1$, $x_{-1} \geq 1$. *Then*

$$\{x_n\}_{n=1}^{\infty} = \left\{x_{-2}, x_{-1}, \frac{1}{x_0}, x_{-2}, x_{-1}, \frac{1}{x_0}, ...\right\}.$$

c) If $x_{-2} \leq -1$, $x_0 \geq -1$, $x_{-1} \geq 1$. *Then*

$$\{x_n\}_{n=1}^{\infty} = \left\{\frac{1}{x_{-2}}, x_{-1}, x_0, \frac{1}{x_{-2}}, x_{-1}, x_0, ...\right\}.$$

d) *If* $x_{-2},\ x_0 \geq -1,\ x_{-1} \geq 1$. *Then*

$$\{x_n\}_{n=1}^{\infty} = \{x_{-2}, x_{-1}, x_0, x_{-2}, x_{-1}, x_0, ...\}.$$

e) *If* $x_{-2},\ x_0 \leq -1,\ x_{-1} \leq 1$. *Then*

$$\{x_n\}_{n=1}^{\infty} = \left\{\frac{1}{x_{-2}}, \frac{1}{x_{-1}}, \frac{1}{x_0}, \frac{1}{x_{-2}}, \frac{1}{x_{-1}}, \frac{1}{x_0}, ...\right\}.$$

f) *If* $x_{-2} \geq -1,\ x_0 \leq -1,\ x_{-1} \leq 1$. *Then*

$$\{x_n\}_{n=1}^{\infty} = \left\{x_{-2}, \frac{1}{x_{-1}}, \frac{1}{x_0}, x_{-2}, \frac{1}{x_{-1}}, \frac{1}{x_0}, ...\right\}.$$

g) *If* $x_{-2} \leq -1,\ x_0 \geq -1,\ x_{-1} \leq 1$. *Then*

$$\{x_n\}_{n=1}^{\infty} = \left\{\frac{1}{x_{-2}}, \frac{1}{x_{-1}}, x_0, \frac{1}{x_{-2}}, \frac{1}{x_{-1}}, x_0, ...\right\}.$$

h) *If* $x_{-2},\ x_0 \geq -1,\ x_{-1} \leq 1$. *Then*

$$\{x_n\}_{n=1}^{\infty} = \left\{x_{-2}, \frac{1}{x_{-1}}, x_0, x_{-2}, \frac{1}{x_{-1}}, x_0, ...\right\}.$$

Theorem 7. *Consider the difference equation (1) for* $x_{-2},\ x_{-1} < 0 < x_0$.
a) *If* $x_{-2},\ x_{-1} \leq -1,\ x_0 \geq 1$. *Then*

$$\{x_n\}_{n=1}^{\infty} = \left\{\frac{1}{x_{-2}}, \frac{1}{x_{-1}}, x_0, \frac{1}{x_{-2}}, \frac{1}{x_{-1}}, x_0, ...\right\}.$$

b) *If* $x_{-2} \geq -1,\ x_{-1} \leq -1,\ x_0 \geq 1$. *Then*

$$\{x_n\}_{n=1}^{\infty} = \left\{x_{-2}, \frac{1}{x_{-1}}, x_0, x_{-2}, \frac{1}{x_{-1}}, x_0, ...\right\}.$$

c) *If* $x_{-2} \leq -1,\ x_{-1} \geq -1,\ x_0 \geq 1$. *Then*

$$\{x_n\}_{n=1}^{\infty} = \left\{\frac{1}{x_{-2}}, x_{-1}, x_0, \frac{1}{x_{-2}}, x_{-1}, x_0, ...\right\}.$$

d) *If* $x_{-2},\ x_{-1} \geq -1,\ x_0 \geq 1$. *Then*

$$\{x_n\}_{n=1}^{\infty} = \{x_{-2}, x_{-1}, x_0, x_{-2}, x_{-1}, x_0, ...\}.$$

e) *If* $x_{-2},\ x_{-1} \leq -1,\ x_0 \leq 1$. *Then*

$$\{x_n\}_{n=1}^{\infty} = \left\{\frac{1}{x_{-2}}, \frac{1}{x_{-1}}, \frac{1}{x_0}, \frac{1}{x_{-2}}, \frac{1}{x_{-1}}, \frac{1}{x_0}, ...\right\}.$$

f) *If* $x_{-2} \geq -1$, $x_{-1} \leq -1$, $x_0 \leq 1$. *Then*

$$\{x_n\}_{n=1}^{\infty} = \left\{x_{-2}, \frac{1}{x_{-1}}, \frac{1}{x_0}, x_{-2}, \frac{1}{x_{-1}}, \frac{1}{x_0}, ...\right\}.$$

g) *If* $x_{-2} \leq -1$, $x_{-1} \geq -1$, $x_0 \leq 1$. *Then*

$$\{x_n\}_{n=1}^{\infty} = \left\{\frac{1}{x_{-2}}, x_{-1}, \frac{1}{x_0}, \frac{1}{x_{-2}}, x_{-1}, \frac{1}{x_0}, ...\right\}.$$

h) *If* x_{-2}, $x_{-1} \geq -1$, $x_0 \leq 1$. *Then*

$$\{x_n\}_{n=1}^{\infty} = \left\{x_{-2}, x_{-1}, \frac{1}{x_0}, x_{-2}, x_{-1}, \frac{1}{x_0}, ...\right\}.$$

Theorem 8. *Consider the difference equation (1) for* x_{-2}, x_{-1}, $x_0 < 0$.
a) *If* x_{-2}, x_{-1}, $x_0 \leq -1$. *Then*

$$\{x_n\}_{n=1}^{\infty} = \left\{\frac{1}{x_{-2}}, \frac{1}{x_{-1}}, \frac{1}{x_0}, \frac{1}{x_{-2}}, \frac{1}{x_{-1}}, \frac{1}{x_0}, ...\right\}.$$

b) *If* x_{-1}, $x_0 \leq -1$, $x_{-2} \geq -1$. *Then*

$$\{x_n\}_{n=1}^{\infty} = \left\{x_{-2}, \frac{1}{x_{-1}}, \frac{1}{x_0}, x_{-2}, \frac{1}{x_{-1}}, \frac{1}{x_0}, ...\right\}.$$

c) *If* x_{-2}, $x_0 \leq -1$, $x_{-1} \geq -1$. *Then*

$$\{x_n\}_{n=1}^{\infty} = \left\{\frac{1}{x_{-2}}, x_{-1}, \frac{1}{x_0}, \frac{1}{x_{-2}}, x_{-1}, \frac{1}{x_0}, ...\right\}.$$

d) *If* x_{-2}, $x_{-1} \leq -1$, $x_0 \geq -1$. *Then*

$$\{x_n\}_{n=1}^{\infty} = \left\{\frac{1}{x_{-2}}, \frac{1}{x_{-1}}, x_0, \frac{1}{x_{-2}}, \frac{1}{x_{-1}}, x_0, ...\right\}.$$

e) *If* $x_0 \leq -1$, x_{-2}, $x_{-1} \geq -1$. *Then*

$$\{x_n\}_{n=1}^{\infty} = \left\{x_{-2}, x_{-1}, \frac{1}{x_0}, x_{-2}, x_{-1}, \frac{1}{x_0}, ...\right\}.$$

f) *If* $x_{-1} \leq -1$, x_{-2}, $x_0 \geq -1$. *Then*

$$\{x_n\}_{n=1}^{\infty} = \left\{x_{-2}, \frac{1}{x_{-1}}, x_0, x_{-2}, \frac{1}{x_{-1}}, x_0, ...\right\}.$$

g) *If* $x_{-2} \leq -1$, x_{-1}, $x_0 \geq -1$. *Then*

$$\{x_n\}_{n=1}^{\infty} = \left\{\frac{1}{x_{-2}}, x_{-1}, x_0, \frac{1}{x_{-2}}, x_{-1}, x_0, ...\right\}.$$

h) *If* x_{-2}, x_{-1}, $x_0 \geq -1$. *Then*

$$\{x_n\}_{n=1}^{\infty} = \{x_{-2}, x_{-1}, x_0, x_{-2}, x_{-1}, x_0, ...\}.$$

Proposition. *It is easy to see by induction that every solution of the following general difference equation*

$$x_{n+1} = \max\left\{x_{n-k}, \frac{1}{x_{n-k}}\right\}, \qquad n = 0, 1, ...$$

is periodic with period $(k+1)$.

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