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A DECOMPOSITION OF $\alpha-$CONTINUITY AND $\alpha gs-$CONTINUITY

Abstract. The main purpose of this paper is to introduce the concepts of η^*-sets, $\eta^{**}-$sets, η^*-continuity and $\eta^{**}-$continuity and to obtain decomposition of $\alpha-$continuity and $\alpha gs-$continuity in topological spaces.

Key words: $\alpha gs-$closed set, η^*-sets, $\eta^{**}-$sets, η^*-continuity, $\eta^{**}-$continuity.

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1. Introduction and preliminaries

Tong [16] introduced the notions of $A-$sets and $A-$continuity in topological spaces and established a decomposition of continuity. In[17], he also introduced the notions of $B-$sets and $B-$continuity and used them to obtain a new decomposition of continuity and Ganster and Reilly [3] improved Tong's decomposition result. Quit recently, Noiri and Sayed [10] introduced the notions of $\eta-$sets and obtained some decompositions of continuity.

In this paper, we introduce the notions of η^*-sets, $\eta^{**}-$sets, η^*-continuity and $\eta^{**}-$continuity and obtain decompositions of $\alpha-$continuity and $\alpha gs-$continuity.

Throughout the present paper, spaces mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a space X, the closure, the interior, the pre-closure, the pre-interior, the $\alpha-$closure and the $\alpha-$interior of A in X are denoted by $Cl(A)$, $Int(A)$, $pCl(A)$, $pInt(A)$, $\alpha Cl(A)$ and $\alpha Int(A)$, respectively.

Definition 1. *A subset A of a space X is called:*

a) *a pre-open set [6] if $A \subset Int(Cl(A))$ and a pre-closed set if $Cl(Int(A)) \subset A$,*

b) *a semi-open set [5] if $A \subset Cl(Int(A))$ and a semi-closed set if $Int(Cl(A)) \subset A$,*

c) *an $\alpha-$open set [8] if $A \subset Int(Cl(Int(A)))$ and an $\alpha-$closed set if $Cl(Int(Cl(A))) \subset A$,*

d) a t−set [11]) if $Int(Cl(A)) = Int(A)$*,*
e) an α^*- *set [4] if* $Int(A) = Int(Cl(Int(A)))$*,*
f) an A−set [16] if $A = V \cap T$ *where* V *is open and* T *is a regular-closed set,*
g) a B−set [12] if $A = V \cap T$ *where* V *is open and* T *is a t−set,*
h) an $\alpha B-$*set [1] if* $A = V \cap T$ *where* V *is* $\alpha-$*open and* T *is a t−set,*
i) an $\eta-$*set [10] if* $A = V \cap T$ *where* V *is open and* T *is an* $\alpha-$*closed set,*
j) an α *generalized semi-open [13] (written as* $\alpha gs-$*open) set in* X *if* $U \subset \alpha Int(A)$ *whenever* $U \subset A$ *and* U *is semi-closed in* X*,*
k) a pre generalized semi-open [15] (written as $pgs-$*open) set in* X *if* $U \subset pInt(A)$ *whenever* $U \subset A$ *and* U *is semi-closed in* X*.*

Remark 1.
1) Every $\alpha gs-$closed set is $pgs-$closed but not conversely [15].
2) Every $\alpha gs-$continuous map is $pgs-$continuous but not conversely [15].

2. η^*-sets and $\eta^{**}-$sets

In this section, we introduce and study the notions of η^*-sets and $\eta^{**}-$sets in topological spaces.

Definition 2. *A subset* A *of a space* X *is said to be*
a) an η^*-*set if* $A = U \cap T$ *where* U *is semi-open and* T *is* $\alpha-$*closed in* X*.*
b) an $\eta^{**}-$*set if* $A = U \cap T$ *where* U *is* $\alpha gs-$*open and* T *is a t−set in* X*.*
The collection of all η^*-*sets (resp.* $\eta^{**}-$*sets) in* X *will be denoted by* $\eta^*(X)$ *(resp.* $\eta^{**}(X)$*).*

Theorem 1. *For a subset* A *of a space* X*, the following are equivalent:*
a) A *is an* η^*-*set.*
b) $A = U \cap \alpha Cl(A)$ *for some semi-open set* U*.*

Proof. *a)* $\rightarrow$ *b).* Since A is an η^*-set, then $A = U \cap T$, where U is semi-open and T is $\alpha-$closed. So, $A \subset U$ and $A \subset T$. Hence $\alpha Cl(A) \subset \alpha Cl(T)$. Therefore, $A \subset U \cap \alpha Cl(A) \subset U \cap \alpha Cl(T) = U \cap T = A$. Thus, $A = U \cap \alpha Cl(A)$.
b) $\rightarrow$ *a).* It is obvious because $\alpha Cl(A)$ is $\alpha-$closed. ■

Remark 2. In a space X, the intersection of two $\eta^{**}-$sets is an $\eta^{**}-$set.

Remark 3. Observe that since the union of t-sets need not be a t-set, then the union of two $\eta^{**}-$sets need not be an $\eta^{**}-$set as seen from the following example.

Example 1. Let $X = \{a, b, c\}, \tau = \{\emptyset, \{a, b\}, X\}$. The sets $\{a\}, \{c\}$ are $\eta^{**}-$sets in (X, τ), but their union $\{a, c\}$ is not an $\eta^{**}-$set in (X, τ).

Remark 4. We have the following implications.

$$\begin{array}{ccccccc}
A(X) & \longrightarrow & LC(X) & & & & \\
\downarrow & & \downarrow & & & & \\
 & & \eta(X) & \longrightarrow & \eta^*(X) & & \\
 & & \downarrow & & & & \\
B(X) & \longrightarrow & \alpha B(X) & \longrightarrow & \eta^{**}(X) & & \\
 & & & & \uparrow & & \\
 & & & & \alpha GSO(X) & \longrightarrow & PGSO(X)
\end{array}$$

where none of these implications is reversible as shown by [10] and the following examples.

Example 2. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, X\}$. Then $\{a, b\}$ is an η^*-set but not an $\eta-$set in (X, τ). Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a, b\}, X\}$. Then $\{c\}$ is an $\eta^{**}-$set but not an $\alpha gs-$open set and the set $\{a\}$ is an $\eta^{**}-$set but not an $\alpha B-$set in (X, τ).

Remark 5.
1. The notions of η^*-sets and $\alpha gs-$closed sets are independent.
2. The notions of $\eta^{**}-$sets and $pgs-$closed sets are independent.

Example 3. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a, b\}, X\}$. Then $\{b, c\}$ is $\alpha gs-$closed but not an η^*-set in (X, τ).

Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. Then $\{a, b\}$ is an η^*-set but not $\alpha gs-$closed in (X, τ).

Example 4. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a, b\}, X\}$. Then the set $\{c\}$ is an $\eta^{**}-$set but not a $pgs-$open set and also the set $\{a, c\}$ is a $pgs-$open set but not an $\eta^{**}-$set in (X, τ).

Theorem 2. *For a subset A of a space X, the following are equivalent:*
a) A is $\alpha-$closed.
b) A is an η^*-set and $\alpha gs-$closed.

Proof. $a) \rightarrow b)$: Obvious.

$b) \rightarrow a)$: Since A is an η^*-set, then $A = U \cap \alpha cl(A)$, where U is semi-open in X. So, $A \subset U$ and since A is $\alpha gs-$closed, then $\alpha cl(A) \subset U$. Therefore, $\alpha Cl(A) \subset U \cap \alpha Cl(A) = A$. Hence A is $\alpha-$closed. ■

Proposition 1 ([9]). *Let A and B are subsets of a space X. If B is an α^*- set, then $\alpha Int(A \cap B) = \alpha Int(A) \cap Int(B)$.*

Theorem 3. *For a subset S of a space X, the following are equivalent:*
a) S is $\alpha gs-$open.
b) S is an $\eta^{**}-$set and pgs$-$open.

Proof. Necessity: Trivial.
Sufficiency: Assume that S is pgs-open and an $\eta^{**}-$set in X.
Then $S = A \cap B$, where A is $\alpha gs-$ open and B is a $t-$set in X.
Let $F \subset S$, where F is semi-closed in X. Since S is pgs-open in X, $F \subset pInt(S) = S \cap Int(Cl(S)) = (A \cap B) \cap Int(Cl(A \cap B)) \subset A \cap B \cap Int(Cl(A)) \cap Int(Cl(B)) = A \cap B \cap Int(Cl(A)) \cap Int(B)$, since B is a $t-$set. This implies, $F \subset Int(B)$. Note that A is $\alpha gs-$open and that $F \subset A$. So, $F \subset \alpha Int(A)$. Therefore, $F \subset \alpha Int(A) \cap Int(B) = \alpha Int(S)$ by Proposition 1. Hence S is $\alpha gs-$open. ■

3. η^*-continuity and $\eta^{**}-$continuity

Definition 3. *A function $f : X \to Y$ is said to be η^*-continuous (resp. $\eta^{**}-$continuous) if $f^{-1}(V)$ is an η^*-set (resp. an $\eta^{**}-$set) in X for every open subset V of Y.*

Definition 4. *A function $f : X \to Y$ is said to be $C\eta^*-$continuous if $f^{-1}(V)$ is an η^*-set in X for every closed subset V of Y.*

We shall recall the definitions of some functions used in the sequel.

Definition 5. *A function $f : X \to Y$ is said to be*
a) $A-$continuous [16] if $f^{-1}(V)$ is an $A-$set in X for every open set V of Y,
b) $B-$continuous [12] if $f^{-1}(V)$ is a $B-$set in X for every open set V of Y,
c) $\alpha-$continuous [7] if $f^{-1}(V)$ is an $\alpha-$open set in X for every open set V of Y,
d) $LC-$continuous [2] (resp. $\alpha B-$continuous [1]) if $f^{-1}(V)$ is an $LC-$set (resp. $\alpha B-$set) in X for every open set V of Y,
e) $\eta-$continuous [10] if $f^{-1}(V)$ is an $\eta-$ set in X for every open set V of Y,
f) $\alpha gs-$continuous [14] (resp. pgs$-$continuous [15]) if $f^{-1}(V)$ is an $\alpha gs-$ open set (resp. pgs$-$ open set) in X for every open set V of Y.

Remark 6. It is clear that, a function $f : X \to Y$ is $\alpha-$continuous if and only if $f^{-1}(V)$ is an $\alpha-$closed set in X for every closed set V of Y, From the definitions stated above, we obtain the following diagram.

$$
\begin{array}{llll}
A-continuity \longrightarrow & LC-continuity & & \\
\downarrow & \downarrow & & \\
 & \eta-\text{continuity} \longrightarrow & \eta^{*}-\text{continuity} & \\
 & \downarrow & & \\
B-\text{continuity} \longrightarrow & \alpha B-\text{continuity} \longrightarrow & \eta^{**}-\text{continuity} & \\
 & & \uparrow & \\
 & & \alpha gs-\text{continuity} \longrightarrow & pgs-\text{continuity}
\end{array}
$$

Remark 7. None of the implications is reversible as shown by the following examples.

Example 5. Let $X = Y = \{a, b, c\}, \tau = \{\emptyset, \{a\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, Y\}$. Then the identity function $f : X \to Y$ is $\eta^{*}-$continuous but not $\eta-$continuous.

Example 6. Let $X = Y = \{a, b, c\}, \tau = \{\emptyset, \{c\}, \{b, c\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, Y\}$. Then the identity function $f : X \to Y$ is $\eta^{**}-$continuous but not $\alpha B-$continuous.

Example 7. Let$X = Y = \{a, b, c\}, \tau = \{\emptyset, \{a, b\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{c\}, \{a, b\}, Y\}$. Then the identity function $f : X \to Y$ is $\eta^{**}-$continuous but not $\alpha gs-$continuous.

Remark 8. The following examples show that the concept of
1. $\alpha gs-$continuity and $\eta^{*}-$continuity are independent.
2. $\alpha gs-$continuity and $C\eta^{*}-$continuity are independent.
3. $\eta^{*}-$continuity and $C\eta^{*}-$continuity are independent.

Example 8. Let $X = Y = \{a, b, c\}, \tau = \{\emptyset, \{a, b\}, X\}$ and $\sigma = \{\emptyset, \{a\}, Y\}$. Let $f : X \to Y$ be the identity function on X. Then f is $\alpha gs-$continuous but not $\eta^{*}-$continuous.

Example 9. Let $X = Y = \{a, b, c\}, \tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, Y\}$. Let $f : X \to Y$ be the identity function on X. Then f is $\eta^{*}-$continuous but not $\alpha gs-$continuous.

Example 10. Let $X = Y = \{a, b, c\}, \tau = \{\emptyset, \{a, b\}, X\}$ and $\sigma = \{\emptyset, \{a\}, Y\}$.Then the identity function $f : X \to Y$ is $\alpha gs-$continuous but not $C\eta^{*}-$continuous.

Example 11. Let $X = Y = \{a, b, c\}, \tau = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{b, c\}, Y\}$. Then the identity function $f : X \to Y$ is $C\eta^{*}-$continuous but not $\alpha gs-$continuous.

Example 12. Let X, Y, τ as in Example 10 and $\sigma = \{\emptyset, \{b\}, \{c\}, \{b, c\}, Y\}$. Let $f : X \to Y$ be the identity function on X. Then f is $C\eta^*-$continuous but not η^*-continuous.

Example 13. Let X, Y, τ as in Example 10 and $\sigma = \{\emptyset, \{a, c\}, Y\}$. Let $f : X \to Y$ be the identity function on X. Then f is η^*-continuous but not $C\eta^*-$continuous.

Remark 9. The following examples show that the concept of $pgs-$continuity and $\eta^{**}-$continuity are independent.

Example 14. Let $X = Y = \{a, b, c\}, \tau = \{\emptyset, \{a, b\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, Y\}$. Then the identity function $f : X \to Y$ is $pgs-$continuous. But it is not $\eta^{**}-$continuous.

Example 15. Let $X = Y = \{a, b, c\}, \tau = \{\emptyset, \{c\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, Y\}$. Let $f : X \to Y$ be defined by $f(a) = c, f(b) = b$ and $f(c) = a$. Then f is $\eta^{**}-$continuous but it is not $pgs-$continuous.

Theorem 4. *For a function $f : X \to Y$, the following are equivalent:*
a) f is $\alpha-$continuous.
b) f is $C\eta^*-$continuous and $\alpha gs-$continuous.

Proof. The proof follows from Definitions 4, 5f), Remark 6 and Theorem 2. ■

Theorem 5. *For a function $f : X \to Y$, the following are equivalent:*
a) f is $\alpha gs-$continuous.
b) f is $\eta^{**}-$continuous and $pgs-$continuous.

Proof. The proof follows from Theorem 3. ■

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