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Baburao G. Pachpatte

ON SOME FUNDAMENTAL ASPECTS OF CERTAIN SUM-DIFFERENCE EQUATIONS

Abstract. In this paper we study the boundedness, uniqueness and continuous dependence of solutions of certain Volterra type sum-difference equations involving functions of one and two variables. Finite difference inequalities with explicit estimates are used to establish the results.

Key words: Volterra type, sum-difference equations, finite difference inequalities, explicit estimates, numerical analysis, boundedness, uniqueness, continuous dependence.

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1. Introduction

Let R^n denotes the real n-dimensional Euclidean space with appropriate norm denoted by $|\cdot|$. Let $R_+ = [0, \infty)$, $N_0 = \{0, 1, 2, ...\}$ be the given subsets of R, the set of real numbers and $E_1 = \left\{(n, \sigma) \in N_0^2 : 0 \leq \sigma \leq n < \infty\right\}$, $E_2 = \left\{(m, n, \sigma, \tau) \in N_0^4 : 0 \leq \sigma \leq m < \infty, 0 \leq \tau \leq n < \infty\right\}$. For the functions $z(m)$, $w(m, n)$, $m, n \in N_0$ we define the operators Δ, Δ_1, Δ_2 by $\Delta z\,(m) = z\,(m+1) - z\,(m)$, $\Delta_1 w\,(m, n) = w\,(m+1, n) - w\,(m, n)$, $\Delta_2 w(m, n) = w\,(m, n+1) - w\,(m, n)$ and $\Delta_2 \Delta_1 w\,(m, n) = \Delta_2(\Delta_1 w(m, n))$. Let $D\,(S_1, S_2)$ denotes the class of functions from the set S_1 to the set S_2. We use the usual conventions that the empty sums and products are taken to be 0 and 1 respectively. Consider the Volterra type sum-difference equations in one variable of the forms:

$$x\,(n) = f\left(n, x\,(n)\,, \sum_{\sigma=0}^{n-1} k\,(n, \sigma, x\,(\sigma))\right), \tag{1}$$

and

$$\Delta x\,(n) = g\left(n, x\,(n)\,, \sum_{\sigma=0}^{n-1} k\,(n, \sigma, x\,(\sigma))\right), \qquad x\,(0) = x_0, \tag{2}$$

for $n \in N_0$, where $k \in D(E_1 \times R^n, R^n)$, $f, g \in D(N_0 \times R^n \times R^n, R^n)$, and also consider the Volterra type sum-difference equations in two variables of the forms:

$$u(m,n) = F\left(m, n, u(m,n), \sum_{\sigma=0}^{m-1}\sum_{\tau=0}^{n-1} L(m,n,\sigma,\tau,u(\sigma,\tau))\right), \tag{3}$$

and

$$\Delta_2\Delta_1 u(m,n) = G\left(m, n, u(m,n), \sum_{\sigma=0}^{m-1}\sum_{\tau=0}^{n-1} L(m,n,\sigma,\tau,u(\sigma,\tau))\right), \tag{4}$$

with the given initial boundary conditions

$$u(m,0) = \sigma(m), \qquad u(0,n) = \tau(n), \qquad u(0,0) = 0, \tag{5}$$

for $m, n \in N_0$, where $L \in D(E_2 \times R^n, R^n)$, $F, G \in D\left(N_0^2 \times R^n \times R^n, R^n\right)$, $\sigma, \tau \in D(N_0, R^n)$.

The theory of finite difference equations, the methods used in their solutions and their wide applications in numerical analysis has drawn much attention in recent years, see [1, 2, 4-7, 9, 10] and the references cited therein. In [3] Kwapisz has studied the existence of solutions for certain finite difference equations by using fixed point techniques employed in the theory of ordinary differential equations. One can formulate existence results for the solutions of the above equations by modifying the idea employed in [3], see also [8]. The main purpose of this paper is to study the boundedness, uniqueness and continuous dependence of solutions of the above equations under various assumptions on the functions involved therein. The main tool employed in the analysis is based on the application of the finite difference inequalities with explicit estimates given in [6, 7]. We believe that the results obtained here present some useful basic results for future reference, by using elementary analysis.

2. Statement of results

In this section we state our results to be proved in this paper. We need the following versions of the inequalities given in [6, 7]. We shall state them here for completeness.

Lemma 1 (6, Theorem 1.3.4, p. 21 *or* **7, Theorem 4.3.1, p. 206).** *Let $z(n) \in D(N_0, R_+)$, $r(n,s), \Delta_1 r(n,s) \in D(E_1, R_+)$ and $c \geq 0$ is a real constant. If*

$$z(n) \leq c + \sum_{s=0}^{n-1} r(n,s)\, z(s), \tag{6}$$

for $n \in N_0,$ *then*

$$z(n) \le c \prod_{s=0}^{n-1} \left[1 + A(s)\right], \tag{7}$$

for $n \in N_0,$ *where*

$$A(n) = r(n+1, n) + \sum_{\sigma=0}^{n-1} \Delta_1 r(n, \sigma). \tag{8}$$

Lemma 2 (**7, Theorem 4.4.1, part** (a_1)**, p. 214**). *Let* $z(n), p(n) \in D(N_o, R_+),$ $r(n,s), \Delta_1 r(n,s) \in D(E_1, R_+)$ *and* $c \ge 0$ *is a real constant. If*

$$z(n) \le c + \sum_{s=0}^{n-1} p(s) \left[z(s) + \sum_{\sigma=0}^{s-1} r(s,\sigma) z(\sigma)\right], \tag{9}$$

for $n \in N_0,$ *then*

$$z(n) \le c \left[1 + \sum_{s=0}^{n-1} p(s) \prod_{\tau=0}^{s-1} \left[1 + p(\tau) + A(\tau)\right]\right], \tag{10}$$

for $n \in N_0,$ *where* $A(n)$ *is given by (8).*

Lemma 3 (**6, Theorem 4.2.6, p. 304** *or* **7, Theorem 5.2.2, part** (b_1)**, p.246**). *Let* $w(m,n) \in D\left(N_0^2, R_+\right),$ $e(m,n,\sigma,\tau),$ $\Delta_1 e(m,n,\sigma,\tau),$ $\Delta_2 e(m,n,\sigma,\tau),$ $\Delta_2\Delta_1 e(m,n,\sigma,\tau) \in D(E_2, R_+)$ *and* $c \ge 0$ *is a real constant. If*

$$w(m,n) \le c + \sum_{\sigma=0}^{m-1} \sum_{\tau=0}^{n-1} e(m,n,\sigma,\tau)\, w(\sigma,\tau), \tag{11}$$

for $m, n \in N_0,$ *then*

$$w(m,n) \le c \prod_{s=0}^{m-1} \left[1 + \sum_{t=0}^{n-1} E(s,t)\right], \tag{12}$$

for $m, n \in N_0,$ *where*

$$\begin{aligned} E(m,n) = {} & e(m+1, n+1, m, n) + \sum_{\sigma=0}^{m-1} \Delta_1 e(m, n+1, \sigma, n) \\ & + \sum_{\tau=0}^{n-1} \Delta_2 e(m+1, n, m, \tau) + \sum_{\sigma=0}^{m-1} \sum_{\tau=0}^{n-1} \Delta_2\Delta_1 e(m,n,\sigma,\tau). \end{aligned} \tag{13}$$

Lemma 4 (7, Theorem 5.3.2 part (b_1)**, p. 258).** *Let* $w(m,n)$, $p(m,n) \in D\left(N_0^2, R_+\right)$, $e(m,n,\sigma,\tau)$, $\Delta_1 e(m,n,\sigma,\tau)$, $\Delta_2 e(m,n,\sigma,\tau)$, $\Delta_2\Delta_1 e(m,n,\sigma,\tau) \in D(E_2, R_+)$ *and* $c \geq 0$ *is a real constant. If*

$$w(m,n) \leq c + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} p(s,t)\left[w(s,t) + \sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{t-1} e(s,t,\sigma,\tau)\, w(\sigma,\tau)\right], \tag{14}$$

for $m,n \in N_0$*, then*

$$w(m,n) \leq c\left[1 + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} p(s,t) \times \prod_{\xi=0}^{s-1}\left[1 + \sum_{\eta=0}^{t-1}\left[p(\xi,\eta) + E(\xi,\eta)\right]\right]\right], \tag{15}$$

for $m,n \in N_0$*, where* $E(m,n)$ *is given by (13).*

Our main results are given in the following theorems.

Theorem 1. (a_1) *Suppose that the functions* f*,* k *in equation (1) satisfy the conditions*

$$|f(n,u,v) - f(n,\bar{u},\bar{v})| \leq N\left[|u-\bar{u}| + |v-\bar{v}|\right], \tag{16}$$

$$|k(n,\sigma,u) - k(n,\sigma,v)| \leq r(n,\sigma)\,|u-v|, \tag{17}$$

where $0 \leq N < 1$ *is a constant and* $r(n,\sigma), \Delta_1 r(n,\sigma) \in D(E_1, R_+)$*. Let*

$$c_1 = \sup_{n\in N_0}\left|f\left(n, 0, \sum_{\sigma=0}^{n-1} k(n,\sigma,0)\right)\right| < \infty, \tag{18}$$

$$B(n) = \frac{N}{1-N} A(n), \tag{19}$$

in which $A(n)$ *is given by (8) and assume that*

$$\prod_{s=0}^{\infty}\left[1 + B(s)\right] < \infty. \tag{20}$$

Then any solution $x(n)$*,* $n \in N_0$ *of equation (1) is bounded and*

$$|x(n)| \leq \left(\frac{c_1}{1-N}\right)\prod_{s=0}^{n-1}\left[1 + B(s)\right], \tag{21}$$

for $n \in N_0$.

(a_2) *Suppose that the function* g *in equation (2) satisfies the condition*

$$|g(n,u,v) - g(n,\bar{u},\bar{v})| \leq p(n)\left[|u-\bar{u}| + |v-\bar{v}|\right], \tag{22}$$

where $p(n) \in D(N_0, R_+)$ *and the function* k *in equation (2) satisfies the condition (17). Let*

$$c_2 = \sup_{n\in N_0}\left|x_0 + \sum_{s=0}^{n-1} g\left(s, 0, \sum_{\sigma=0}^{s-1} k(s,\sigma,0)\right)\right| < \infty, \tag{23}$$

and assume that

$$\sum_{s=0}^{\infty} p(s)\prod_{\tau=0}^{s-1}\left[1 + p(\tau) + A(\tau)\right] < \infty, \tag{24}$$

where $A(n)$ *is given by (8). Then any solution* $u(n)$, $n \in N_0$ *of equation (2) is bounded and*

$$|x(n)| \leq c_2\left[1 + \sum_{s=0}^{n-1} p(s)\prod_{\tau=0}^{s-1}\left[1 + p(\tau) + A(\tau)\right]\right], \tag{25}$$

for $n \in N_0$.

Theorem 2. (a_3) *Suppose that the functions* f, k *in equation (1) satisfy the conditions (16), (17) respectively. Assume that the condition (20) holds. Then the equation (1) has at most one solution on* N_0.

(a_4) *Suppose that the functions* g ,k *in equation (2) satisfy the conditions (22), (17) respectively. Assume that the condition (24) holds. Then the equation (2) has at most one solution on* N_0.

In order to study the continuous dependence of solutions of equations (1) and (2) on the functions involved therein, we consider the following corresponding equations

$$y(n) = \bar{f}\left(n, y(n), \sum_{\sigma=0}^{n-1}\bar{k}(n,\sigma,y(\sigma))\right), \tag{26}$$

and

$$\Delta y(n) = \bar{g}\left(n, y(n), \sum_{\sigma=0}^{n-1}\bar{k}(n,\sigma,y(\sigma))\right), \qquad y(0) = y_0, \tag{27}$$

for $n \in N_0$, where $\bar{k} \in D(E_1 \times R^n, R^n)$, $\bar{f}, \bar{g} \in D(N_0 \times R^n \times R^n, R^n)$.

Theorem 3. (a_5) *Suppose that the functions f and k in equation (1) satisfy the conditions (16) and (17) respectively. Furthermore, suppose that*

$$\left| f\left(n, y(n), \sum_{\sigma=0}^{n-1} k(n,\sigma,y(\sigma))\right) - \bar{f}\left(n, y(n), \sum_{\sigma=0}^{n-1} \bar{k}(n,\sigma,y(\sigma))\right)\right| \leq \varepsilon_1, \tag{28}$$

where f, k and $\bar{f}$, $\bar{k}$ are the functions involved in equations (1) and (26), $\varepsilon_1 > 0$ is an arbitrary small constant and $y(n)$ is a solution of equation (26). Then the solution $x(n)$, $n \in N_0$ of equation (1) depends continuously on the functions involved on the right hand side of equation (1).

(a_6) *Suppose that the functions g and k in equation (2) satisfy the conditions (22) and (17) respectively. Furthermore, suppose that*

$$|x_0 - y_0| + \sum_{s=0}^{n-1}\left| g\left(s, y(s), \sum_{\sigma=0}^{s-1} k(s,\sigma,y(\sigma))\right) - \bar{g}\left(s, y(s), \sum_{\sigma=0}^{s-1} \bar{k}(s,\sigma,y(\sigma))\right)\right| \leq \varepsilon_2, \tag{29}$$

where x_0, g, k and y_0, $\bar{g}$, $\bar{k}$ are the functions involved in equations (2) and (27), $\varepsilon_2 > 0$, is an arbitrary small constant and $y(n)$ is a solution of equation (27). Then the solution $x(n)$, $n \in N_0$ of equation (2) depends continuously on the functions involved on the right hand side of equation (2).

Next, in order to study the continuous dependence of solutions on parameters, we consider the following systems of Volterra type sum-difference equations

$$z(n) = h\left(n, z(n), \sum_{\sigma=0}^{n-1} q(n,\sigma,z(\sigma)), \mu\right), \tag{30}$$

$$z(n) = h\left(n, z(n), \sum_{\sigma=0}^{n-1} q(n,\sigma,z(\sigma)), \mu_0\right), \tag{31}$$

and

$$\Delta z(n) = h\left(n, z(n), \sum_{\sigma=0}^{n-1} q(n,\sigma,z(\sigma)), \mu\right), \qquad z(0) = z_0, \tag{32}$$

$$\Delta z(n) = h\left(n, z(n), \sum_{\sigma=0}^{n-1} q(n,\sigma,z(\sigma)), \mu_0\right), \qquad z(0) = z_0, \tag{33}$$

for $n \in N_0$, where $q \in D(E_1 \times R^n, R^n)$, $h \in D(N_0 \times R^n \times R^n \times R, R^n)$ and μ, μ_0 are parameters.

Theorem 4. (a_7) *Suppose that the functions h, q in equations (30), (31) satisfy the conditions*

$$|h(n,u,v,\mu) - h(n,\bar{u},\bar{v},\mu)| \le \bar{N}\left[|u-\bar{u}| + |v-\bar{v}|\right], \tag{34}$$

$$|h(n,u,v,\mu) - h(n,u,v,\mu_0)| \le \alpha(n)|\mu-\mu_0|, \tag{35}$$

$$|q(n,\sigma,u) - q(n,\sigma,v)| \le \bar{r}(n,\sigma)|u-v|, \tag{36}$$

where $0 \le \bar{N} < 1$ is a constant, $\alpha(n) \in D(N_0, R_+)$ such that $\alpha(n) \le Q < \infty$, Q is a constant and $\bar{r}(n,\sigma)$, $\Delta_1\bar{r}(n,\sigma) \in D(E_1, R_+)$. Let $z_1(n)$ and $z_2(n)$ be the solutions of equations (30) and (31) respectively. Then

$$|z_1(n) - z_2(n)| \le \left(\frac{Q|\mu-\mu_0|}{1-\bar{N}}\right)\prod_{s=0}^{n-1}\left[1+\bar{B}(s)\right], \tag{37}$$

for $n \in N_0$, where

$$\bar{B}(n) = \frac{\bar{N}}{1-\bar{N}}\bar{A}(n), \tag{38}$$

in which

$$\bar{A}(n) = \bar{r}(n+1,n) + \sum_{\sigma=0}^{n-1}\Delta_1\bar{r}(n,\sigma). \tag{39}$$

(a_8) *Suppose that the functions h, q in equations (32), (33) satisfy the conditions (34)-(36) with $\bar{p}(n)$ in place of $\bar{N}$ in (34), where $\bar{p}(n) \in D(N_0, R_+)$ and the function $\alpha(n)$ in (35) be such that $\sum_{s=0}^{n-1}\alpha(s) \le \bar{Q} < \infty$, $\bar{Q}$ is a constant. Let $z_1(n)$ and $z_2(n)$ be the solutions of equations (32) and (33) respectively. Then*

$$|z_1(n) - z_2(n)| \le \left(\bar{Q}|\mu-\mu_0|\right)\left[1+\sum_{s=0}^{n-1}\bar{p}(s) \times \prod_{\tau=0}^{s-1}\left[1+\bar{p}(\tau)+\bar{A}(\tau)\right]\right], \tag{40}$$

for $n \in N_0$, where $\bar{A}(n)$ is given by (39).

The following theorems deal with the boundedness, uniqueness and continuous dependence of solutions of equations (3) and (4)-(5).

Theorem 5. (b_1) *Suppose that the functions F, L in equation (3) satisfy the conditions*

$$|F(m,n,u,v) - F(m,n,\bar{u},\bar{v})| \le M\left[|u-\bar{u}| + |v-\bar{v}|\right], \tag{41}$$

$$|L(m,n,\sigma,\tau,u) - L(m,n,\sigma,\tau,v)| \le e(m,n,\sigma,\tau)\,|u-v|, \tag{42}$$

where $0 \le M < 1$ is a constant and $e(m,n,\sigma,\tau)$, $\Delta_1 e(m,n,\sigma,\tau)$, $\Delta_2 e(m,n,\sigma,\tau)$, $\Delta_2\Delta_1 e(m,n,\sigma,\tau) \in D(E_2,R_+)$. Let

$$d_1 = \sup_{(m,n)\in N_0^2}\left|F\left(m,n,0,\sum_{\sigma=0}^{m-1}\sum_{\tau=0}^{n-1} L(m,n,\sigma,\tau,0)\right)\right| < \infty, \tag{43}$$

$$\bar{E}(m,n) = \frac{M}{1-M}E(m,n), \tag{44}$$

in which $E(m,n)$ is given by (13) and suppose that

$$\prod_{s=0}^{\infty}\left[1 + \sum_{t=0}^{\infty}\bar{E}(s,t)\right] < \infty. \tag{45}$$

Then any solution $u(m,n)$, $m,n \in N_0$ of equation (3) is bounded and

$$|u(m,n)| \le \left(\frac{d_1}{1-M}\right)\prod_{s=0}^{m-1}\left[1 + \sum_{t=0}^{n-1}\bar{E}(s,t)\right], \tag{46}$$

for $m,n \in N_0$.

(b_2) *Suppose that the function G in equation (4) satisfies the condition*

$$|G(m,n,u,v) - G(m,n,\bar{u},\bar{v})| \le p(m,n)\left[|u-\bar{u}| + |v-\bar{v}|\right], \tag{47}$$

where $p(m,n) \in D\left(N_0^2,R_+\right)$ and the function L in equation (4) satisfies the condition (42). Let

$$d_2 = \sup_{(m,n)\in N_0^2}\Bigg|\sigma(m) + \tau(n) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} G\left(s,t,0,\sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{t-1} L(s,t,\sigma,\tau,0)\right)\Bigg| < \infty, \tag{48}$$

and suppose that

$$\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}p(s,t)\prod_{\xi=0}^{s-1}\left[1+\sum_{\eta=0}^{t-1}\left[p(\xi,\eta)+E(\xi,\eta)\right]\right]<\infty, \tag{49}$$

where $E(m,n)$ is given by (13). Then any solution $u(m,n)$, $m,n\in N_0$, of equations (4)-(5) is bounded and

$$|u(m,n)|\le d_2\left[1+\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}p(s,t)\right.$$
$$\left.\times\prod_{\xi=0}^{s-1}\left[1+\sum_{\eta=0}^{t-1}\left[p(\xi,\eta)+E(\xi,\eta)\right]\right]\right], \tag{50}$$

for $m,n\in N_0$.

Theorem 6. (b_3) *Suppose that the functions F, L in equation (3) satisfy the conditions (41), (42) respectively. Assume that the condition (45) holds. Then the equation (3) has at most one solution on N_0^2*

(b_4) *Suppose that the functions G, L in equation (4) satisfy the conditions (47), (42) respectively. Assume that the condition (49) holds. Then the equations (4)-(5) has at most one solution on N_0^2.*

Consider the equations (3) and (4)-(5) and the corresponding equations

$$v(m,n)=\bar F\left(m,n,v(m,n),\sum_{\sigma=0}^{m-1}\sum_{\tau=0}^{n-1}\bar L(m,n,\sigma,\tau,v(\sigma,\tau))\right), \tag{51}$$

and

$$\Delta_2\Delta_1 v(m,n)=\bar G\left(m,n,v(m,n),\sum_{\sigma=0}^{m-1}\sum_{\tau=0}^{n-1}\bar L(m,n,\sigma,\tau,v(\sigma,\tau))\right), \tag{52}$$

with the given initial boundary conditions

$$v(m,0)=\bar\sigma(m),\qquad v(0,n)=\bar\tau(n),\qquad v(0,0)=0, \tag{53}$$

for $m,n\in N_0$ where $\bar L\in D(E_2\times R^n,R^n)$, $\bar F,\bar G\in D\left(N_0^2\times R^n\times R^n,R^n\right)$, $\bar\sigma,\bar\tau\in D(N_0,R^n)$.

Theorem 7. (b_5) *Suppose that the functions F, L in equation (3) satisfy the conditions (41), (42) respectively. Furthermore suppose that*

$$\left|F\left(m,n,v(m,n),\sum_{\sigma=0}^{m-1}\sum_{\tau=0}^{n-1}L(m,n,\sigma,\tau,v(\sigma,\tau))\right)\right.$$
$$\left.-\bar F\left(m,n,v(m,n),\sum_{\sigma=0}^{m-1}\sum_{\tau=0}^{n-1}\bar L(m,n,\sigma,\tau,v(\sigma,\tau))\right)\right|\le\varepsilon_3, \tag{54}$$

where F, L and $\bar{F}$, $\bar{L}$ are the functions involved in equations (3) and (51), $\varepsilon_3 > 0$ is an arbitrary small constant and $v(m,n)$ is a solution of equation (51). Then the solution $u(m,n)$, $m,n \in N_0$ of equation (3) depends continuously on the functions involved on the right hand side of equation (3).

(b_6) *Suppose that the functions G, L in equation (4) satisfy the conditions (47), (42) respectively. Furthermore, suppose that*

$$
\begin{aligned}
(55) \quad & |\sigma(m) + \tau(n) - \bar{\sigma}(m) - \bar{\tau}(n)| \\
& + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\left| G\left(s,t,v(s,t),\sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{t-1} L(s,t,\sigma,\tau,v(\sigma,\tau))\right)\right. \\
& \left. - \bar{G}\left(s,t,v(s,t),\sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{t-1} \bar{L}(s,t,\sigma,\tau,v(\sigma,\tau))\right)\right| \le \varepsilon_4,
\end{aligned}
$$

where σ, τ, F, L and $\bar{\sigma}$, $\bar{\tau}$, $\bar{F}$, $\bar{L}$ are the functions involved in equations (4)-(5) and (52)-(53), $\varepsilon_4 > 0$ is an arbitrary small constant and $v(m,n)$ is a solution of equations (52)-(53). Then the solution $u(m,n)$, $m,n \in N_0$ of equations (4)-(5) depends continuously on the functions involved in equations (4)-(5).

We next consider the following systems of Volterra sum-difference equations

$$(56) \qquad z(m,n) = H\left(m,n,z(m,n),\sum_{\sigma=0}^{m-1}\sum_{\tau=0}^{n-1} P(m,n,\sigma,\tau,z(\sigma,\tau)),\mu\right),$$

$$(57) \qquad z(m,n) = H\left(m,n,z(m,n),\sum_{\sigma=0}^{m-1}\sum_{\tau=0}^{n-1} P(m,n,\sigma,\tau,z(\sigma,\tau)),\mu_0\right),$$

and

$$(58) \quad \Delta_2\Delta_1 z(m,n) = H\left(m,n,z(m,n),\sum_{\sigma=0}^{m-1}\sum_{\tau=0}^{n-1} P(m,n,\sigma,\tau,z(\sigma,\tau)),\mu\right),$$

$$(59) \quad \Delta_2\Delta_1 z(m,n) = H\left(m,n,z(m,n),\sum_{\sigma=0}^{m-1}\sum_{\tau=0}^{n-1} P(m,n,\sigma,\tau,z(\sigma,\tau)),\mu_0\right),$$

with the given initial boundary conditions

$$(60) \qquad z(m,0) = \sigma_0(m), \quad z(0,n) = \tau_0(n), \quad z(0,0) = 0,$$

for $m,n \in N_0$, where $P \in D(E_2 \times R^n, R^n)$, $H \in D\left(N_0^2 \times R^n \times R^n \times R, R^n\right)$, $\sigma_0, \tau_0 \in D(N_0, R^n)$ and μ, μ_0 are parameters.

Theorem 8. (b_7) *Suppose that the functions H, P in equations (56), (57) satisfy the conditions*

$$|H(m,n,u,v,\mu) - H(m,n,\bar{u},\bar{v},\mu)| \leq \bar{N}\left[|u-\bar{u}| + |v-\bar{v}|\right], \tag{61}$$

$$|H(m,n,u,v,\mu) - H(m,n,u,v,\mu_0)| \leq \beta(m,n)|\mu-\mu_0|, \tag{62}$$

$$|P(m,n,\sigma,\tau,u) - P(m,n,\sigma,\tau,v)| \leq e(m,n,\sigma,\tau)|u-v|, \tag{63}$$

where $0 \leq \bar{N} < 1$ is a constant, $\beta(m,n) \in D\left(N_0^2, R_+\right)$ such that $\beta(m,n) \leq M_0 < \infty$, M_0 is a constant and $e(m,n,\sigma,\tau)$, $\Delta_1 e(m,n,\sigma,\tau)$, $\Delta_2 e(m,n,\sigma,\tau)$, $\Delta_2\Delta_1 e(m,n,\sigma,\tau) \in D(E_2, R_+)$. Let $z_1(m,n)$ and $z_2(m,n)$ be the solutions of equations (56) and (57) respectively. Then

$$|z_1(m,n) - z_2(m,n)| \leq \left(\frac{M_0|\mu-\mu_0|}{1-\bar{N}}\right)\prod_{s=0}^{m-1}\left[1 + \sum_{t=0}^{n-1}\bar{E}(s,t)\right], \tag{64}$$

for $m, n \in N_0$, where $\bar{E}(m,n)$ is given by (44).

(b_8) *Suppose that the functions H, P in equations (58), (59) satisfy the conditions (61)-(63) with $\bar{p}(m,n)$ in place of $\bar{N}$ in (61), where $\bar{p}(m,n) \in D\left(N_0^2, R_+\right)$ and the function $\beta(m,n)$ in (62) be such that $\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\beta(s,t) \leq \bar{M}_0 < \infty$, $\bar{M}_0$ is a constant. Let $z_1(m,n)$ and $z_2(m,n)$ be the solutions of equations (58), (60) and (59), (60) respectively. Then*

$$\begin{aligned}|z_1(m,n) - z_2(m,n)| \leq \left(\bar{M}_0|\mu-\mu_0|\right)&\left[1 + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\bar{p}(s,t)\right.\\ &\left.\times\prod_{\xi=0}^{s-1}\left[1 + \sum_{\eta=0}^{t-1}\left[\bar{p}(\xi,\eta) + E(\xi,\eta)\right]\right]\right],\end{aligned} \tag{65}$$

for $m, n \in N_0$, where $E(m,n)$ is given by (13).

3. Proofs of theorems 1-8

Since the proofs resemble one another, we give the details for (a_1), (a_3), (a_5), (a_7) and (b_2), (b_4), (b_6), (b_8) only. The proofs of (a_2), (a_4), (a_6), (a_8) and (b_1), (b_3), (b_5), (b_7) can be completed by following the proofs of the above mentioned results, by making use of Lemmas 2 and 3.

(a_1) Using the fact that $x(n)$ is a solution of equation (1) and the hypotheses we have

$$(66)\quad |x(n)| \leq \left|f\left(n, x(n), \sum_{\sigma=0}^{n-1} k(n,\sigma,x(\sigma))\right) - f\left(n, 0, \sum_{\sigma=0}^{n-1} k(n,\sigma,0)\right)\right| + \left|f\left(n, 0, \sum_{\sigma=0}^{n-1} k(n,\sigma,0)\right)\right|$$
$$\leq c_1 + N\left[|x(n)| + \sum_{\sigma=0}^{n-1} r(n,\sigma)|x(\sigma)|\right].$$

From (66) and using the assumption $0 \leq N < 1$, we observe that

$$(67)\qquad |x(n)| \leq \left(\frac{c_1}{1-N}\right) + \frac{N}{1-N}\sum_{\sigma=0}^{n-1} r(n,\sigma)|x(\sigma)|.$$

Now an application of Lemma 1 to (67) yields (21), which in view of the assumption (20) implies the boundedness of solution $x(n)$ of equation (1) on N_0.

(a_3) Let $x_1(n)$ and $x_2(n)$ be two solutions of equation (1) on N_0. Using this fact and the hypotheses we have

$$(68)\quad |x_1(n) - x_2(n)| \leq \left|f\left(n, x_1(n), \sum_{\sigma=0}^{n-1} k(n,\sigma,x_1(\sigma))\right) - f\left(n, x_2(n), \sum_{\sigma=0}^{n-1} k(n,\sigma,x_2(n))\right)\right|$$
$$\leq N\left[|x_1(n) - x_2(n)| + \sum_{\sigma=0}^{n-1} r(n,\sigma)|x_1(\sigma) - x_2(\sigma)|\right].$$

From (68) we observe that

$$(69)\qquad |x_1(n) - x_2(n)| \leq \frac{N}{1-N}\sum_{\sigma=0}^{n-1} r(n,\sigma)|x_1(\sigma) - x_2(\sigma)|.$$

Now a suitable application of Lemma 1 to (69) yields $|x_1(n) - x_2(n)| \leq 0$, which implies $x_1(n) = x_2(n)$ for $n \in N_0$. Thus there is at most one solution to equation (1) on N_0.

(a_5) Let $u(n) = |x(n) - y(n)|$, $n \in N_0$. Using the facts that $x(n)$ and $y(n)$ are the solutions of equations (1) and (26) and hypotheses we have

$$(70)\qquad u(n) \leq \left|f\left(n, x(n), \sum_{\sigma=0}^{n-1} k(n,\sigma,x(\sigma))\right) - f\left(n, y(n), \sum_{\sigma=0}^{n-1} k(n,\sigma,y(\sigma))\right)\right|$$

$$
\begin{aligned}
&+\left|f\left(n, y(n), \sum_{\sigma=0}^{n-1} k(n, \sigma, y(\sigma))\right)\right. \\
&\qquad\qquad \left.-\bar{f}\left(n, y(n), \sum_{\sigma=0}^{n-1} \bar{k}(n, \sigma, y(\sigma))\right)\right| \\
\leq\ & \varepsilon_1+N\left[u(n)+\sum_{\sigma=0}^{n-1} r(n, \sigma) u(\sigma)\right].
\end{aligned}
$$

From (70) we observe that

$$
u(n) \leq \frac{\varepsilon_1}{1-N}+\frac{N}{1-N} \sum_{\sigma=0}^{n-1} r(n, \sigma) u(\sigma). \tag{71}
$$

Now an application of Lemma 1 to (71) yields

$$
|x(n)-y(n)| \leq\left(\frac{\varepsilon_1}{1-N}\right) \prod_{s=0}^{n-1}[1+B(s)], \tag{72}
$$

where $B(n)$ is given by (19). From (72) it follows that the solution of equation (1) depends continuously on the functions involved on the right hand side of equation (1).

(a_7) Let $z(n)=|z_1(n)-z_2(n)|$, $n \in N_0$. Using the facts that $z_1(n)$ and $z_2(n)$ are the solutions of equations (30) and (31) and hypotheses we have

$$
\begin{aligned}
z(n) \leq\ & \left|h\left(n, z_1(n), \sum_{\sigma=0}^{n-1} q(n, \sigma, z_1(\sigma)), \mu\right)\right. \\
&\qquad\qquad \left.-h\left(n, z_2(n), \sum_{\sigma=0}^{n-1} q(n, \sigma, z_2(\sigma)), \mu\right)\right| \\
&+\left|h\left(n, z_2(n), \sum_{\sigma=0}^{n-1} q(n, \sigma, z_2(\sigma)), \mu\right)\right. \\
&\qquad\qquad \left.-h\left(n, z_2(n), \sum_{\sigma=0}^{n-1} q(n, \sigma, z_2(\sigma)), \mu_0\right)\right| \\
\leq\ & \bar{N}\left[z(n)+\sum_{\sigma=0}^{n-1} \bar{r}(n, \sigma) z(\sigma)\right]+Q|\mu-\mu_0|.
\end{aligned} \tag{73}
$$

From (73) we observe that

$$
z(n) \leq \frac{Q|\mu-\mu_0|}{1-\bar{N}}+\frac{\bar{N}}{1-\bar{N}} \sum_{\sigma=0}^{n-1} \bar{r}(n, \sigma) z(\sigma). \tag{74}
$$

Now an application of Lemma 1 to (74) yields (37), which shows the dependency of solutions of equations (30), (31) on parameters.

(b_2) Using the fact that $u(m,n)$ is a solution of equations (4)-(5) and the hypotheses we have

$$(75)\ |u(m,n)| \le \Bigg|\sigma(m)+\tau(n) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} G\left(s,t,0,\sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{t-1} L(s,t,\sigma,\tau,0)\right)\Bigg| + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\Bigg| G\left(s,t,u(s,t),\sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{t-1} L(s,t,\sigma,\tau,u(\sigma,\tau))\right) - G\left(s,t,0,\sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{t-1} L(s,t,\sigma,\tau,0)\right)\Bigg|$$
$$\le d_2 + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} p(s,t)\left[|u(s,t)| + \sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{t-1} e(s,t,\sigma,\tau)|u(\sigma,\tau)|\right].$$

Now an application of Lemma 4 to (75) yields (50), which in view of the assumption (49) implies the boundedness of solution of equations (4)-(5) on N_0^2.

(b_4) Let $u_1(m,n)$ and $u_2(m,n)$ be two solutions of equations (4)-(5). Using this fact and hypotheses we have

$$(76)\ |u_1(m,n)-u_2(m,n)| \le \sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\Bigg| G\left(s,t,u_1(s,t),\sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{t-1} L(s,t,\sigma,\tau,u_1(\sigma,\tau))\right) - G\left(s,t,u_2(s,t),\sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{t-1} L(s,t,\sigma,\tau,u_2(\sigma,\tau))\right)\Bigg|$$
$$\le \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} p(s,t)\left[|u_1(s,t)-u_2(s,t)| + \sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{t-1} e(s,t,\sigma,\tau)|u_1(\sigma,\tau)-u_2(\sigma,\tau)|\right].$$

Now a suitable application of Lemma 4 to (76) yields $|u_1(m,n)-u_2(m,n)| \le 0$, which implies $u_1(m,n)=u_2(m,n)$ for $m,n\in N_0$. Thus there is at most one solution to equations (4)-(5).

(b_6) Let $z(m,n) = |u(m,n) - v(m,n)|$, $m,n \in N_0$. Using the facts that $u(m,n)$ and $v(m,n)$ are the solutions of equations (4)-(5) and (52)-(53) and the hypotheses we have

$$\begin{aligned}
(77)\qquad z(m,n) \le{}& |\sigma(m) + \tau(n) - \bar{\sigma}(m) - \bar{\tau}(n)| \\
&+ \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} \left| G\left(s,t,u(s,t), \sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{t-1} L(s,t,\sigma,\tau,u(\sigma,\tau))\right)\right. \\
&\left. - G\left(s,t,v(s,t), \sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{t-1} L(s,t,\sigma,\tau,v(\sigma,\tau))\right)\right| \\
&+ \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} \left| G\left(s,t,v(s,t), \sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{t-1} L(s,t,\sigma,\tau,v(\sigma,\tau))\right)\right. \\
&\left. - \bar{G}\left(s,t,v(s,t), \sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{t-1} \bar{L}(s,t,\sigma,\tau,v(\sigma,\tau))\right)\right| \\
\le{}& \varepsilon_4 + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} p(s,t)\left[z(s,t) + \sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{t-1} e(s,t,\sigma,\tau)\, z(\sigma,\tau)\right].
\end{aligned}$$

Now an application of Lemma 4 to (77) yields

$$\begin{aligned}
(78)\qquad & |u(m,n) - v(m,n)| \\
&\le \varepsilon_4\left[1 + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} p(s,t)\prod_{\xi=0}^{s-1}\left[1 + \sum_{\eta=0}^{t-1}[p(\xi,\eta) + E(\xi,\eta)]\right]\right],
\end{aligned}$$

for $m,n \in N_0$, where $E(m,n)$ is given by (13). From (78) it follows that the solution of equations (4)-(5) depends continuously on the functions involved therein.

(b_8) Let $w(m,n) = |z_1(m,n) - z_2(m,n)|$, $m,n \in N_0$. Using the facts that $z_1(m,n)$ and $z_2(m,n)$ are the solutions of equations (58), (60) and (59), (60) and the hypotheses we have

$$\begin{aligned}
(79)\; w(m,n) \le{}& \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} \left| H\left(s,t,z_1(s,t), \sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{t-1} P(s,t,\sigma,\tau,z_1(\sigma,\tau)), \mu\right)\right. \\
&\left. - H\left(s,t,z_2(s,t), \sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{t-1} P(s,t,\sigma,\tau,z_2(\sigma,\tau)), \mu\right)\right| \\
&+ \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} \left| H\left(s,t,z_2(s,t), \sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{t-1} P(s,t,\sigma,\tau,z_2(\sigma,\tau)), \mu\right)\right. \\
&\left. - H\left(s,t,z_2(s,t), \sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{t-1} P(s,t,\sigma,\tau,z_2(\sigma,\tau)), \mu_0\right)\right|
\end{aligned}$$

$$\leq \sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\bar{p}(s,t)\left[w(s,t)+\sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{t-1}e(s,t,\sigma,\tau)\,w(\sigma,\tau)\right]$$
$$+\sum_{\sigma=0}^{m-1}\sum_{\tau=0}^{n-1}\beta(s,t)\,|\mu-\mu_0|$$
$$\leq \bar{M}_0\,|\mu-\mu_0|+\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\bar{p}(s,t)\left[w(s,t)+\sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{t-1}e(s,t,\sigma,\tau)\,w(\sigma,\tau)\right].$$

Now an application of Lemma 4 to (79) yields (65), which shows the dependency of solutions of equations (58), (60) and (59), (60) on parameters.

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Baburao G. Pachpatte
57 Shri Niketan Colony, Near Abhinay Talkies
Aurangabad 431 001 (Maharashtra), India
e-mail: bgpachpatte@gmail.com

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