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Baburao G. Pachpatte

GROWTH ESTIMATES ON MIXED VOLTERRA-FREDHOLM TYPE INTEGRAL INEQUALITIES

Abstract. In this paper we establish explicit estimates on some mixed Volterra-Fredholm type integral inequalities which can be used as tools in certain applications. Some applications are also given to illustrate the usefulness of the results.

Key words: growth estimates, Volterra-Fredholm type, integral inequalities, partial differential equations, parabolic type, epidemic models.

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1. Introduction

In the study of various properties of solutions of differential and integral equations the method of integral inequalities with explicit estimates is a very powerful tool. Many authors have established explicit bounds on the Volterra type integral inequalities

$$u(x,y) \le f(x,y) + \int_0^x \int_0^y k(x,y,s,t)\, u(s,t)\, dtds, \tag{1}$$

and used as tools in various applications (see [6,7]). Some initial boundary value problems for partial differential equations of the parabolic type and certain epidemic models (see [1-5]) are reducible to the integral equation of the form

$$u(x,t) = f(x,t) + \int_0^t \int_G k(x,t,y,s)\, u(y,s)\, dyds, \tag{2}$$

where G is a compact subset of R^n and f depends on the given initial boundary conditions. The integral equation (2) appears to be Volterra type in t, and of Fredholm type with respect to x and hence it can be viewed as a mixed Volterra-Fredholm type integral equation. It is easy to observe that the bounds obtained on the integral inequalities of the form (1) are not directly applicable to studying integral equations of the form (2). Motivated by the desire to apply integral inequalities which provide explicit estimates on unknown functions to study various properties of solutions of equations of the form (2), in the present paper we offer some basic mixed Volterra-Fredholm type integral inequalities which can be used as powerful tools for handling equations of the form (2). Some applications are also given to convey the importance of our results to the literature.

2. Statement of results

Let R denote the set of real numbers, $R_+ = [0,\infty)$ be the given subset of R and B be a bounded domain in R^n, the n-dimensional Euclidean space defined by $B = \prod_{i=1}^{n} [a_i, b_i]$ $(a_i < b_i)$. Let $x = (x_1, ..., x_n)$, $(x_i \in R)$ is a variable point in B, $dx = dx_1 ... dx_n$ and $'$ the derivative of a function with respect to $t \in R_+$. For any continuous function $z : B \to R$, we denote by $\int_B z(x)\,dx$ the n-fold integral $\int_{a_n}^{b_n} ... \int_{a_1}^{b_1} z(x_1, ..., x_n)\,dx_1 ... dx_n$. Let $\Delta = B \times R_+$ and denote by $C(S_1, S_2)$ the class of continuous functions from the set S_1 to the set S_2.

Our main results on Volterra-Fredholm type integral inequalities are given in the following theorems.

Theorem 1. *Let $u, p, q \in C(\Delta, R_+)$ and $L \in C(\Delta \times R_+, R_+)$ be such that*

$$0 \le L(x,t,u) - L(x,t,v) \le M(x,t,v)(u - v), \tag{3}$$

for $u \ge v \ge 0$, where $M \in C(\Delta \times R_+, R_+)$. If

$$u(x,t) \le p(x,t) + q(x,t) \int_0^t \int_B L(y,s,u(y,s))\,dyds, \tag{4}$$

for $(x,t) \in \Delta$, then

$$\begin{aligned} u(x,t) \le p(x,t) + q(x,t) \int_0^t \int_B & L(y,s,p(y,s)) \\ & \times \exp\left(\int_s^t \int_B M(z,\tau,p(z,\tau))\,q(z,\tau)\,dzd\tau \right) dyds, \end{aligned} \tag{5}$$

for $(x,t) \in \Delta$.

As an immediate consequence of Theorem 1 when $L(y,s,u(y,s)) = f(y,s)\,u(y,s)$, we have the following corollary.

Corollary 1. *Let $u,p,q,f \in C(\Delta, R_+)$. If*

$$u(x,t) \le p(x,t) + q(x,t)\int_0^t\int_B f(y,s)\,u(y,s)\,dyds, \tag{6}$$

for $(x,t)\in\Delta$, then

$$u(x,t) \le p(x,t) + q(x,t)\int_0^t\int_B f(y,s)\,p(y,s) \times \exp\left(\int_s^t\int_B f(z,\tau)\,q(z,\tau)\,dzd\tau\right) dyds, \tag{7}$$

for $(x,t)\in\Delta$.

Theorem 2. *Let $u,p,q,r,f,g \in C(\Delta, R_+)$ and suppose that*

$$u(x,t) \le p(x,t) + q(x,t)\int_0^t\int_B f(y,s)\,u(y,s)\,dyds + r(x,t)\int_0^\infty\int_B g(y,s)\,u(y,s)\,dyds, \tag{8}$$

for $(x,t)\in\Delta$. If

$$d = \int_0^\infty\int_B g(y,s)\,K_2(y,s)\,dyds < 1, \tag{9}$$

then

$$u(x,t) \le K_1(x,t) + DK_2(x,t), \tag{10}$$

for $(x,t)\in\Delta$, where

$$K_1(x,t) = p(x,t) + q(x,t)\int_0^t\int_B f(y,s)\,p(y,s) \times \exp\left(\int_s^t\int_B f(z,\tau)\,q(z,\tau)dzd\tau\right) dyds, \tag{11}$$

$$K_2(x,t) = r(x,t) + q(x,t) \int_0^t \int_B f(y,s)\, r(y,s) \times \exp\left(\int_s^t \int_B f(z,\tau)\, q(z,\tau) dz d\tau \right) dyds, \tag{12}$$

and

$$D = \frac{1}{1-d} \int_0^\infty \int_B g(y,s)\, K_1(y,s)\, dyds. \tag{13}$$

By taking $g = 0$, it is easy to observe that the inequality in Theorem 2 reduces to the inequality obtained in Corollary 1. If we choose $f = 0$ in Theorem 2, then we have the following corollary.

Corollary 2. *Let $u, p, r, g \in C(\Delta, R_+)$ and suppose that*

$$u(x,t) \le p(x,t) + r(x,t) \int_0^\infty \int_B g(y,s)\, u(y,s)\, dyds, \tag{14}$$

for $(x,t) \in \Delta$. If

$$d_0 = \int_0^\infty \int_B g(y,s)\, r(y,s)\, dyds < 1, \tag{15}$$

then

$$u(x,t) \le p(x,t) + r(x,t) \left\{ \frac{1}{1-d_0} \int_0^\infty \int_B g(y,s)\, p(y,s)\, dyds \right\}, \tag{16}$$

for $(x,t) \in \Delta$.

3. Proofs of Theorems 1 and 2

Introduce the notation

$$e(s) = \int_B L(y,s,u(y,s))\, dy. \tag{17}$$

Then the inequality (4) can be restated as

$$u(x,t) \le p(x,t) + q(x,t) \int_0^t e(s)\, ds, \tag{18}$$

for $(x,t)\in\Delta$. Define

$$m(t)=\int_0^t e(s)\,ds, \tag{19}$$

then $m(0)=0$ and

$$u(x,t)\le p(x,t)+q(x,t)\,m(t). \tag{20}$$

From (19), (17), (20) and (3) we observe that

$$\begin{aligned}
m'(t) &= e(t)=\int_B L(y,t,u(y,t))\,dy \\
&\le \int_B L(y,t,p(y,t)+q(y,t)\,m(t))\,dy \\
&= \int_B \{L(y,t,p(y,t)+q(y,t)\,m(t)) \\
&\qquad\qquad - L(y,t,p(y,t))+L(y,t,p(y,t))\}\,dy \\
&\le \int_B M(y,t,p(y,t))\,q(y,t)\,m(t)\,dy+\int_B L(y,t,p(y,t))\,dy \\
&= m(t)\int_B M(y,t,p(y,t))\,q(y,t)\,dy+\int_B L(y,t,p(y,t))\,dy.
\end{aligned} \tag{21}$$

The inequality (21) implies (see [6, Theorem 1.3.2])

$$\begin{aligned}
m(t) \le &\int_0^t\int_B L(y,s,p(y,s)) \\
&\times\exp\left(\int_s^t\int_B M(z,\tau,p(z,\tau))\,q(z,\tau)dzd\tau\right)dyds,
\end{aligned} \tag{22}$$

for $(x,t)\in\Delta$. Using (22) in (20) we get the required inequality in (5). This completes the proof of Theorem 1.

In order to prove Theorem 2, let

$$w(t)=\int_0^t\int_B f(y,s)\,u(y,s)\,dyds, \tag{23}$$

$$\lambda=\int_0^\infty\int_B g(y,s)\,u(y,s)\,dyds. \tag{24}$$

Then (8) can be restated as

$$u(x,t) \le p(x,t) + q(x,t)\, w(t) + r(x,t)\,\lambda. \tag{25}$$

Introducing the notation

$$E(s) = \int_B f(y,s)\, u(y,s)\, dy, \tag{26}$$

in (23) we get

$$w(t) = \int_0^t E(s)\, ds. \tag{27}$$

From (27) and (25) we have

$$\begin{aligned} w'(t) &= E(t) = \int_B f(y,t)\, u(y,t)\, dy \\ &\le \int_B f(y,t)\left[p(y,t) + q(y,t)\, w(t) + r(y,t)\,\lambda\right] dy \\ &= w(t)\int_B f(y,t)\, q(y,t)\, dy + \int_B f(y,t)\left[p(y,t) + r(y,t)\,\lambda\right] dy. \end{aligned} \tag{28}$$

The inequality (28) implies (see [6, Theorem 1.3.2])

$$\begin{aligned} w(t) &\le \int_0^t \int_B f(y,s)\left[p(y,s) + r(y,s)\,\lambda\right] \\ &\qquad \times \exp\left(\int_s^t \int_B f(z,\tau)\, q(z,\tau)\, dz d\tau\right) dy ds \\ &= \int_0^t \int_B f(y,s)\, p(y,s) \exp\left(\int_s^t \int_B f(z,\tau)\, q(z,\tau)\, dz d\tau\right) dy ds \\ &\quad + \lambda \int_0^t \int_B f(y,s)\, r(y,s) \exp\left(\int_s^t \int_B f(z,\tau)\, q(z,\tau)\, dz d\tau\right) dy ds. \end{aligned} \tag{29}$$

From (25) and (29) we get

$$
\begin{aligned}
(30)\ u(x,t) &\le p(x,t) \\
&+ q(x,t)\left\{\int_0^t\int_B f(y,s)\,p(y,s)\exp\left(\int_s^t\int_B f(z,\tau)\,q(z,\tau)\,dzd\tau\right)dyds\right. \\
&\left.+\lambda\int_0^t\int_B f(y,s)\,r(y,s)\exp\left(\int_s^t\int_B f(z,\tau)\,q(z,\tau)\,dzd\tau\right)dyds\right\} \\
&+ r(x,t)\,\lambda = K_1(x,t)+\lambda K_2(x,t).
\end{aligned}
$$

From (24) and (30) we observe that

$$
\begin{aligned}
\lambda &= \int_0^\infty\int_B g(y,s)\,u(y,s)\,dyds \\
&\le \int_0^\infty\int_B g(y,s)\left[K_1(y,s)+\lambda K_2(y,s)\right]dyds,
\end{aligned}
$$

which implies

$$\lambda \le D. \tag{31}$$

Using (31) in (30) we get (10) and the proof of Theorem 2 is complete.

4. Some applications

Consider the following mixed Volterra-Fredholm type integral equation

$$u(x,t) = h(x,t) + \int_0^t\int_B F(x,t,y,s,u(y,s))\,dyds, \tag{32}$$

for $(x,t)\in\Delta$, where $h\in C(\Delta,R)$, $F\in C\left(\Delta^2\times R,R\right)$, which occur in a natural way in a wide variety of applications (see [1-5,10]). For the existence and uniqueness of solutions of equation (32), see [8]. In this section we apply the inequality established in Corollary 1 to obtain explicit estimates on the solution of equation (32).

Theorem 3. *Suppose that the function F in equation (32) satisfies the condition*

$$|F(x,t,y,s,u)| \le q(x,t)\,f(y,s)\,|u|, \tag{33}$$

where $q, f \in C(\Delta, R_+)$. If $u(x,t)$ is any solution of equation (32) on Δ, then

$$|u(x,t)| \le |h(x,t)| + q(x,t) \int_0^t \int_B f(y,s) |h(y,s)| \times \exp\left(\int_s^t \int_B f(z,\tau) q(z,\tau) \, dz d\tau \right) dyds, \tag{34}$$

for $(x,t) \in \Delta$.

Proof. Let $u \in C(\Delta, R)$ be a solution of equation (32). Then from the hypotheses, we have

$$\begin{aligned} |u(x,t)| &\le |h(x,t)| + \int_0^t \int_B |F(x,t,y,s,u(y,s))| \, dyds \\ &\le |h(x,t)| + q(x,t) \int_0^t \int_B f(y,s) |u(y,s)| \, dyds. \end{aligned} \tag{35}$$

Now an application of Corollary 1 to (35) gives the desired estimate in (34). ∎

The following theorem deals with a slight variant of Theorem 3, assuming that the function F in equation (32) satisfies Lipschitz type condition.

Theorem 4. *Suppose that the function F in equation (32) satisfies the condition*

$$|F(x,t,y,s,u) - F(x,t,y,s,v)| \le q(x,t) f(y,s) |u - v|, \tag{36}$$

where $q, f \in C(\Delta, R_+)$. If $u(x,t)$ is any solution of equation (32) on Δ, then

$$|u(x,t) - h(x,t)| \le k(x,t) + q(x,t) \int_0^t \int_B f(y,s) k(y,s) \times \exp\left(\int_s^t \int_B f(z,\tau) q(z,\tau) \, dz d\tau \right) dyds, \tag{37}$$

for $(x,t) \in \Delta$, where

$$k(x,t) = \int_0^t \int_B |F(x,t,y,s,h(y,s))| \, dyds, \tag{38}$$

for $(x,t) \in \Delta$.

Proof. Let $u \in C(\Delta, R)$ be a solution of equation (32). Then from the hypotheses, we have

$$|u(x,t) - h(x,t)| \le \int_0^t \int_B |F(x,t,y,s,u(y,s))|\, dyds \tag{39}$$
$$\le \int_0^t \int_B |F(x,t,y,s,u(y,s)) - F(x,t,y,s,h(y,s))|\, dyds$$
$$+ \int_0^t \int_B |F(x,t,y,s,h(y,s))|\, dyds$$
$$\le k(x,t) + q(x,t) \int_0^t \int_B f(y,s)\, |u(y,s) - h(y,s)|\, dyds,$$

for $(x,t) \in \Delta$. Now an application of Corollary 1 to (39) gives the required estimate in (37). ■

We note that the inequality given in Theorem 2 can be used to establish similar results as in Theorems 3 and 4 given above for the following general mixed Volterra-Fredholm type integral equation

$$u(x,t) = h(x,t) + \int_0^t \int_B F(x,t,y,s,u(y,s))\, dyds \tag{40}$$
$$+ \int_0^\infty \int_B G(x,t,y,s,u(y,s))\, dyds,$$

for $(x,t) \in \Delta$, where $h \in C(\Delta, R)$, $F, G \in C\left(\Delta^2 \times R, R\right)$. Moreover, Corollary 1 and Theorem 2 can be used to establish results on the continuous dependence of solutions of equations (32), (40) by closely looking at the results recently given in [9]. Here, we omit the details.

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BABURAO G. PACHPATTE
57 SHRI NIKETAN COLONY, NEAR ABHINAY TALKIES
AURANGABAD 431 001 (MAHARASHTRA), INDIA
e-mail: bgpachpatte@gmail.com

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