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# ON THE DYNAMICS OF THE DIFFERENCE EQUATION $x_{n+1} = \frac{ax_{n-k}}{b+cx_n^p}$

Abstract. This paper studies the global behavior of the difference equation

$$x_{n+1} = \frac{ax_{n-k}}{b + cx_n^p}, \quad n = 0, 1, 2, ...$$

with non-negative parameters and non-negative initial conditions where $k$ is an odd number .

Key words: Difference equation, Global asymptotic stability, Oscillation, Period (k+1) solutions, Semicycles.

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## 1. Introduction

Difference equations appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations having applications in biology, ecelogy, physics, etc. [9]

Recenlty there has been an increasing interest in the study of global behavior of rational difference equations. Although difference equations' forms very simple, it is extremely difficult to understand thoroughly the global behaviors of their solutions. For example see Refs. [1-8].

El-Owaidy et al. [2] studied the difference equation $x_{n+1} = \frac{\alpha x_{n-1}}{\beta+\gamma x_{n-2}^p}$ with non-negative parameters and non-negative initial conditions.

In this paper, we investigate the global asymptotic behavior and the periodic character of the solutions of the difference equation

$$x_{n+1} = \frac{ax_{n-k}}{b + cx_n^p}, \; n = 0, 1, 2, \ldots \tag{1}$$

where the parameters $a$, $b$, $c$ and $p$ are non-negative real numbers, $k$ is an odd number and the initial conditions $x_i (i = 0, -1, -2, \ldots, -k)$ are non-negative real numbers such that $b + cx_n^p > 0, \; n = 0, 1, 2, \ldots$

The following cases can be obtained (for $k = 1, 3, 5, \ldots$ and $n = 0, 1, 2, \ldots$).

When $a = 0$ :

$$x_{n+1} = 0\,. \tag{2}$$

When $b = 0$ :

$$x_{n+1} = \frac{a x_{n-k}}{c x_n^p}\,. \tag{3}$$

When $p = 0$ :

$$x_{n+1} = \frac{a x_{n-k}}{b + c}\,. \tag{4}$$

When $c = 0$ :

$$x_{n+1} = \frac{a x_{n-k}}{b}\,. \tag{5}$$

When $b = p = 0$ :

$$x_{n+1} = \frac{a x_{n-k}}{c}\,. \tag{6}$$

In each of the above five equations, we assume that all parameters in the equations are positive. Eq. (2) is trivial and Eqs. (4)-(6) are linear. Eq. (3) can be also reduced to a linear difference equation by the change of variables $x_n = e^{y_n}$.

We need the following definitions and theorem [8]:

**Definition 1.** *Let $I$ be an interval of real numbers and let $f : I^{k+1} \to I$ be a continuously differentiable function. Consider the difference equation*

$$x_{n+1} = f\left(x_n, x_{n-1}, ..., x_{n-k}\right), \quad n = 0, 1, 2, ... \tag{7}$$

*with $x_{-k}, ..., x_0 \in I$ . Let $\overline{x}$ be the equilibrium point of Eq. (7).*

*The linearized equation of Eq. (7) about the equilibrium point $\overline{x}$ is*

$$y_{n+1} = c_1 y_n + c_2 y_{n-1+} ... + c_{(k+1)} y_{n-k}, \quad n = 0, 1, 2, \ldots \tag{8}$$

*Where*

$$c_1 = \frac{\partial f}{\partial x_n}(\overline{x}, ..., \overline{x}), c_2 = \frac{\partial f}{\partial x_{n-1}}(\overline{x}, ..., \overline{x}), ..., c_{(k+1)} = \frac{\partial f}{\partial x_{n-k}}(\overline{x}, ..., \overline{x})\,.$$

*The characteristic equation of Eq. (8) is*

$$\lambda^{(k+1)} - c_1 \lambda^k - c_2 \lambda^{(k-1)} - ... - c_k \lambda - c_{(k+1)} = 0\,. \tag{9}$$

**Definition 2.** *A positive semicycle of a solution* $\{x_n\}_{n=-k}^{\infty}$*of Eq.(7) consists of a "string" of terms* $\{x_l, x_{l+1}, ..., x_m\}$, *all greater than or equal to equilibrium* $\overline{x}$ *with* $l \geq -k$ *and* $m \leq \infty$ *such that either* $l = -k$ *or* $l > -k$ *and* $x_{l-1} < \overline{x}$ *and either* $m = \infty$ *or* $m \leq \infty$ *and* $x_{m+1} < \overline{x}$.

*A negative semicycle of a solution* $\{x_n\}_{n=-k}^{\infty}$ *of Eq. (7) consists of a "string" of terms* $\{x_l, x_{l+1}, ..., x_m\}$ *all less than* $\overline{x}$ *with* $l \geq -k$ *and* $m \leq \infty$ *such that either* $l = -k$ *or* $l > -k$ *and* $x_{l-1} \geq \overline{x}$ *and either* $m = \infty$ *or* $m \leq \infty$ *and* $x_{m+1} \geq \overline{x}$.

**Definition 3.** *A solution* $\{x_n\}_{n=-k}^{\infty}$ *of Eq. (7) is called nonoscillatory if there exists* $N \geq -k$ *such that either*

$$x_n > \overline{x} \text{ for } \forall n \geq N \quad \text{or} \quad x_n < \overline{x} \text{ for } \forall n \geq N,$$

*and it is called oscillatory if it is not nonoscillatory.*

**Theorem 1.** $(i)$ *If all roots of Eq. (9) have absolute values less than one, then the equilibrium point* $\overline{x}$ *of Eq. (7) is locally asymptotically stable.*

$(ii)$ *If at least one of the roots of Eq. (9) has absolute value greater than one, then the equilibrium point* $\overline{x}$*of Eq. (7) is unstable.*

## 2. Dynamics of equation (1)

In this section, we investigate the dynamics of Eq. (1) under the assumptions that all parameters in the equation are positive and the initial conditions are non-negative.

The change of variables $x_n = (\frac{b}{c})^{\frac{1}{p}} y_n$ reduces Eq. (1) to the difference equation

$$y_{n+1} = \frac{r y_{n-k}}{1 + y_n^p} \quad \text{for } k = 1, 3, 5, ... \text{ and } n = 0, 1, 2, \ldots \tag{10}$$

where $r = \frac{a}{b} > 0$. Note that $\overline{y_1} = 0$ is always an equilibrium point of Eq. (10). When $r > 1$, Eq. (10) also possesses the unique positive equilibrium $\overline{y_2} = (r-1)^{\frac{1}{p}}$.

**Lemma 1.** *The following statements are true:*

$(i)$ *If* $r < 1$, *then the equilibrium point* $\overline{y_1} = 0$ *of Eq. (10) is locally asymptotically stable,*

$(ii)$ *If* $r > 1$, *then the equilibrium point* $\overline{y_1} = 0$ *of Eq. (10) is unstable.*

**Proof.** The linearized equation of Eq. (10) about the equilibrium point $\overline{y_1} = 0$ is

$$z_{n+1} = r z_{n-k} \quad \text{for } k = 1, 3, 5, ... \text{ and } n = 0, 1, 2, ...$$

so the characteristic equation of Eq. (10) about the equilibrium point $\overline{y_1} = 0$ is

$$\lambda^{k+1} - r = 0$$

and hence the proof of $(i)$ and $(ii)$ follows from Theorem 1. This completes the proof. ■

**Theorem 2.** *Assume that $r > 1$ and let $\{y_n\}_{n=-k}^{\infty}$ be a solution of Eq. (10) such that*

$$y_{-(k-1)}, y_{-(k-3)}, ..., y_0 \geq \overline{y_2} \quad \text{and} \quad y_{-k}, y_{-(k-2)}, ..., y_{-1} < \overline{y_2} \tag{11}$$

*or*

$$y_{-(k-1)}, y_{-(k-3)}, ..., y_0 < \overline{y_2} \quad \text{and} \quad y_{-k}, y_{-(k-2)}, ..., y_{-1} \geq \overline{y_2}\,. \tag{12}$$

*Then, $\{y_n\}_{n=-k}^{\infty}$ oscillates about $\overline{y_2} = (r-1)^{\frac{1}{p}}$ with semicycles of length one.*

**Proof.** Assume that (11) holds. Then,

$$y_1 = \frac{ry_{-k}}{1+y_0^p} < \overline{y_2} \quad \text{and} \quad y_2 = \frac{ry_{-(k-1)}}{1+y_1^p} > \overline{y_2}$$

then, the proof follows by induction. The case where (12) holds is similar and will be omitted. ■

**Theorem 3.** *Assume that $r < 1$, then the equilibrium point $\overline{y_1} = 0$ of Eq. (10) is globally asymptotically stable.*

**Proof.** We know by Lemma 1 that the equilibrium point $\overline{y_1} = 0$ of Eq. (10) is locally asymptotically stable. So, let $\{y_n\}_{n=-k}^{\infty}$ be a solution of Eq. (10). It suffices to show that

$$\lim_{n\to\infty} y_n = 0\,.$$

Since

$$0 \leq y_{n+1} = \frac{ry_{n-k}}{1+y_n^p} \leq ry_{n-k}\,.$$

We obtain

$$y_{n+1} \leq ry_{n-k}\,.$$

Then, we can write,

$$\begin{aligned}
&y_{t(k+1)+1} \leq r^{(t+1)} y_{-k} \\
&y_{t(k+1)+2} \leq r^{(t+1)} y_{-(k-1)} \\
&\cdots \\
&y_{t(k+1)+(k+1)} \leq r^{(t+1)} y_0 \quad \text{for} \quad t = 0, 1, \ldots
\end{aligned}$$

If $r < 1$, then $\lim_{t\to\infty} r^{(t+1)} = 0$. So, $\lim_{n\to\infty} y_n = 0$. This completes the proof. ■

**Theorem 4.** *If Eq. (10) possesses the prime period $(k+1)$ solution, all of which aren't equal with each other at the same time, then both $r = 1$ and these solutions are at least in number $\frac{k+1}{2}$ equal to $0$ and at least in number $1$ greater than $0$.*

**Proof.** Let $\varphi_0, \varphi_1, ..., \varphi_k$, all of which aren't equal with each other at the same time, be the solutions of Eq. (10)'s prime period. That's to say,

$$..., \varphi_0, \varphi_1, ..., \varphi_k, \varphi_0, \varphi_1, ..., \varphi_k, ...$$

be a period (k+1) solution of Eq. (10). Then,

$$\varphi_k = \frac{r\varphi_k}{1+\varphi_{(k-1)}^p}, \varphi_{k-1} = \frac{r\varphi_{k-1}}{1+\varphi_{(k-2)}^p}, ..., \varphi_1 = \frac{r\varphi_1}{1+\varphi_0^p}, \varphi_0 = \frac{r\varphi_0}{1+\varphi_k^p}.$$

So, if $\varphi_k = 0$ and $r \neq 1$ then,

$$\varphi_0 = \varphi_1 = ... = \varphi_k = 0$$

which is impossible ($\varphi_k = 0$ and $r \neq 1$ is a conradiction).

If $\varphi_k \neq 0$ and $r \neq 1$ then,

$$\varphi_0 = \varphi_1 = ... = \varphi_k = \overline{y_2}$$

which is impossible ($\varphi_k \neq 0$ and $r \neq 1$ is a conradiction). This result in $r = 1$.

To complete the proof, we use $r = 1$ at above equalities

$$\varphi_k = \frac{\varphi_k}{1+\varphi_{(k-1)}^p},\ \varphi_{k-1} = \frac{\varphi_{k-1}}{1+\varphi_{(k-2)}^p},\ ...,\ \varphi_1 = \frac{\varphi_1}{1+\varphi_0^p},\ \varphi_0 = \frac{\varphi_0}{1+\varphi_k^p}.$$

Let's do the proof with induction. Assume that $k = 1$,

$$\varphi_1 = \frac{\varphi_1}{1+\varphi_0^p},\ \varphi_0 = \frac{\varphi_0}{1+\varphi_1^p}.$$

So one of the solutions is certainly equal to 0. ($\varphi_1 = 0$ or $\varphi_0 = 0$)

Assume that $k = t-2$ ($t \geqslant 5$ is an odd number),

$$\varphi_{t-2} = \frac{\varphi_{t-2}}{1+\varphi_{(t-3)}^p}, \varphi_{t-3} = \frac{\varphi_{t-3}}{1+\varphi_{(t-4)}^p}, ..., \varphi_1 = \frac{\varphi_1}{1+\varphi_0^p}, \varphi_0 = \frac{\varphi_0}{1+\varphi_{(t-2)}^p}$$

these solutions must be at least in number $\frac{(t-2)+1}{2} = \frac{t-1}{2}$ equal to 0.

Assume that $k = t$,

$$\varphi_t = \frac{\varphi_t}{1 + \varphi_{(t-1)}^p},\ \varphi_{t-1} = \frac{\varphi_{t-1}}{1 + \varphi_{(t-2)}^p},\ ...,\ \varphi_1 = \frac{\varphi_1}{1 + \varphi_0^p},\ \varphi_0 = \frac{\varphi_0}{1 + \varphi_t^p}.$$

We separete and then search the above equalities, with result of $k = t$ assumption. Hence, if $k = t$ we get these solutions are at least in number $\frac{t+1}{2}$ equal to 0. Now let's indicate that one of these solutions is greater than 0. All the solutions will be positive so it is equal or greater than 0. Let none of them be greater than 0. If they aren't greater than 0, then all the solutions equal to 0. This conrasts that all of the solutions which aren't equal to with each other at the same time hypothesis. Then at least one solution certainly greater than 0. This completes the proof. ■

**Theorem 5.** *Assume that $r > 1$, then Eq. (10) possesses an unbounded solution.*

**Proof.** From Theorem 2, we can assume that (11) holds, without loss of generality that the solution $\{y_n\}_{n=-k}^{\infty}$ of Eq. (10) is such that

$$y_{2n+1} < \overline{y_2} \text{ and } y_{2n+2} > \overline{y_2} \text{ for } n \geqslant 0.$$

The

$$y_{2n+1} = \frac{ry_{2n-k}}{1 + y_{2n}^p} < y_{2n-k}$$

and

$$y_{2n+2} = \frac{ry_{2n-(k-1)}}{1 + y_{2n+1}^p} > y_{2n-(k-1)}$$

and so $\{y_{2n}\}$ increases to $\infty$ and $\{y_{2n+1}\}$ decreases to 0. Similarly, we can assume that (12) holds, then $\{y_{2n}\}$ decreases to 0 and $\{y_{2n+1}\}$ increases to $\infty$ .The proof is complete. ■

## 3. Numerical results

**Example 1.** *Let $x_{n+1} = \frac{ax_{n-k}}{b+cx_n^p}$, $n = 0, 1, 2, ..., 199$ and $k = a = b = c = 1$, $p = 2$, $x_{-1} = 3$, $x_0 = 2$. Then we have the following results:*

| $n$ | $x_n$ | $n$ | $x_n$ |
|---|---|---|---|
| 1 | $0,6$ | 149 | $0,2989994484.10^{-35}$ |
| 2 | $1,470588235$ | 150 | $1,413199333$ |
| 49 | $0,2046286738.10^{-11}$ | 199 | $0,3614284505.10^{-47}$ |
| 50 | $1,413199333$ | 200 | $1,413199333$ |

**Example 2.** *Let* $x_{n+1} = \frac{ax_{n-k}}{b+cx_n^p}$, $n = 0, 1, 2, ..., 199$ *and* $k = 3, a = 5, b = 1, c = 2, p = 3, x_{-3} = 0.2, x_{-2} = 3, x_{-1} = 0.1, x_0 = 2$. *Then we have the following results:*

| $n$ | $x_n$ | $n$ | $x_n$ |
|---|---|---|---|
| 1 | 0 | 149 | 0 |
| 2 | 15 | 150 | $0,1091396342.10^{28}$ |
| 49 | 0 | 199 | 0 |
| 50 | $0,3662109375.10^{10}$ | 200 | $0,1776356838.10^{36}$ |

**Example 3.** Let $x_{n+1} = \frac{ax_{n-k}}{b+cx_n^p}$, $n = 0, 1, 2, ..., 199$ and $k = 5, a = 5, b = 1, c = 3, p = 2, x_{-5} = 1, x_{-4} = 5, x_{-3} = 1.5, x_{-2} = 3, x_{-1} = 1, x_0 = 2$. Then we have the following results:

| $n$ | $x_n$ | $n$ | $x_n$ |
|---|---|---|---|
| 1 | $0,3846153846$ | 149 | $0,2296104668.10^{-472}$ |
| 2 | $17,31557377$ | 150 | $0,5959486030.10^{18}$ |
| 49 | $0,1648660085.10^{-53}$ | 199 | $0,6264846310.10^{-797}$ |
| 50 | $0,6763067275.10^{7}$ | 200 | $0,2015551112.10^{25}$ |

**A note** A slightly different version of the paper, was submitted for publication in the *Chaos, Solitons & Fractals* in September 2007. However, it was withdrawn since we had not received any reply in a reasonable long period of time.

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