F A S C I C U L I M A T H E M A T I C I

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ON THE DIFFERENCE EQUATION $x_{n+1} = \alpha + \frac{x_{n-2}}{x_n^k}$

Abstract. In this paper, we investigate the global behavior of the difference equation of order three

$$x_{n+1} = \alpha + \frac{x_{n-2}}{x_n^k}, \quad n = 0, 1, \ldots$$

where the parameters $\alpha, k \in (0, \infty)$ and the initial values x_{-2}, x_{-1} and x_0 are arbitrary positive real numbers.

Key words: difference equation, global asymptotic stability, equilibrium point, periodicity, semicycle, boundedness.

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1. Introduction

Although difference equations are relatively simple in form, it is, unfortunately, extremely difficult to understand thoroughly the global behavior of their solutions. See, for example, [1-13] and the relevant references cited therein.

Difference equations appear naturally as a discrete analogues and as a numerical solutions of differential and delay differential equations having applications various scientific branches, such as in ecology, economy, physics, technics, sociology, biology, etc.

Hamza and Morsy in [8] investigated the global behavior of the difference equation

$$x_{n+1} = \alpha + \frac{x_{n-1}}{x_n^k}, \quad n = 0, 1, \ldots \tag{1}$$

where the parameters $\alpha, k \in (0, \infty)$ and the initial values x_{-1} and x_0 are arbitrary positive real numbers.

Eq. (1) was investigated when $k = 1$ where $\alpha \in (0, \infty)$, see [1] and [6]. There are some other examples of the research regarding Eq. (1). For examples [7] and [11].

In this paper, we consider the following difference equation of order three

$$x_{n+1} = \alpha + \frac{x_{n-2}}{x_n^k}, \quad n = 0, 1, \ldots \tag{2}$$

where the parameters $\alpha, k \in (0, \infty)$ and the initial values x_{-2}, x_{-1} and x_0 are arbitrary positive real numbers.

Here, we review some results which will be useful in our investigation of the behaviour of Eq. (2) solutions.(c.f. [13])

Definition 1. *Let I be an interval of real numbers and let $f : I^{k+1} \to I$ be a continuously differentiable function where k is a non-negative integer. Consider the difference equation*

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k}), \quad n = 0, 1, \ldots, \tag{3}$$

with the initial values $x_{-k}, \ldots, x_0 \in I$. A point $\overline{x}$ called an equilibrium point *of Eq. (3) if*

$$\overline{x} = f(\overline{x}, \overline{x}, \ldots, \overline{x}).$$

Definition 2. *Let $\overline{x}$ be an equilibrium point of Eq. (3).*

(a) *The equilibrium $\overline{x}$ is called* locally stable *if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $x_0, \ldots, x_{-k} \in I$ and $|x_0 - \overline{x}| + \cdots + |x_{-k} - \overline{x}| < \delta$, then*

$$|x_n - \overline{x}| < \varepsilon, \ \text{for all} \quad n \geq -k.$$

(b) *The equilibrium $\overline{x}$ is called* locally asymptotically stable *if it is locally stable and if there exists $\gamma > 0$ such that if $x_0, \ldots, x_{-k} \in I$ and $|x_0 - \overline{x}| + \cdots + |x_{-k} - \overline{x}| < \gamma$, then*

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(c) *The equilibrium $\overline{x}$ is called* global attractor *if for every $x_0, \ldots, x_{-k} \in I$ we have*

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(d) *The equilibrium $\overline{x}$ is called* globally asymptotically stable *if it is locally stable and is a global attractor.*

(e) *The equilibrium $\overline{x}$ is called* unstable *if is not stable.*

Definition 3. *Let $a_i = \dfrac{\partial f}{\partial u_i}(\overline{x}, \ldots, \overline{x})$ for each $i = 0, 1, \ldots, k$, denote the partial derivatives of $f(u_0, u_1, \ldots, u_k)$ evaluated at an equilibrium $\overline{x}$ of Eq. (3). Then the equation*

$$z_{n+1} = a_0 z_n + a_1 z_{n-1+} \cdots + a_k z_{n-k}, \quad n = 0, 1, \ldots \tag{4}$$

is called the linearized equation *of Eq. (3) about the equilibrium point $\overline{x}$.*

Theorem 1 (**Clark's Theorem**). *Consider the difference equation* (4). *Then,*

$$\sum_{i=0}^{k} |a_i| < 1$$

is a sufficient condition for the locally asymptotically stability of Eq. (3).

Definition 4. *The sequence* $\{x_n\}$ *is said to be* periodic *with period* p *if* $x_{n+p} = x_n$ *for* $n = 0, 1, \ldots$ (*c.f.*[5]).

2. Main results

In this section we investigate the global behavior, the boundedness and some periodicity of Eq. (2).

A point $\overline{x} \in \mathbb{R}$ is an equilibrium point of Eq. (2) if and only if it is a root for the function,

$$g(x) = x - x^{1-k} - \alpha \tag{5}$$

that is

$$\overline{x} - \overline{x}^{1-k} - \alpha = 0.$$

Lemma 1. *Eq. (2) has a unique equilibrium point* $\overline{x} > 1$.

Proof. Case 1: Assume that $k = 1$, then Eq. (2) has a unique equilibrium point $\overline{x} = \alpha + 1 > 1$.

Case 2: Assume that $0 < k < 1$. The function g defined by Eq. (5) is decreasing on $[0, (1-k)^{1/k}]$ and increasing on $[(1-k)^{1/k}, \infty)$. Since $g(1) = -\alpha$ and $\lim_{x\to\infty} g(x) = \infty$, then g has a unique root $\overline{x} > 1$.

Case 3: Assume that $1 < k$. Since g increasing on $[0, \infty)$, $g(1) = -\alpha$ and $\lim_{x\to\infty} g(x) = \infty$, then g has a unique root $\overline{x} > 1$,

Therefore, the proof is complete. ■

Theorem 2. *Assume that* $\overline{x}$ *is the equilibrium point of Eq (2). If* $k(k+1)^{\frac{1-k}{k}} < \alpha$, *then* $\overline{x}$ *is locally asymptotically stable.*

Proof. From Equations (3)-(4), we see that

$$f(u_0, u_1, u_2) = \alpha + u_0^{-k} u_2,$$

then

$$a_0 = \frac{-k}{\overline{x}^k}, \quad a_1 = 0, \quad a_2 = \frac{1}{\overline{x}^k}.$$

By using Clark's Theorem, we get that $\overline{x}$ is locally asymptotically stable if $\overline{x}^k > k + 1$.

Let $k(k+1)^{\frac{1-k}{k}} < \alpha$, a simple calculations shows that

$$g((k+1)^{1/k}) = k(k+1)^{\frac{1-k}{k}} - \alpha < 0$$

where g is defined by Eq. (5). Then, since $\lim_{x\to\infty} g(x) = \infty$, $\overline{x} > (k+1)^{1/k}$ and $\overline{x}^k > k+1$. Therefore, the proof is complete. ■

Lemma 2. *If $\alpha > 1$, then every solution of Eq. (2) is bounded and persists.*

Proof. We get that

$$\alpha < x_{n+1} < \alpha + \beta x_{n-2}$$

where $\beta = \frac{1}{\alpha^k}$.

By induction we obtain

$$\alpha < x_{3n+i} < \alpha \frac{1-\beta^n}{1-\beta} + \beta^n x_i \quad \text{for} \quad i \in \{-1,0,1\}.$$

Also, we see that if $\alpha > 1$,

$$\alpha < x_{3n+i} < \frac{\alpha}{1-\beta} + x_i \quad \text{for} \quad i \in \{-1,0,1\}.$$

Therefore, the proof is complete. ■

Theorem 3. *Assume that $\overline{x}$ is the equilibrium point of Eq. (2). If $\alpha > k^{1/k} \geq 1$, then $\overline{x}$ is globally asymptotically stable.*

Proof. We must show that the equilibrium point $\overline{x}$ of Eq. (2) is both locally asymptotically stable and $\lim_{n\to\infty} x_n = \overline{x}$.

Firstly, since $k \geq 1$, then $k \geq k(k+1)^{\frac{1-k}{k}}$ and since $\alpha > k^{1/k}$, we get $\alpha > k(k+1)^{\frac{1-k}{k}}$. By Theorem 2, $\overline{x}$ is locally asymptotically stable.

Let $\{x_n\}_{n=-2}^{\infty}$ be a solution of Eq. (2). By Lemma 2, $\{x_n\}_{n=-2}^{\infty}$ is bounded.

Let us introduce

$$\Lambda_1 = \lim_{n\to\infty} \inf x_n \quad \text{and} \quad \Lambda_2 = \lim_{n\to\infty} \sup x_n.$$

Then for all $\varepsilon \in (0, \Lambda_1)$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we get

$$\Lambda_1 - \varepsilon \leq x_n \leq \Lambda_2 + \varepsilon.$$

This implies that

$$\alpha + \frac{\Lambda_1 - \varepsilon}{(\Lambda_2 + \varepsilon)^k} \leq x_{n+1} \leq \alpha + \frac{\Lambda_2 + \varepsilon}{(\Lambda_1 - \varepsilon)^k} \quad \text{for} \quad n \geq n_0 + 1.$$

Then we obtain

$$\alpha + \frac{\Lambda_1 - \varepsilon}{(\Lambda_2 + \varepsilon)^k} \leq \Lambda_1 \leq \Lambda_2 \leq \alpha + \frac{\Lambda_2 + \varepsilon}{(\Lambda_1 - \varepsilon)^k},$$

and from the above inequality

$$\alpha + \frac{\Lambda_1}{{\Lambda_2}^k} \leq \Lambda_1 \leq \Lambda_2 \leq \alpha + \frac{\Lambda_2}{{\Lambda_1}^k},$$

which implies that

$$(\alpha {\Lambda_2}^k {\Lambda_1}^{k-1} + {\Lambda_1}^k) \leq \Lambda_1^k \Lambda_2^k \leq (\alpha {\Lambda_2}^{k-1} {\Lambda_1}^k + {\Lambda_2}^k).$$

Consequently, we obtain

$$\alpha {\Lambda_2}^{k-1} {\Lambda_1}^{k-1} (\Lambda_2 - \Lambda_1) \leq (\Lambda_2^k - \Lambda_1^k).$$

Suppose that $\Lambda_1 \neq \Lambda_2$ we get that

$$\alpha {\Lambda_2}^{k-1} {\Lambda_1}^{k-1} \leq \frac{\Lambda_2^k - \Lambda_1^k}{\Lambda_2 - \Lambda_1}.$$

There exists $\gamma \in (\Lambda_1, \Lambda_2)$ such that

$$\frac{\Lambda_2^k - \Lambda_1^k}{\Lambda_2 - \Lambda_1} = k\gamma^{k-1} \leq k\Lambda_2^{k-1}.$$

This implies that $\alpha^k \leq k$, which is a contradiction. Hence, $\Lambda_1 = \Lambda_2 = \overline{x}$. So, we have shown that

$$\lim_{n \to \infty} x_n = \overline{x}.$$

Therefore, the proof is complete. ■

Theorem 4. *Eq. (2) has a period three solution (not necessary prime)* $\{x_n\}_{n=-2}^{\infty}$ *if and only if* (x_{-2}, x_{-1}, x_0) *is a solution of the system*

$$x = \alpha + \tfrac{x}{z^k}, \qquad y = \alpha + \tfrac{y}{x^k}, \qquad z = \alpha + \tfrac{z}{y^k}. \tag{6}$$

Moreover, if at least one of the initial values of Eq. (2) is different from the others, then $\{x_n\}_{n=-2}^{\infty}$ *has a prime period three solution.*

Proof. First, assume that $\{x_n\}_{n=-2}^{\infty}$ is a period three solution of Eq. (2), then

$$x_{-2} = x_1 = \alpha + \frac{x_{-2}}{x_0^k},$$

$$x_{-1} = x_2 = \alpha + \frac{x_{-1}}{x_1^k} = \alpha + \frac{x_{-1}}{x_{-2}^k},$$

and

$$x_0 = x_3 = \alpha + \frac{x_0}{x_2^k} = \alpha + \frac{x_0}{x_{-1}^k}.$$

Then (x_{-2}, x_{-1}, x_0) is a solution of the system (6).

Second, assume that (x_{-2}, x_{-1}, x_0) is a solution of the system (6) then

$$x_1 = \alpha + \frac{x_{-2}}{x_0^k} = x_{-2},$$

$$x_2 = \alpha + \frac{x_{-1}}{x_1^k} = \alpha + \frac{x_{-1}}{x_{-2}^k} = x_{-1},$$

and

$$x_3 = \alpha + \frac{x_0}{x_2^k} = \alpha + \frac{x_0}{x_{-1}^k} = x_0.$$

By induction we see that

$$x_{n+3} = x_n \quad \text{for all } n \geq -2.$$

In the case where at least one of the initial values of Eq. (2) is different from the others, clearly $\{x_n\}_{n=-2}^{\infty}$ is a prime period three solution. ∎

Conclusion. The author believe that these results in this paper can be conveniently extended the following higher order diffrence equation

$$x_{n+1} = \alpha + \frac{x_{n-m}}{x_n^k}, \quad \text{for} \quad m > 2.$$

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