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ON LACUNARY STRONG $\sigma-$CONVERGENCE WITH RESPECT TO A SEQUENCE OF φ-FUNCTIONS

Abstract. In this paper, we introduce some new sequence spaces combining with lacunary sequence, $\sigma-$convergence, a sequence of φ-functions and a sequence of modulus functions. We establish some inclusion relations between these spaces under some conditions. Also we studied connections between lacunary (A, φ_k, σ) $-$statistically convergence with these spaces.

Key words: $\sigma-$convergence, modulus function, φ-function, lacunary sequence, statistical convergence.

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1. Introduction

Let w be the set of all sequences of real or complex numbers and l_∞, c and c_0 be, respectively, the Banach spaces of bounded, convergent and null sequences $x = (x_k)$ with the usual norm $\|x\| = \sup_k |x_k|$.

A sequence $x \in l_\infty$ is said to be almost convergent if all of its Banach limits coincide. Let $\hat{c}$ denote the space of all almost convergent sequences. Lorentz [1] has shown that

$$\hat{c} = \left\{ x \in l_\infty : \lim_m t_{mn}(x) \text{ exists uniformly in } n \right\}$$

where

$$t_{mn}(x) = \frac{x_n + x_{n+1} + \ldots + x_{n+m}}{m+1}.$$

The space $[\hat{c}]$ of strongly almost convergent sequences was introduced by Maddox [2] as follows:

$$[\hat{c}] = \left\{ x \in l_\infty : \lim_m t_{mn}(|x - le|) = 0, \text{ uniformly in } n, \text{ for some } l \right\}.$$

Let σ be one-to-one mapping of the set of positive integers into itself such that $\sigma^k(n) = \sigma(\sigma^{k-1}(n))$, $k = 1, 2, 3, \ldots$and $\sigma^0(n) := n$. A continuous linear functional Φ on l_∞ is said to be an invariant mean or a $\sigma-$mean if and only if

(1) $\Phi(x) \geq 0$ when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n
(2) $\Phi(e) = 1$ where $e = (1, 1, \ldots)$ and
(3) $\Phi(x_{\sigma(n)}) = \Phi(x)$ for all $x \in l_\infty$.

For a certain kinds of mapping σ every invariant mean Φ extends the limit functional on space c, in the sense that $\Phi(x) = \lim x$ for all $x \in c$. Consequently, $c \subset V_\sigma$ where V_σ is the set of bounded sequences all of whose $\sigma-$means are equal.

It can be shown [3] that

$$V_\sigma = \left\{ x \in l_\infty : \lim_k t_{km}(x) = L \text{ uniformly in } m \text{ , for some } L = \sigma - \lim x \right\}$$

where

$$t_{km}(x) = \frac{x_m + x_{\sigma(m)} + \ldots + x_{\sigma^k(m)}}{k+1}.$$

We say that a bounded sequence $x = (x_k)$ is $\sigma-$convergent if and only if $x \in V_\sigma$ such that $\sigma^k(m) \neq m$ for all $m \geq 0$, $k \geq 1$.

$[V_\sigma]$ denotes the set of all strongly σ-convergent sequences which has been defined by Mursaleen [5], as

$$[V_\sigma] = \left\{ x \in l_\infty : \lim_m \frac{1}{m} \sum_{k=1}^{m} \left| x_{\sigma^k(n)} - l \right| = 0 \text{ uniformly in } n \right\}.$$

Taking $\sigma(n) = n + 1$, we obtain $[V_\sigma] = [\hat{c}]$ so that strong σ-convergence generalizes the concept of strong almost convergence.

By a lacunary $\theta = (k_r)$; $r = 0, 1, 2, \ldots$ where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers with $k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and $h_r = k_r - k_{r-1}$. The ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r. In [4], the space of lacunary strongly convergent sequences N_θ was defined as follows:

$$N_\theta = \left\{ x = (x_i) : \lim_{r\to\infty} h_r^{-1} \sum_{i \in I_r} |x_i - s| = 0 \text{ for some } s \right\}.$$

A modulus function f is a function acting from $[0, \infty)$ to $[0, \infty)$ such that
(i) $f(x) = 0$ if and only if $x = 0$,
(ii) $f(x + y) \leq f(x) + f(y)$ for all $x, y \geq 0$,
(iii) f increasing,
(iv) f is right continuous at zero.

Since $|f(x)-f(y)| \leq f(|x-y|)$, it follows from condition (iv) that f is continuous on $[0,\infty)$. Furthermore, we have $f(nx) \leq nf(x)$ for all $n \in \mathbb{N}$, from condition (ii) and so

$$f(x) = f(nx\frac{1}{n}) \leq nf(\frac{x}{n}).$$

Hence, for all $n \in \mathbb{N}$

$$\frac{1}{n}f(x) \leq f(\frac{x}{n}).$$

A modulus may be bounded or unbounded. For example, $f(x)=x^p$, for $0<p\leq 1$ is unbounded, but $f(x)=\frac{x}{1+x}$ is bounded. Ruckle [6] used the idea of a modulus function f to construct a class of FK spaces

$$L(f) = \left\{ x=(x_k) : \sum_{k=1}^{\infty} f(|x_k|) < \infty \right\}.$$

The space $L(f)$ is closely related to the space l_1 which is a $L(f)$ space with $f(x)=x$ for all real $x \geq 0$.

Furthermore, modulus function has been discussed in [7], [8], [9], [10], [11],[12], [15] and many others.

By a φ-function we understand a continuous non-decreasing function $\varphi(v)$ defined for $v \geq 0$ and such that $\varphi(0)=0, \varphi(v)>0$ for $v>0$ and $\varphi(v) \to \infty$ as $v \to \infty$.

In [12], [13], [14] and [16] some sequence spaces were studied using φ-function.

Let $\varphi=(\varphi_k)$ and $\psi=(\psi_k)$ be sequences of φ-functions.

A sequence of φ-functions φ is called non weaker than a sequence of φ-function ψ and we write $\psi \prec \varphi$ (or $\psi_k \prec \varphi_k$ for all k) if there are constants $c,b,n,l>0$ such that $c\psi_k(lv) \prec b\varphi_k(nv)$ (for all, large or small v, respectively).

Two sequences of φ-functions φ and ψ are called equivalent and we write $\varphi \sim \psi$ (or $\psi_k \prec \varphi_k$ for all k) if there are positive constants b_1, b_2, c, k_1, k_2, l such that $b_1\varphi_k(k_1v) \leq c\psi_k(lv) \leq b_2\varphi_k(k_2v)$ (for all, large or small v, respectively).

A sequence of φ-functions φ is said to satisfy the Δ_2-condition (for all, large or small v, respectively) if for some constant $l>1$ there is satisfied the inequality $\varphi_k(2v) \leq l\varphi_k(v)$ for all k. For a φ-function satisfying the Δ_2-condition, there is $L>0$ such that

$$\varphi_k(cv) \leq L\varphi_k(v) \tag{1}$$

for v large enough.

Indeed, for every $c>0$ there is an integer s such that $c \leq 2^s$ and

$$\varphi_k(cv) \leq \varphi_k(2^s v) \leq l^s\varphi_k(v) \tag{2}$$

for v large enough.

Let $A=(a_{nk})$ be an infinite matrix such that;

a) A is non-negative, i.e. $a_{nk}\geq 0$ for $n,k=1,2,\ldots$,

b) for an arbitrary positive integer n (or k) there exists a positive integer k_0 (or n_0) such that $a_{nk_0}\neq 0$ (or $a_{n_0k}\neq 0$), respectively,

c) there exists $\lim_n a_{nk}=0$ for $k=1,2,\ldots$,

d) $\sup_n \sum_{k=1}^{\infty} a_{nk}<\infty$,

e) $\sup_n a_{nk}\to 0$ as $k\to\infty$.

In the present paper, we introduce and study some properties of the following sequence space that is defined by using the sequence of φ -functions and the sequence of modulus functions.

2. Main results

Let $\theta=(k_r)$ be a lacunary sequence, $\varphi=(\varphi_k)$ and $f=(f_n)$ be a sequence of φ-functions and a sequence of modulus functions, respectively. Moreover, let a matrix $A=(a_{nk})$ be given as above. Then we define,

$$V_\theta^0\left(\left(A,\varphi_k,\sigma\right),f_n\right)=\Bigg\{x=(x_k)\in w:$$
$$\lim_r \frac{1}{h_r}\sum_{n\in I_r} f_n\left(\sum_{k=1}^{\infty} a_{nk}\varphi_k\left(\left|x_{\sigma^k(m)}\right|\right)\right)=0,\ \text{uniformly in } m\Bigg\}.$$

Throughout this paper, the sequence of modulus functions $f=(f_n)$ satisfy the condition $\lim_{v\to 0^+}\sup_n f_n(v)=0$.

If $x\in V_\theta^0\left(\left(A,\varphi_k,\sigma\right),f_n\right)$ then the sequence x is said to be lacunary strong (A,φ_k,σ)- convergent to zero with respect to a sequence of modulus f.

If we take $\theta=(2^r)$ then we have

$$V^0\left(\left(A,\varphi_k,\sigma\right),f_n\right)=\Bigg\{x\in w:$$
$$\lim_k \frac{1}{k}\sum_{n=1}^{k} f_n\left(\sum_{k=1}^{\infty} a_{nk}\varphi_k\left(\left|x_{\sigma^k(m)}\right|\right)\right)=0,\ \text{uniformly in } m\Bigg\}.$$

When $\varphi_k(x)=x$ for all x and k, we obtain

$$V_\theta^0\left(\left(A,\sigma\right),f_n\right)=\Bigg\{x\in w:$$
$$\lim_r \frac{1}{h_r}\sum_{n\in I_r} f_n\left(\sum_{k=1}^{\infty} a_{nk}\left(\left|x_{\sigma^k(m)}\right|\right)\right)=0,\ \text{uniformly in } m\Bigg\}.$$

If $f_n(x) = x$ for all x and n, we write

$$V_\theta^0(A, \varphi_k, \sigma) = \Bigg\{ x \in w : \lim_r \frac{1}{h_r} \sum_{n \in I_r} \left(\sum_{k=1}^{\infty} a_{nk} \varphi_k \left(\left| x_{\sigma^k(m)} \right| \right) \right) = 0, \text{ uniformly in } m \Bigg\}.$$

When $A = I$, we get the following sequence space,

$$V_\theta^0((I, \varphi_k, \sigma), f_n) = \Bigg\{ x \in w : \lim_r \frac{1}{h_r} \sum_{n \in I_r} f_n\left(\varphi_n\left(\left| x_{\sigma^n(m)} \right| \right) \right) = 0, \text{ uniformly in } m \Bigg\}.$$

If we take $A = I$, $\varphi_k(x) = x$ for all x and k, then we have

$$V_\theta^0((I, \sigma), f_n) = \Bigg\{ x \in w : \lim_r \frac{1}{h_r} \sum_{n \in I_r} f_n\left(\left| x_{\sigma^n(m)} \right| \right) = 0, \text{ uniformly in } m \Bigg\}.$$

If we take $A = I$, $\varphi_k(x) = x$ for all x and k and $f_n(x) = f(x)$ for all x and n then we have

$$V_\theta^0((I, \sigma), f) = \Bigg\{ x \in w : \lim_r \frac{1}{h_r} \sum_{n \in I_r} f\left(\left| x_{\sigma^n(m)} \right| \right) = 0, \text{ uniformly in } m \Bigg\}.$$

If we take $A = I$, $\varphi_k(x) = x$ for all x and k, $f_n(x) = x$ for all x and n then we have

$$V_\theta^0(I, \sigma) = \Bigg\{ x \in w : \lim_r \frac{1}{h_r} \sum_{n \in I_r} \left| x_{\sigma^n(m)} \right| = 0, \text{ uniformly in } m \Bigg\}.$$

If we define the matrix $A = (a_{nk})$ as follows:

$$a_{nk} = \frac{1}{n} \text{ for } n \geq k \quad \text{and} \quad a_{nk} = 0 \text{ for } n < k,$$

then we have the sequence space,

$$V_\theta^0((C, \varphi_k, \sigma), f_n) = \Bigg\{ x \in w : \lim_r \frac{1}{h_r} \sum_{n \in I_r} f_n \left(\frac{1}{n} \sum_{k=1}^{n} \varphi_k \left(\left| x_{\sigma^k(m)} \right| \right) \right) = 0, \text{ uniformly in } m \Bigg\}.$$

If we take $\sigma(m) = m+1$, the sequence spaces $V_\theta^0\left(\left(A, \varphi_k, \sigma\right), f_n\right)$, $V^0\left(\left(A, \varphi_k, \sigma\right), f_n\right)$, $V_\theta^0\left(\left(A, \sigma\right), f_n\right)$, $V_\theta^0\left(A, \varphi_k, \sigma\right)$, $V_\theta^0\left(\left(I, \varphi_k, \sigma\right), f_n\right)$, $V_\theta^0\left(\left(I, \sigma\right), f_n\right)$, $V_\theta^0\left(\left(I, \sigma\right), f\right)$, $V_\theta^0\left(I, \sigma\right)$ and $V_\theta^0\left(\left(C, \varphi_k, \sigma\right), f_n\right)$ reduce to the following spaces of sequences, respectively.

$$V_\theta^0\left(\left(A, \varphi_k\right), f_n\right) = \Bigg\{ x = (x_k) \in w : \lim_r \frac{1}{h_r} \sum_{n \in I_r} f_n\left(\sum_{k=1}^{\infty} a_{nk} \varphi_k\left(|x_{k+m}|\right)\right) = 0, \text{ uniformly in } m \Bigg\},$$

$$V^0\left(\left(A, \varphi_k\right), f_n\right) = \Bigg\{ x \in w : \lim_k \frac{1}{k} \sum_{n=1}^{k} f_n\left(\sum_{k=1}^{\infty} a_{nk} \varphi_k\left(|x_{k+m}|\right)\right) = 0, \text{ uniformly in } m \Bigg\},$$

$$V_\theta^0\left(\left(A\right), f_n\right) = \Bigg\{ x \in w : \lim_r \frac{1}{h_r} \sum_{n \in I_r} f_n\left(\sum_{k=1}^{\infty} a_{nk}\left(|x_{k+m}|\right)\right) = 0, \text{ uniformly in } m \Bigg\},$$

$$V_\theta^0\left(A, \varphi_k\right) = \Bigg\{ x \in w : \lim_r \frac{1}{h_r} \sum_{n \in I_r} \left(\sum_{k=1}^{\infty} a_{nk} \varphi_k\left(|x_{k+m}|\right)\right) = 0, \text{ uniformly in } m \Bigg\},$$

$$V_\theta^0\left(\left(I, \varphi_k\right), f_n\right) = \Bigg\{ x \in w : \lim_r \frac{1}{h_r} \sum_{n \in I_r} f_n\left(\varphi_n\left(|x_{n+m}|\right)\right) = 0, \text{ uniformly in } m \Bigg\},$$

$$V_\theta^0\left(\left(I\right), f_n\right) = \Bigg\{ x \in w : \lim_r \frac{1}{h_r} \sum_{n \in I_r} f_n\left(|x_{n+m}|\right) = 0, \text{ uniformly in } m \Bigg\},$$

$$V_{\theta}^{0}\left(\left(I\right),f\right)=\left\{x\in w:\ \lim_{r}\frac{1}{h_{r}}\sum_{n\in I_{r}}f\left(\left|x_{n+m}\right|\right)=0,\ \text{uniformly in } m\right\},$$

$$V_{\theta}^{0}\left(I\right)=\left\{x\in w:\ \lim_{r}\frac{1}{h_{r}}\sum_{n\in I_{r}}\left(\left|x_{n+m}\right|\right)=0,\ \text{uniformly in } m\right\},$$

$$V_{\theta}^{0}\left(\left(C,\varphi_{k}\right),f_{n}\right)=\left\{x\in w:\ \lim_{r}\frac{1}{h_{r}}\sum_{n\in I_{r}}f_{n}\left(\frac{1}{n}\sum_{k=1}^{n}\varphi_{k}\left(\left|x_{k+m}\right|\right)\right)=0,\ \text{uniformly in } m\right\}.$$

We can note that the space $[V_{\sigma}]$ (for l=0) is equivalent to space $V^{0}\left(I,\sigma\right)$ which has been noticed by the referee.

Now we have,

Theorem 1. *Let us suppose that $\varphi=(\varphi_{k})$ and $\psi=(\psi_{k})$ be two sequences of φ-functions and $\varphi=(\varphi_{k}\left(v\right))$ satisfies the Δ_{2}-condition for large v.*

(i)If $\psi\prec\varphi$ then $V_{\theta}^{0}\left(\left(A,\varphi_{k},\sigma\right),f_{n}\right)\subset V_{\theta}^{0}\left(\left(A,\psi_{k},\sigma\right),f_{n}\right)$.

(ii) If two sequences of φ-functions $(\varphi_{k}\left(v\right))$ and $(\psi_{k}\left(v\right))$ are equivalent for large v and they satisfy the Δ_{2}-condition for large v then $V_{\theta}^{0}\left(\left(A,\varphi_{k},\sigma\right),f_{n}\right)=V_{\theta}^{0}\left(\left(A,\psi_{k},\sigma\right),f_{n}\right)$.

Proof. (i) Let $x=(x_{k})\in V_{\theta}^{0}\left(\left(A,\varphi_{k},\sigma\right),f_{n}\right)$.

Then $\lim_{r}\frac{1}{h_{r}}\sum_{n\in I_{r}}f_{n}\left(\sum_{k=1}^{\infty}a_{nk}\varphi_{k}\left(\left|x_{\sigma^{k}(m)}\right|\right)\right)=0$, uniformly in m. By assumption, $\psi\prec\varphi$, we have

$$\psi_{k}\left(\left|x_{k}\right|\right)\leq b\varphi_{k}\left(c\left|x_{k}\right|\right) \tag{3}$$

for $b,c,v_{0}>0$, all k, and $\left|x_{k}\right|>v_{0}$. Let us denotes $x=x'+x''$, where for all m, $x'=\left(x'_{\sigma^{k}(m)}\right)$ and $x'_{\sigma^{k}(m)}=x_{\sigma^{k}(m)}$ for $\left|x_{\sigma^{k}(m)}\right|<v_{0}$ and $x'_{\sigma^{k}(m)}=0$ for remaining values of k and m. It is easy to see that $x'\in V_{\theta}^{0}\left(\left(A,\psi_{k},\sigma\right),f_{n}\right)$. Furthermore, by the assumptions and the inequality (3) we get

$$\begin{aligned}\frac{1}{h_{r}}\sum_{n\in I_{r}}f_{n}\left(\sum_{k=1}^{\infty}a_{nk}\psi_{k}\left(\left|x''_{\sigma^{k}(m)}\right|\right)\right)&\leq\frac{1}{h_{r}}\sum_{n\in I_{r}}f_{n}\left(b\sum_{k=1}^{\infty}a_{nk}\varphi_{k}\left(c\left|x''_{\sigma^{k}(m)}\right|\right)\right)\\&\leq\frac{1}{h_{r}}\sum_{n\in I_{r}}f_{n}\left(bL\sum_{k=1}^{\infty}a_{nk}\varphi_{k}\left(\left|x''_{\sigma^{k}(m)}\right|\right)\right)\\&\leq\frac{K}{h_{r}}\sum_{n\in I_{r}}f_{n}\left(\sum_{k=1}^{\infty}a_{nk}\varphi_{k}\left(\left|x''_{\sigma^{k}(m)}\right|\right)\right)\end{aligned}$$

where the constants K and L is connected with properties of f and φ functions. We recall that a $\varphi-$function satisfying the Δ_2-condition imply (1) and (2).

Finally we obtain $x'' = (x''_k) \in V_\theta^0\left(\left(A, \psi_k, \sigma\right), f_n\right)$ and in consequence $x \in V_\theta^0\left(\left(A, \psi_k, \sigma\right), f_n\right)$.

(ii) The identity $V_\theta^0\left(\left(A, \varphi_k, \sigma\right), f_n\right) = V_\theta^0\left(\left(A, \psi_k, \sigma\right), f_n\right)$ is proved by using the same argument. ■

Theorem 2. *Let the sequence $\varphi = (\varphi_k(v))$ of φ-functions satisfies the Δ_2-condition for all k and for large v then $V_\theta^0\left(\left(A, \varphi_k, \sigma\right), f_n\right)$ is linear space.*

Proof. Firstly we prove that if $x = (x_k) \in V_\theta^0\left(\left(A, \varphi_k, \sigma\right), f_n\right)$ and α is an arbitrary number then $\alpha x \in V_\theta^0\left(\left(A, \varphi_k, \sigma\right), f_n\right)$. Let us remark that for $0 < \alpha < 1$ we get

$$\frac{1}{h_r}\sum_{n\in I_r} f_n\left(\sum_{k=1}^{\infty} a_{nk}\varphi_k\left(\left|\left(\alpha x_{\sigma^k(m)}\right)\right|\right)\right) \le \frac{1}{h_r}\sum_{n\in I_r} f_n\left(\sum_{k=1}^{\infty} a_{nk}\varphi_k\left(\left|x_{\sigma^k(m)}\right|\right)\right).$$

Moreover, if $\alpha > 1$ then we may find a positive number s such that $\alpha < 2^s$ and we obtain

$$\begin{aligned}\frac{1}{h_r}\sum_{n\in I_r} f_n\left(\sum_{k=1}^{\infty} a_{nk}\varphi_k\left(\left|\left(\alpha x_{\sigma^k(m)}\right)\right|\right)\right) &\le \frac{1}{h_r}\sum_{n\in I_r} f_n\left(d^s\sum_{k=1}^{\infty} a_{nk}\varphi_k\left(\left|x_{\sigma^k(m)}\right|\right)\right)\\ &\le \frac{K}{h_r}\sum_{n\in I_r} f_n\left(\sum_{k=1}^{\infty} a_{nk}\varphi_k\left(\left|x_{\sigma^k(m)}\right|\right)\right)\end{aligned}$$

where d and K are constants connected with the properties of φ and f functions. We recall that a $\varphi-$function satisfying the Δ_2-condition imply (1) and (2). Hence we obtain $\alpha x \in V_\theta^0\left(\left(A, \varphi_k, \sigma\right), f_n\right)$.

Secondly, let $x, y \in V_\theta^0\left(\left(A, \varphi_k, \sigma\right), f_n\right)$ and α, β arbitrary numbers. We will show that $\alpha x + \beta y \in V_\theta^0\left(\left(A, \varphi_k, \sigma\right), f_n\right)$.

$$\begin{aligned}\frac{1}{h_r}\sum_{n\in I_r} f_n\left(\sum_{k=1}^{\infty} a_{nk}\varphi_k\left(\left|\left(\alpha x_{\sigma^k(m)} + \beta y_{\sigma^k(m)}\right)\right|\right)\right) &\le \frac{K_1}{h_r}\sum_{n\in I_r} f_n\left(\sum_{k=1}^{\infty} a_{nk}\varphi_k\left(\left|x_{\sigma^k(m)}\right|\right)\right)\\ &+ \frac{K_2}{h_r}\sum_{n\in I_r} f_n\left(\sum_{k=1}^{\infty} a_{nk}\varphi_k\left(\left|y_{\sigma^k(m)}\right|\right)\right)\end{aligned}$$

where the constants K_1 and K_2 are defined as above. In consequence, $\alpha x+\beta y\in V_\theta^0\left(\left(A,\varphi_k,\sigma\right),f_n\right)$. ■

Now, we give the following Proposition that is necessary for proof of the Theorem 3.

Proposition 1 (15)**.** *Let f be a modulus and let $0<\delta<1$. Then for each $v\geq\delta$ we have $f(v)\leq 2f(1)\delta^{-1}v$.*

Theorem 3. *Let $\varphi=(\varphi_k)$ and $f=(f_n)$ be a given sequence of φ-functions and a sequence of modulus functions, respectively, and $\sup\limits_n f_n(1)<\infty$. Then $V_\theta^0\left(A,\varphi_k,\sigma\right)\subset V_\theta^0\left(\left(A,\varphi_k,\sigma\right),f_n\right)$.*

Proof. Let $x\in V_\theta^0\left(A,\varphi_k,\sigma\right)$ and put $\sup\limits_n f_n(1)=M$. For a given $\varepsilon>0$ we choose $0<\delta<1$ such that $f_n(x)<\varepsilon$ for every $x\in[0,\delta]$ and for all n. We can write

$$\frac{1}{h_r}\sum_{n\in I_r}f_n\left(\sum_{k=1}^{\infty}a_{nk}\varphi_k\left(\left|x_{\sigma^k(m)}\right|\right)\right)=S_1+S_2$$

where

$$S_1=\frac{1}{h_r}\sum_{n\in I_r}f_n\left(\sum_{k=1}^{\infty}a_{nk}\varphi_k\left(\left|x_{\sigma^k(m)}\right|\right)\right)$$

and this sum is taken over

$$\sum_{k=1}^{\infty}a_{nk}\varphi_k\left(\left|x_{\sigma^k(m)}\right|\right)\leq\delta$$

and

$$S_2=\frac{1}{h_r}\sum_{n\in I_r}f_n\left(\sum_{k=1}^{\infty}a_{nk}\varphi_k\left(\left|x_{\sigma^k(m)}\right|\right)\right)$$

and this sum is taken over

$$\sum_{k=1}^{\infty}a_{nk}\varphi_k\left(\left|x_{\sigma^k(m)}\right|\right)>\delta$$

By the definition of the modulus f we have

$$S_1\leq\frac{1}{h_r}\sum_{n\in I_r}f_n\left(\delta\right)<\frac{1}{h_r}\left(h_r\varepsilon\right)=\varepsilon$$

and moreover

$$S_2\leq 2M\frac{1}{\delta}\frac{1}{h_r}\sum_{n\in I_r}\sum_{k=1}^{\infty}a_{nk}\varphi_k\left(\left|x_{\sigma^k(m)}\right|\right)$$

by Proposition 1. Finally we have $x \in V_{\theta}^{0}\left(\left(A, \varphi_{k}, \sigma\right), f_{n}\right)$.

This completes the proof. ∎

Theorem 4. *Let $\varphi=\left(\varphi_{k}\right)$ and $f=\left(f_{n}\right)$ be a given sequence of φ-functions and a sequence of modulus functions, respectively. If $\lim\limits_{v \rightarrow \infty} \inf\limits_{n} \frac{f_{n}(v)}{v}>0$ $V_{\theta}^{0}\left(\left(A, \varphi_{k}, \sigma\right), f_{n}\right)=V_{\theta}^{0}\left(A, \varphi_{k}, \sigma\right)$.*

Proof. If $\lim\limits_{v \rightarrow \infty} \inf\limits_{n} \frac{f_{n}(v)}{v}>0$ then there exists a number $c>0$ such that $f_{n}(v)>c v$ for $v>0$ and $n \in \mathbb{N}$. Let $x \in V_{\theta}^{0}\left(\left(A, \varphi_{k}, \sigma\right), f_{n}\right)$. Clearly

$$
\begin{aligned}
\frac{1}{h_{r}} \sum_{n \in I_{r}} f_{n}\left(\sum_{k=1}^{\infty} a_{n k} \varphi_{k}\left(\left|x_{\sigma^{k}(m)}\right|\right)\right) & \geq \frac{1}{h_{r}} \sum_{n \in I_{r}} c\left(\sum_{k=1}^{\infty} a_{n k} \varphi_{k}\left(\left|x_{\sigma^{k}(m)}\right|\right)\right) \\
& =\frac{c}{h_{r}} \sum_{n \in I_{r}}\left(\sum_{k=1}^{\infty} a_{n k} \varphi_{k}\left(\left|x_{\sigma^{k}(m)}\right|\right)\right).
\end{aligned}
$$

Therefore $x \in V_{\theta}^{0}\left(A, \varphi_{k}, \sigma\right)$. We complete the proof using Theorem 3. ∎

Theorem 5. *Let $\theta=\left(k_{r}\right)$ be a lacunary sequence and $f=\left(f_{n}\right)$ be a sequence of modulus functions.*

(i) If $\liminf q_{r}>1$ then $V^{0}\left(\left(A, \varphi_{k}, \sigma\right), f_{n}\right) \subset V_{\theta}^{0}\left(\left(A, \varphi_{k}, \sigma\right), f_{n}\right)$.

(ii) If $\limsup q_{r}<\infty$ then $V_{\theta}^{0}\left(\left(A, \varphi_{k}, \sigma\right), f_{n}\right) \subset V^{0}\left(\left(A, \varphi_{k}, \sigma\right), f_{n}\right)$.

(iii) If $1<\liminf q_{r} \leq \limsup q_{r}<\infty$ then $V_{\theta}^{0}\left(\left(A, \varphi_{k}, \sigma\right), f_{n}\right)=V^{0}\left(\left(A, \varphi_{k}, \sigma\right), f_{n}\right)$.

Proof. This can be proved by using the same techniques as in [9] and hence we omit the proof. ∎

The next result follows from Theorem 4 and Theorem 5.

Corollary 1. *If $\lim\limits_{v \rightarrow \infty} \inf\limits_{n} \frac{f_{n}(v)}{v}>0$ and $1<\liminf q_{r} \leq \limsup q_{r}<\infty$ then $V_{\theta}^{0}\left(A, \varphi_{k}, \sigma\right)=V^{0}\left(\left(A, \varphi_{k}, \sigma\right), f_{n}\right)$.*

3. $S_{\theta}^{0}\left(A, \varphi_{k}, \sigma\right)$ -statistical convergence

Let the matrix $A=\left(a_{n k}\right)$ be given as previously, $\theta=\left(k_{r}\right)$ be a lacunary sequence, the sequence of φ-functions $\varphi=\left(\varphi_{k}\right)$ and a positive number $\varepsilon>0$ be given. We write,

$$
K_{\theta}^{r}\left(\left(\left(A, \varphi_{k}, \sigma\right), \varepsilon\right)\right)=\left\{n \in I_{r}: \sum_{k=1}^{\infty} a_{n k} \varphi_{k}\left(\left|x_{\sigma^{k}(m)}\right|\right) \geq \varepsilon\right\}.
$$

The sequence x is said to be lacunary (A,φ_k,σ)- statistically convergent to a number zero if for every $\varepsilon>0$

$$\lim_r \frac{1}{h_r}\mu\left(K_\theta^r\left(\left(\left(A,\varphi_k,\sigma\right),\varepsilon\right)\right)\right)=0$$

where $\mu\left(K_\theta^r\left(\left(\left(A,\varphi_k,\sigma\right),\varepsilon\right)\right)\right)$ denotes the number of elements belonging to $K_\theta^r\left(\left(\left(A,\varphi_k,\sigma\right),\varepsilon\right)\right)$. We denote by $S_\theta^0\left(A,\varphi_k,\sigma\right)$, the set of sequences $x=(x_k)$ which are lacunary (A,φ_k,σ)-statistically convergent to a number zero. We write

$$S_\theta^0\left(A,\varphi_k,\sigma\right)=\left\{x=\left(x_k\right):\lim_r \frac{1}{h_r}\mu\left(K_\theta^r\left(\left(\left(A,\varphi_k,\sigma\right),\varepsilon\right)\right)\right)=0\right\}.$$

When we take $\theta=(2^r)$, $S_\theta^0\left(A,\varphi_k,\sigma\right)$ reduces to $S^0\left(A,\varphi_k,\sigma\right)$.

If we take $A=I$ and $\varphi_k(x)=x$ for all k and x, then $S_\theta^0\left(A,\varphi_k,\sigma\right)$ reduces to $S_\theta^0\left(\sigma\right)$ defined by

$$S_\theta^0\left(\sigma\right)=\left\{x=\left(x_k\right):\lim_r \frac{1}{h_r}\left|\left\{k\in I_r:\left(\left|x_{\sigma^k(m)}\right|\right)\geq\varepsilon\right\}\right|=0\right\}.$$

Now we have,

Theorem 6. *Let $\theta=(k_r)$ be a lacunary sequence, $\varphi=\left(\varphi_k\left(v\right)\right)$ and $\psi=\left(\psi_k\left(v\right)\right)$ are two sequences of φ-functions.*

(i) If $\psi\prec\varphi$ and φ_k satisfies the Δ_2-condition for large v and for all k then $S_\theta^0\left(A,\psi_k,\sigma\right)\subset S_\theta^0\left(A,\varphi_k,\sigma\right)$.

(ii) If $\varphi\sim\psi$ and φ_k and ψ_k satisfy the Δ_2-condition for large v and for all k then $S_\theta^0\left(A,\psi_k,\sigma\right)=S_\theta^0\left(A,\varphi_k,\sigma\right)$.

Proof. (i) Let $x\in S_\theta^0\left(A,\psi_k,\sigma\right)$. By assumption we have $\psi_k\left(\left|x_{\sigma^k(m)}\right|\right)\leq b\varphi_k\left(c\left|x_{\sigma^k(m)}\right|\right)$ and we have for all m,

$$\sum_{k=1}^{\infty}a_{nk}\psi_k\left(\left|x_{\sigma^k(m)}\right|\right)\leq b\sum_{k=1}^{\infty}a_{nk}\varphi_k\left(c\left|x_{\sigma^k(m)}\right|\right)\leq K\sum_{k=1}^{\infty}a_{nk}\varphi_k\left(\left|x_{\sigma^k(m)}\right|\right)$$

for $b,c>0$, where the constant K is connected with properties of φ functions. Thus the condition $\sum_{k=1}^{\infty}a_{nk}\psi_k\left(\left|x_{\sigma^k(m)}\right|\right)\geq\varepsilon$ implies the condition $\sum_{k=1}^{\infty}a_{nk}\varphi_k\left(\left|x_{\sigma^k(m)}\right|\right)\geq\varepsilon$ and in consequence we get

$$K_\theta^r\left(\left(\left(A,\varphi_k,\sigma\right),\varepsilon\right)\right)\subset K_\theta^r\left(\left(\left(A,\psi_k,\sigma\right),\varepsilon\right)\right)$$

and

$$\lim_r \frac{1}{h_r}\mu\left(K_\theta^r\left(\left(\left(A,\varphi_k,\sigma\right),\varepsilon\right)\right)\right)\leq\lim_r \frac{1}{h_r}\mu\left(K_\theta^r\left(\left(\left(A,\psi_k,\sigma\right),\varepsilon\right)\right)\right).$$

This completes the proof.

(*ii*) The identity $S_\theta^0(A,\psi_k,\sigma)=S_\theta^0(A,\varphi_k,\sigma)$ is proved by using the same argument. ■

Theorem 7. *Let $f=(f_n)$ be given a sequence of modulus functions. If $\inf_n f_n(v)>0$, for $v>0$, then*

$$V_\theta^0\left(\left(A,\varphi_k,\sigma\right),f_n\right)\subset S_\theta^0\left(A,\varphi_k,\sigma\right).$$

Proof. If $\inf_n f_n(v)>0$ then there exists a number $\alpha>0$ such that $f_n(v)\geq\alpha$ for $v>0$ and $n\in\mathbb{N}$. Let $x\in V_\theta^0\left(\left(A,\varphi_k,\sigma\right),f_n\right)$.

$$\begin{aligned}\frac{1}{h_r}\sum_{n\in I_r}f_n\left(\sum_{k=1}^{\infty}a_{nk}\varphi_k\left(\left|x_{\sigma^k(m)}\right|\right)\right)&\\ &\geq\frac{1}{h_r}\sum_{\substack{n\in I_r\\ \sum_{k=1}^{\infty}a_{nk}\varphi_k\left(\left|x_{\sigma^k(m)}\right|\right)\geq\varepsilon}}f_n\left(\sum_{k=1}^{\infty}a_{nk}\varphi_k\left(\left|x_{\sigma^k(m)}\right|\right)\right)\\ &\geq\frac{\alpha}{h_r}\left|\left\{n\in I_r:\sum_{k=1}^{\infty}a_{nk}\varphi_k\left(\left|x_{\sigma^k(m)}\right|\right)\geq\varepsilon\right\}\right|\end{aligned}$$

and it follows that $x\in S_\theta^0(A,\varphi_k,\sigma)$. ■

Theorem 8. *Let $f=(f_n)$ be given a sequence of modulus functions. If $\sup_v\sup_n f_n(v)<\infty$ then*

$$S_\theta^0\left(A,\varphi_k,\sigma\right)\subset V_\theta^0\left(\left(A,\varphi_k,\sigma\right),f_n\right).$$

Proof. We suppose $T(v)=\sup_n f_n(v)$ and $T=\sup_v T(v)$. Let $x\in S_\theta^0(A,\varphi_k,\sigma)$. Since $f_n(v)\leq T$ for $n\in\mathbb{N}$ and $v>0$, we have

$$\begin{aligned}\frac{1}{h_r}\sum_{n\in I_r}f_n\left(\sum_{k=1}^{\infty}a_{nk}\varphi_k\left(\left|x_{\sigma^k(m)}\right|\right)\right)&\\ &=\frac{1}{h_r}\sum_{\substack{n\in I_r\\ \sum_{k=1}^{\infty}a_{nk}\varphi_k\left(\left|x_{\sigma^k(m)}\right|\right)\geq\varepsilon}}f_n\left(\sum_{k=1}^{\infty}a_{nk}\varphi_k\left(\left|x_{\sigma^k(m)}\right|\right)\right)\\ &\quad+\frac{1}{h_r}\sum_{\substack{n\in I_r\\ \sum_{k=1}^{\infty}a_{nk}\varphi_k\left(\left|x_{\sigma^k(m)}\right|\right)<\varepsilon}}f_n\left(\sum_{k=1}^{\infty}a_{nk}\varphi_k\left(\left|x_{\sigma^k(m)}\right|\right)\right)\end{aligned}$$

$$\leq \frac{1}{h_r} T \left| \left\{ n \in I_r : \sum_{k=1}^{\infty} a_{nk} \varphi_k \left(\left| x_{\sigma^k(m)} \right| \right) \geq \varepsilon \right\} \right| + T(\varepsilon).$$

Taking the limit as $\varepsilon \to 0$, it follows that $x \in V_\theta^0 \left(\left(A, \varphi_k, \sigma \right), f_n \right)$. ■

Corollary 2. *Let $f = (f_n)$ be a given sequence of modulus functions. If $\inf_n f_n(v) > 0 (v > 0)$ and $\sup_v \sup_n f_n(v) < \infty$ then $S_\theta^0 \left(A, \varphi_k, \sigma \right) = V_\theta^0 \left(\left(A, \varphi_k, \sigma \right), f_n \right)$.*

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