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COMMON FIXED POINT THEOREMS FOR COMPATIBLE MAPS OF TYPE (P) AND KIND OF WEAKLY COMMUTING MAPS

Abstract. In this article, the existence of a unique common fixed point of two families of compatible maps of type (P) on a complete metric space and a common fixed point theorem for four mappings on a metric space are proved. These theorems are an improvement over the theorems generalizes Banach Fixed Point Theorems [1], Kannan Fixed Point Theorem [12], Edelstein Fixed Point Theorem [6], Boyd and Wong's Fixed Point Theorem [2], Ćirić's Fixed Point Theorems [3], Das and Naik's [5], Fixed Point Theorems for at least a pair of maps of the Jungck [7], Fixed Point Theorem and Theorem 3.1 [16].

Key words: common fixed point, compatible maps of type (P), weakly compatible maps, metric space.

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1. Introduction

Studies of common fixed points of commuting maps were initiated by Jungck [7]. Jungck [8] further made a generalization of commuting maps by introducing the notion of compatible mappings. In [16], Pathak ct al defined compatible mappings of type (P). The notion of compatible mappings developed in many direction such as see [[4], [9], [10], [11], [14] and [15]]. The purpose of this article is to prove Theorem 3.1 [16] for two families of compatible mappings of type (P) on a complete metric space and a common fixed point theorem for four mappings in a metric space.

Definition 1 ([16]). *Let $S, T : (X, d) \to (X, d)$ be mappings. S and T are said to be compatible of type (P) if*

$$\lim_{n \longrightarrow \infty} d(SSx_n, TTx_n) = 0$$

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whenever $\{x_n\}$ *is a sequence in* X *such that* $Sx_n, Tx_n \to z$, *for some* $z \in X$, *as* $n \to \infty$.

Proposition 1 ([16]). *Let* $S, T : (X, d) \to (X, d)$ *be mappings. Let* S *and* T *are compatible of type* (P) *and let* $Sx_n, Tx_n \to z$ *as* $n \to \infty$ *for some* $z \in X$. *Then we have the following:*
(a) $\lim_{n\to\infty} TTx_n = Sz$ *if* S *is continuous at* z,
(b) $\lim_{n\to\infty} SSx_n = Tz$ *if* T *is continuous at* z,
(c) $STz = TSz$ *and* $Sz = Tz$ *if* S *and* T *are continuous at* z.

Proposition 2 ([16]). *Let* $S, T : (X, d) \to (X, d)$ *be mappings. If* S *and* T *are compatible of type* (P) *and* $Sz = Tz$ *for some* $z \in X$, *then* $SSz = STz = TTz$.

Lemma 1 ([17]). *For any* $t > 0$, $\gamma(t) < t$, *iff* $\lim_{n\to\infty} \gamma^n(t) = 0$ *where* γ^n *denotes the* n*-times composition of* γ.

In this article a common fixed point theorem for a family of compatible maps of type (P) in a compatible metric space is given. Finally, the existence of a common fixed point for four mappings in a metric space is proved.

2. Main results

In this section we shall prove a common fixed point theorem for any even number of compatible maps of type (P) in a complete metric space.

Let ϱ be the family of all mappings $\phi : (\Re^+)^5 \to \Re^+$, where $\Re^+ = [0, +\infty)$ and each ϕ satisfies the following conditions:
(a) ϕ is upper semicontinuous on $\Re^+$
(b) ϕ is non-decreasing in each coordinate variable, and
(c) for each $t > 0$,
$\phi(t, t, 0, \alpha t, 0) \le \beta t$ and $\phi(t, t, 0, 0, \alpha t) \le \beta t$ where $\beta = 1$ for $\alpha = 2$ and $\beta < 1$ for $\alpha < 2$.

$$\gamma(t) = \phi(t, t, a_1 t, a_2 t, a_3 t) < t$$

where $\gamma : R^+ \to R^+$ is a mapping and $a_1 + a_2 + a_3 = 4$.

Now we prove our main result.

Theorem 1. *Let* $P_1, P_2, \cdots, P_{2n}$, A *and* B *are self maps on a complete metric space* (X, d), *satisfying conditions:*

(I) $A(X) \subseteq P_1 P_3 \cdots P_{2n-1}(X)$, $B(X) \subseteq P_2 P_4 \cdots P_{2n}(X)$;

(II)

$$\begin{aligned} P_2(P_4 \cdots P_{2n}) &= (P_4 \cdots P_{2n})P_2, \\ P_2P_4(P_6 \cdots P_{2n}) &= (P_6 \cdots P_{2n})P_2P_4, \\ &\vdots \end{aligned}$$

$$\begin{aligned}
P_2\cdots P_{2n-2}(P_{2n}) &= (P_{2n})P_2\cdots P_{2n-2},\\
A(P_4\cdots P_{2n}) &= (P_4\cdots P_{2n})A,\\
A(P_6\cdots P_{2n}) &= (P_6\cdots P_{2n})A,\\
&\vdots\\
AP_{2n} &= P_{2n}A,\\
P_1(P_3\cdots P_{2n-1}) &= (P_3\cdots P_{2n-1})P_1,\\
P_1P_3(P_5\cdots P_{2n-1}) &= (P_5\cdots P_{2n-1})P_1P_3,\\
&\vdots\\
P_1\cdots P_{2n-3}(P_{2n-1}) &= (P_{2n-1})P_1\cdots P_{2n-3},\\
B(P_3\cdots P_{2n-1}) &= (P_3\cdots P_{2n-1})B,\\
B(P_5\cdots P_{2n-1}) &= (P_5\cdots P_{2n-1})B,\\
&\vdots\\
BP_{2n-1} &= P_{2n-1}B;
\end{aligned}$$

(III) One of $\prod_{i=1}^{n} P_{2i-1} = P_2\cdots P_{2n}$, A, B and $\prod_{i=1}^{n} P_{2i} = P_1\cdots P_{2n-1}$ is continuous;

(IV) The pair $(A, P_2\cdots P_{2n})$ and the pair $(B, P_1\cdots P_{2n-1})$ are compatible of type (P);

(V) There exists $\phi \in \varrho$ such that

$$\begin{aligned}
d(Au, Bv) \le \phi(&d(Au, P_2P_4\cdots P_{2n}u), d(Bv, P_1P_3\cdots P_{2n-1}v),\\
&d(Au, P_1P_3\cdots P_{2n-1}v),\\
&d(Bv, P_2P_4\cdots P_{2n}u), d(P_2P_4\cdots P_{2n}u, P_1P_3\cdots P_{2n-1}v))
\end{aligned}$$

for all $u, v \in X$. Then $P_1, P_2, \cdots, P_{2n}$, A and B have a unique common fixed point in X.

Proof. Let $x_0 \in X$. From condition (I) there exists $x_1, x_2 \in X$ such that $Ax_0 = P_1P_3\cdots P_{2n-1}x_1 = y_0$ and $Bx_1 = P_2P_4\cdots P_{2n}x_2 = y_1$. Inductively we can construct sequence $\{x_n\}$ and $\{y_n\}$ in X

$$Ax_{2k} = P_1P_3\cdots P_{2n-1}x_{2k+1} = y_{2k}$$

and

$$Bx_{2k+1} = P_2P_4\cdots P_{2n}x_{2k+2} = y_{2k+1},$$

for $k \in \mathbb{N}$. By a similar proof of Lemma 3.2 and Lemma 3.3 [16], is proved that $\{y_n\}$ is a Cauchy sequence in X. Since X is complete, there exists some $z \in X$ such that $y_n \to z$. Also, for its subsequences we have:

$$\begin{aligned}
Bx_{2k+1} \to z \quad &\text{and} \quad P_1P_3\cdots P_{2n-1}x_{2k+1} \to z,\\
Ax_{2k} \to z \quad &\text{and} \quad P_2P_4\cdots P_{2n}x_{2k} \to z.
\end{aligned}$$

Now, suppose that $P_2P_4\cdots P_{2n}$ is continuous.
Denote $P'_1 = P_2P_4\cdots P_{2n}$ and $P'_2 = P_1P_3\cdots P_{2n-1}$. Since (A, P'_1) is compatible of type (P), by Proposition 1,

$$AAx_{2k} \to P'_1z \quad \text{and} \quad P'_1Ax_{2k} \to P'_1z.$$

a) Putting $u = P_2P_4\cdots P_{2n}x_{2k} = P'_1x_{2k},\ v = x_{2k+1}$, and $P'_2 = P_1P_3\cdots P_{2n-1}$ in condition (V), we have:

$$\begin{aligned} d(AAx_{2k}, Bx_{2k+1}) \leq\ & \phi(d(AAx_{2k}, P'_1Ax_{2k}),\\ & d(Bx_{2k+1}, P'_2x_{2k+1}),\\ & d(AAx_{2k}, P'_2x_{2k+1}),\\ & d(Bx_{2k+1}, P'_1Ax_{2k}),\\ & d(\ P'_1Ax_{2k}, P'_2x_{2k+1})). \end{aligned}$$

Letting $k \to \infty$, we get:

$$\begin{aligned} d(P'_1z, z) &\leq \phi(0, 0, d(P'_1z, z), d(z, P'_1z), d(\ P'_1z, z))\\ &< \gamma(d(P'_1z, z))\\ &< \mathrm{d}(P'_1z, z) \end{aligned}$$

which is a contradiction. Thus $P'_1z = z$, i.e., $P_2P_4\cdots P_{2n}z = z$.

b) Putting $u = z,\ v = x_{2k+1},\ P'_1 = P_2P_4\cdots P_{2n}$ and $P'_2 = P_1P_3\cdots P_{2n-1}$ in condition (V), we have:

$$\begin{aligned} d(Az, Bx_{2k+1}) \leq\ & \phi(d(Az, P'_1z),\\ & d(Bx_{2k+1}, P'_2x_{2k+1}),\\ & d(Az, P'_2x_{2k+1}),\\ & d(Bx_{2k+1}, P'_1z),\\ & d(\ P'_1z, P'_2x_{2k+1})). \end{aligned}$$

Letting $k \to \infty$, we get:

$$\begin{aligned} d(Az, z) &\leq \phi(d(Az, z), 0, d(Az, z), 0, 0)\\ &< \gamma(d(Az, z))\\ &< \mathrm{d}(Az, z). \end{aligned}$$

Hence $d(Az, z) = 0$. Therefore, $Az = P_2P_4\cdots P_{2n}z = z$.

c) Putting $u = P_4 \cdots P_{2n}z$, $v = x_{2k+1}$, $P'_1 = P_2P_4 \cdots P_{2n}$ and $P'_2 = P_1P_3 \cdots P_{2n-1}$ in condition (V), and using the condition $P_2(P_4 \cdots P_{2n}) = (P_4 \cdots P_{2n})P_2$ and $A(P_4 \cdots P_{2n}) = (P_4 \cdots P_{2n})A$ in condition (II), we get:

$$\begin{aligned} d(AP_4 \cdots P_{2n}z, Bx_{2k+1}) \leq\ & \phi(d(AP_4 \cdots P_{2n}z, P'_1P_4 \cdots P_{2n}z), \\ & d(Bx_{2k+1}, P'_2x_{2k+1}), \\ & d(AP_4 \cdots P_{2n}z, P'_2x_{2k+1}), \\ & d(Bx_{2k+1}, P'_1P_4 \cdots P_{2n}z), \\ & d(\ P'_1P_4 \cdots P_{2n}z, P'_2x_{2k+1})). \end{aligned}$$

Letting $k \to \infty$, we get:

$$\begin{aligned} d(P_4 \cdots P_{2n}z, z) &\leq \phi(d(P_4 \cdots P_{2n}z, z), 0, d(P_4 \cdots P_{2n}z, z), 0, 0), \\ &< \gamma(d(P_4 \cdots P_{2n}z, z)) \\ &< \mathrm{d}(P_4 \cdots P_{2n}z, z). \end{aligned}$$

Hence it follows that $P_4 \cdots P_{2n}z = z$. Then $P_2(P_4 \cdots P_{2n}z) = P_2z$ and so $P_2z = P_2P_4 \cdots P_{2n}z = z$.

Continuing this procedure, we obtain

$$Az = P_2z = P_4z = \cdots = P_{2n}z = z.$$

d) As $A(X) \subseteq P_1P_3 \cdots P_{2n-1}(X)$, there exists $v \in X$ such that $z = Az = P_1P_3 \cdots P_{2n-1}v$. Putting $u = x_{2k}, P'_1 = P_2P_4 \cdots P_{2n}$ and $P'_2 = P_1P_3 \cdots P_{2n-1}$ in condition (V), we have:

$$\begin{aligned} d(Ax_{2k}, Bv) \leq\ & \phi(d(Ax_{2k}, P'_1x_{2k}), \\ & d(Bv, P'_2v), \\ & d(Ax_{2k}, P'_2v), \\ & d(Bv, P'_1x_{2k}), \\ & d(\ P'_1x_{2k}, P'_2v)). \end{aligned}$$

Letting $k \to \infty$, we get:

$$\begin{aligned} d(z, Bv) &\leq \phi(0, d(Bv, z), 0, d(Bv, z), 0) \\ &< \gamma(d(Bv, z)) \\ &< \mathrm{d}(Bv, z). \end{aligned}$$

Therefore $Q_1v = z$. Hence, $P_1P_3 \cdots P_{2n-1}v = Bv = z$. As $(B, P_1 \cdots P_{2n-1})$ is compatible mappings type of type (P) and $P_1P_3 \cdots P_{2n-1}v = Bv$, by Proposition 2 $d(B\ P'_2v, P'_2P'_2v)) = 0$. Hence $Bz = BP'_2v = P'_2P'_2v = P'_2z$. Thus $P_1P_3 \cdots P_{2n-1}z = Bz$.

e) Putting $u = x_{2k}$, $v = z$, $P'_1 = P_2P_4 \cdots P_{2n}$ and $P'_2 = P_1P_3 \cdots P_{2n-1}$ in condition (V), we have:

$$\begin{aligned} d(z, Bz) = d(Az, Bz) \leq\ & \phi(d(Az, P'_1z), \\ & d(Bz, P'_2z), \\ & d(Az, P'_2z), \\ & d(Bz, P'_1z), \\ & d(P'_1z, P'_2z)). \end{aligned}$$

Letting $k \to \infty$, we get:

$$\begin{aligned} d(z, Bz) \leq\ & \phi(0, d(Bz, z), 0, d(Bz, z), 0) \\ <\ & \gamma(d(Bz, z)) < \mathrm{d}(Bz, z). \end{aligned}$$

Therefore $Bz = z$. Hence, $P_1P_3 \cdots P_{2n-1}z = Bz = z$.

f) Putting $u = x_{2k}$, $v = P_3 \cdots P_{2n-1}z$, $P'_1 = P_2P_4 \cdots P_{2n}$ and $P'_2 = P_1P_3 \cdots P_{2n-1}$ in condition (V), we have:

$$\begin{aligned} d(Ax_{2k}, BP_3 \cdots P_{2n-1}z) \leq\ & \phi(d(Ax_{2k}, P'_1x_{2k}), \\ & d(BP_3 \cdots P_{2n-1}z, P'_2P_3 \cdots P_{2n-1}z), \\ & d(Ax_{2k}, P'_2P_3 \cdots P_{2n-1}z), \\ & d(BP_3 \cdots P_{2n-1}z, P'_1x_{2k}), \\ & d(\ P'_1x_{2k}, P'_2P_3 \cdots P_{2n-1}z)). \end{aligned}$$

Letting $k \to \infty$, we get:

$$\begin{aligned} d(z, P_3 \cdots P_{2n-1}z) \leq\ & \phi(0, 0, d(z, P_3 \cdots P_{2n-1}z), d(P_3 \cdots P_{2n-1}z, z), \\ & d(z, P_3 \cdots P_{2n-1}z)) \\ <\ & \gamma(d(z, P_3 \cdots P_{2n-1}z)) \\ <\ & \mathrm{d}(z, P_3 \cdots P_{2n-1}z). \end{aligned}$$

Therefore $P_3 \cdots P_{2n-1}z = z$. Hence, $P_1z = z$. Continuing this procedure, we have:

$$Bz = P_1z = P_3z = \cdots = P_{2n-1}z.$$

Thus we proved

$$Az = Bz = P_1z = P_2z = \cdots = P_{2n-1}z = P_{2n}z = z.$$

■

Proof of uniqueness. Let z' be another common fixed point of mentioned maps, then $Az' = Bz' = P_1z' = P_2z' = \cdots = P_{2n}z' = z'$. Putting

$u = z, v = z', P'_1 = P_2P_4 \cdots P_{2n}$ and $P'_2 = P_1P_3 \cdots P_{2n-1}$ in condition (V), we have:

$$\begin{aligned} d(Az, Bz') \leq \phi(&d(Az, P_2P_4 \cdots P_{2n}z), d(Bz', P_1P_3 \cdots P_{2n-1}z'), \\ &d(Az, P_1P_3 \cdots P_{2n-1}z'), \\ &d(Bz', P_2P_4 \cdots P_{2n}z), d(P_2P_4 \cdots P_{2n}z, P_1P_3 \cdots P_{2n-1}z')). \end{aligned}$$

It means that

$$d(z, z') \leq \gamma(d(z, z')).$$

Thus $z = z'$ and this shows that z is a unique common fixed point of the maps.

Similarly, we can also complete the proof when A, B and $P_1 \cdots P_{2n-1}$ is continuous. This complete the proof. ■

Remark 1. Theorem 1 is generalization of Theorem 3.1 [16].

Now we shall prove a common fixed point theorem for four mappings in metric space. Let $\mathfrak{R}^+$ be the set non-negative real numbers and Ψ be the family of mappings $\varphi : (\mathfrak{R}^+)^5 \to \mathfrak{R}^+$ such that

(i) φ is non-decreasing,

(ii) φ is upper-semi-continuous in each coordinate variable

(iii) $\gamma(t) = \varphi(t, t, a_1t, a_2t, t) < t$, where $\gamma : \mathfrak{R}^+ \to \mathfrak{R}^+$ is a mapping with $\gamma(0) = 0$ and $a_1 + a_2 = 2$.

The following theorem is an improvement over the theorems generalizes Banach Fixed Point Theorems [1], Kannan Fixed Point Theorem [12], Edelstein Fixed Point Theorem [6], Boyd and Wong's Fixed Point Theorem [2], Ćirić's Fixed Point Theorems [3], Das and Naik's Fixed Point Theorems [5] for at least a pair of maps of the Jungck Fixed Point Theorem Type [7] in which the least possibility is that at least one self mapping is continuous on the point of convergence. We have oppose to assume any mapping is continuous. Also we have relaxed the completeness of the metric space (X, d). Many corollaries are also given of this theorem. Here any kind of weakly commuting means we can choose the pair from Murthy [13].

Theorem 2. *Let (X, d) be a metric space and let A, B, S and T be mappings of X into itself such that:*

(I) $A(X) \bigcup B(X) \subset S(X) \bigcap T(X)$,

(II) The pairs (A, S) and (B, T) are any kind of weakly commuting maps,

(III) $$\begin{aligned} &[1 + \alpha d(Sx, Ty)].d(Ax, By) \\ &\quad \leq \alpha \max\{d(Sx, Ax)d(Ty, By), d(Sx, By)d(Ty, Ax)\} \\ &\qquad + \varphi(d(Sx, Ty), d(Sx, Ax)d(Ty, By), d(Sx, By)d(Ty, Ax)) \end{aligned}$$

for every $x, y \in X$, where $\alpha \geq 0$ and $\varphi \in \Psi$. If $S(X) \bigcap T(X)$ be a closed subspace of X, then

(i) (A, S) and (B, T) are coinciding at a common point,

(ii) A, B, S and T have a unique common fixed point in X.

Proof. Since $A(X) \bigcup B(X) \subset S(X) \bigcap T(X)$, we can choose $A(X) \subset T(X)$, for any arbitrary point $x_0 \in X$, we can choose a point $x_1 \in X$ such that $y_0 = Ax_0 = Tx_1$. Since $B(X) \subset S(X)$, for the point x_1, we can choose a point $x_2 \in X$, such that $y_1 = Bx_1 = Sx_2$ and so on. Inductively, we can define a sequence $\{x_n\}$ in X such that

$$y_{2n} = Tx_{2n+1} = Ax_{2n}, y_{2n+1} = Sx_{2n+2} = Bx_{2n+1}, \quad \text{for} \quad n = 0, 1, 2, \ldots \tag{1}$$

We shall prove that $\{y_n\}$ be a Cauchy sequence. Putting $x = x_{2n}$ and $y = x_{2n+1}$ in (III), we have

$$\begin{aligned}
&[1 + \alpha d(Sx_{2n+1}, Tx_{2n+1})]d(Ax_{2n}, Bx_{2n+1}) \\
&\quad \leq \alpha \max\{d(Sx_{2n}, Ax_{2n})d(Tx_{2n+1}, Bx_{2n+1}), \\
&\qquad\qquad d(Sx_{2n}, Bx_{2n+1})d(Tx_{2n+1}, Ax_{2n})\} \\
&\quad + \varphi(d(Sx_{2n}, Tx_{2n+1}), d(Sx_{2n}, Ax_{2n})d(Tx_{2n+1}, Bx_{2n+1}), \\
&\qquad d(Sx_{2n}, Bx_{2n+1})d(Tx_{2n+1}, Ax_{2n})),
\end{aligned}$$

i.e;

$$\begin{aligned}
&[1 + \alpha d(y_{2n-1}, y_{2n})]d(y_{2n}, y_{2n+1}) \\
&\quad \leq \alpha \max\{d(y_{2n-1}, y_{2n})d(y_{2n}, y_{2n}), d(y_{2n-1}, y_{2n+1})d(y_{2n}, y_{2n})\} \\
&\quad + \varphi(d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n})d(y_{2n}, y_{2n+1}), \\
&\qquad d(y_{2n-1}, y_{2n+1})d(y_{2n}, y_{2n})),
\end{aligned}$$

i.e;

$$\begin{aligned}
&d(y_{2n}, y_{2n+1}) + \alpha d(y_{2n-1}, y_{2n})d(y_{2n}, y_{2n+1}) \\
&\leq \alpha d(y_{2n-1}, y_{2n})d(y_{2n}, y_{2n+1}) \\
&+ \varphi(d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n})d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n+1})d(y_{2n}, y_{2n})).
\end{aligned}$$

So

$$\begin{aligned}
d(y_{2n}, y_{2n+1}) \leq\ & \varphi(d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n})d(y_{2n}, y_{2n+1}), \\
& d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1}).d(y_{2n}, y_{2n})).
\end{aligned}$$

If $d(y_{2n}, y_{2n+1}) > d(y_{2n-1}, y_{2n})$ for some n, then $d(y_{2n}, y_{2n+1}) \leq \gamma(d(y_{2n}, y_{2n+1})) < d(y_{2n}, y_{2n+1})$, which is a contradiction. Thus we have

$$d(y_{2n}, y_{2n+1}) \leq \gamma(d(y_{2n-1}, y_{2n})).$$

Similarly,

$$d(y_{2n+1}, y_{2n+2}) \leq \gamma(d(y_{2n}, y_{2n+1})).$$

Proceeding in this way, we have

$$d(y_n, y_{n+1}) \leq \gamma(d(y_{n-1}, y_n)) \leq \gamma^2(d(y_{n-2}, y_{n-1})) \leq ... \leq \gamma^n(d(y_0, y_1))$$

by Lemma 1, $\lim_{n\to\infty} \gamma^n(d(y_0, y_1)) = 0$ and in turn it implies that $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$. Then, by a similar proof of Theorem 4.1 [13], is proved that $\{y_n\}$ is a Cauchy sequence in X. Since $A(X) \subset T(X)$ and $B(X) \subset S(X)$, then there exists u, v in X such that $Su = w$ and $Tv = w$, respectively. By (III)

$$\begin{aligned}
&[1 + \alpha d(Su, Tx_{2n+1})]d(Au, Bx_{2n+1}) \\
&\quad \leq \alpha \max\{d(Su, Au)d(Tx_{2n+1}, Bx_{2n+1}), d(Su, Bx_{2n+1})d(Tx_{2n+1}, Au)\} \\
&\qquad + \varphi(d(Su, Tx_{2n+1}), d(Su, Au)d(Tx_{2n+1}, Bx_{2n+1}), \\
&\qquad\qquad d(Su, Bx_{2n+1})d(Tx_{2n+1}, Au)).
\end{aligned}$$

Letting $n \to \infty$, we have

$$\begin{aligned}
d(Au, w) &\leq \varphi(0, d(Au, w), 0, 0, d(Au, w)) \\
&< d(Au, w),
\end{aligned}$$

which means that $Su = Au = w$. Similarly, we can show that $Tv = Bv = w$.

Now we shall assume that (A, S) be a weak compatible pair of type (A), so $ASu = SSu$ implies that $Aw = Sw$. Similarly, $Tw = Bw$ by assuming (B, T) as a weak compatible pair of type (A). Now we shall prove that w is a common fixed point of A and S. let if possible $Aw \neq w$, then by (III), we have

$$\begin{aligned}
&[1 + \alpha d(Sw, Tx_{2n+1})]d(Aw, Bx_{2n+1}) \\
&\quad \leq \alpha \max\{d(Sw, Aw)d(Tx_{2n+1}, Bx_{2n+1}), d(Sw, Bx_{2n+1})d(Tx_{2n+1}, Aw)\} \\
&\qquad + \varphi(d(Sw, Tx_{2n+1}), d(Sw, Aw)d(Tx_{2n+1}, Bx_{2n+1}), \\
&\qquad\qquad d(Sw, Bx_{2n+1})d(Tx_{2n+1}, Aw)).
\end{aligned}$$

Taking $n \to \infty$, we have

$$\begin{aligned}
&[1 + \alpha d(Sw, w)]d(Aw, w) \\
&\quad \leq \alpha \; max\{0, d(Sw, w)d(w, Aw)\} \\
&\qquad + \varphi(d(Sw, w), d(Sw, Aw), d(w, w), d(Sw, w), d(w, Aw)).
\end{aligned}$$

So

$$\begin{aligned}
d(Aw, w) &\leq \varphi(d(Aw, w), 0, 0, d(Aw, w), d(w, Aw)) \\
&< d(Aw, w).
\end{aligned}$$

implies $Aw = w = Sw$. i.e. w is a common fixed point A and S. Similarly, we can show that w is a common fixed point of B and T. By uniqueness of w. w is common fixed point of A, B, S and T. This completes the proof of the Theorem. ■

As an immediate consequence of the Theorem 2. with $A = B$, we have the following:

Corollary 1. *Let A, S and T be mappings from a metric space (X, d) into itself satisfying:*

(I) $A(X) \subset S(X) \bigcap T(X)$,

(II)
$$[1 + \alpha d(Sx, Ty)]d(Ax, Ay) \\ \leq \alpha \max\{d(Sx, Ax)d(Ty, Ay), d(Sx, Ay).d(Ty, Ax)\}, \\ + \varphi(d(Sx, Ty), d(Sx, Ax), d(Ty, Ay), d(Sx, Ay), d(Ty, Ax)).$$

for every $x, y \in X$, where $\varphi \in \Psi$ and $\alpha \geq 0$. If $S(X) \bigcap T(X)$ be a closed subspace of X, then (i) A, S and A, T have a coincidence point. Indeed, if A is one-to-one, then (ii) A, S and T have a coincidence point If A, S and A, T are weakly compatible maps of type (A), then (iii) A, S and T have a unique common fixed point in X.

Proof. Omitted as it follows in the lines of Theorem 2. ■

Corollary 2. *Let A, B, S and T be mappings from a metric space (X, d) into itself satisfying::*

(I) $A(X) \bigcup B(X) \subset S(X) \bigcap T(X)$,

(II)
$$[1 + \alpha d(Sx, Ty)]d(Ax, By) \\ \leq \alpha \max\{d(Sx, Ax)d(Ty, By), d(Sx, By)d(Ty, Ax)\}, \\ + \beta \max\{d(Sx, Ty), d(Sx, Ax), d(Ty, By), d(Sx, By), d(Ty, Ax)\}$$

for every $x, y \in X$, where $\alpha \geq 0$ and $\beta \in (0, 1)$. If $S(X) \bigcap T(X)$ be a closed subspace of X, then (i) A, S, B and T have a coincidence point. If A, S and B, T are weakly compatible maps of type (A), then (ii) A, B, S and T have a unique common fixed point in X.

Proof. Define $\varphi : (\Re^+)^5 \to \Re^+$ by $\varphi(t_1, t_2, t_3, t_4, t_5) = \beta \max(t_1, t_2, t_3, t_4, t_5)$, then proof follows. ■

Corollary 3. *Let A, B, S and T be mappings from a metric space (X, d) into itself satisfying:*

(I) $A(X) \bigcup B(X) \subset S(X) \bigcap T(X)$,

(II)
$$[1 + \alpha d(Sx, Ty)]d(Ax, By) \\ \leq \alpha \max\{d(Sx, Ax)d(Ty, By), d(Sx, By).d(Ty, Ax)\}, \\ + \beta \max\{d(Sx, Ty), d(Sx, Ax), d(Ty, By), \tfrac{1}{2}[d(Sx, By) + d(Ty, Ax)]\}$$

for every $x, y \in X$, where $\alpha \geq 0$ and $\beta \in (0,1)$. If $S(X) \bigcap T(X)$ be a closed subspace of X, then (i) A, S, B and T have a coincidence point. If A, S and B, T are weakly compatible maps of type (A), then (ii) A, B, S and T have a unique common fixed point in X.

Proof. Define $\varphi : (\Re^+)^5 \to \Re^+$ by $\varphi(t_1, t_2, t_3, t_4, t_5) = \beta \max(t_1, t_2, t_3, \frac{1}{2}[t_4 + t_5])$, then proof follows. ∎

Corollary 4. *Let A, B, S and T be mappings from a metric space (X, d) into itself satisfying:*

(I) $A(X) \bigcup B(X) \subset S(X) \bigcap T(X)$,

(II)
$$[1 + \alpha d(Sx, Ty)]d(Ax, By) \leq \alpha \max\{d(Sx, Ax)d(Ty, By), d(Sx, By)d(Ty, Ax)\}, + \beta_1[d(Sx, Ty)] + \beta_2[d(Sx, Ax) + d(Ty, By)] + \beta_3[d(Sx, By) + d(Ty, Ax)]$$

for every $x, y \in X$, where $\alpha \geq 0$ and $\beta_1, \beta_2, \beta_3 \geq 0$ and $\beta_1 + 2\beta_2 + 2\beta_3 < 1$. If $S(X) \bigcap T(X)$ be a closed subspace of X, then (i) A, S, B and T have a coincidence point. If A, S and B, T are weakly compatible maps of type (A), then (ii) A, B, S and T have a unique common fixed point in X.

Proof. Define $\varphi : (\Re^+)^5 \to \Re^+$ by $\varphi(t_1, t_2, t_3, t_4, t_5) = \beta_1 t_1 + \beta_2(t_2 + t_3) + \beta_3(t_4 + t_5)$, then proof follows. ∎

Corollary 5. *Let A, B, S and T be mappings from a metric space (X, d) into itself satisfying:*

(I) $A(X) \bigcup B(X) \subset S(X) \bigcap T(X)$,

(II)
$$[1 + \alpha d(Sx, Ty)]d(Ax, By) \leq \alpha \max\{d(Sx, Ax)d(Ty, By), d(Sx, By)d(Ty, Ax)\}, + f(\max\{d(Sx, Ty), d(Sx, Ax), d(Ty, By), d(Sx, By), d(Ty, Ax)\})$$

for every $x, y \in X$, where $\alpha \geq 0$ and $f : \Re^+ \to \Re^+$ is a function satisfying (i) f is non-decreasing; (ii) f is upper semi-continuous, and (iii) $f(t) < t$ for each $t > 0$. If $S(X) \bigcap T(X)$ be a closed subspace of X, then (i) A, S, B and T have a coincidence point. If A, S and B, T are weakly compatible maps of type (A), then (ii) A, B, S and T have a unique common fixed point in X.

Proof. Define $\varphi : (\Re^+)^5 \to \Re^+$ by $\varphi(t_1, t_2, t_3, t_4, t_5) = f \max(t_1, t_2, t_3, \frac{1}{2}[t_4 + t_5])$, then proof follows. ∎

If we put $\alpha = 0$, then we have the following corollary:

Corollary 6. *Let A, S and T be mappings from a metric space (X, d) into itself satisfying:*

(*I*) $A(X) \subset S(X) \bigcap T(X)$,

(*II*) $d(Ax, Ay) \leq \alpha \max\{d(Sx, Ax)d(Ty, Ay), d(Sx, Ay)d(Ty, Ax)\},$
$$+ \varphi(d(Sx, Ty), d(Sx, Ax), d(Ty, Ay), d(Sx, Ay), d(Ty, Ax))$$

for every $x, y \in X$, *where* $\varphi \in \Psi$. *If* $S(X) \bigcap T(X)$ *be a closed subspace of* X, *then (i)* A, S *and* A, T *have a coincidence point. Indeed, if* A *is one-to-one, then* (*ii*) A, S *and* T *have a coincidence point If* A, S *and* A, T *are weakly compatible maps of type* (A), *then* (*iii*) A, S *and* T *have a unique common fixed point in* X.

Corollary 7. *Let* A, B, S *and* T *be mappings from a metric space* (X, d) *into itself satisfying:*

(*I*) $A(X) \bigcup B(X) \subset S(X) \bigcap T(X)$,

(*II*) $d(Ax, By) \leq \alpha \max\{d(Sx, Ax)d(Ty, By), d(Sx, By)d(Ty, Ax)\},$
$$+ \beta(\max\{d(Sx, Ty), d(Sx, Ax), d(Ty, By), d(Sx, By), d(Ty, Ax)\})$$

for every $x, y \in X$, *where* $\beta \in (0, 1)$. *If* $S(X) \bigcap T(X)$ *be a closed subspace of* X, *then* (*i*) A, S, B *and* T *have a coincidence point. If* A, S *and* B, T *are weakly compatible maps of type* (A), *then* (*ii*) A, B, S *and* T *have a unique common fixed point in* X.

Corollary 8. *Let* A, B, S *and* T *be mappings from a metric space* (X, d) *into itself satisfying:*

(*I*) $A(X) \bigcup B(X) \subset S(X) \bigcap T(X)$,

(*II*) $d(Ax, By) \leq \alpha \max\{d(Sx, Ax)d(Ty, By), d(Sx, By)d(Ty, Ax)\},$
$$+ \beta(\max\{d(Sx, Ty), d(Sx, Ax), d(Ty, By), \tfrac{1}{2}[d(Sx, By) + d(Ty, Ax)]\})$$

for every $x, y \in X$, *where* $\beta \in (0, 1)$. *If* $S(X) \bigcap T(X)$ *be a closed subspace of* X, *then* (*i*) A, S, B *and* T *have a coincidence point. If* A, S *and* B, T *are weakly compatible maps of type* (A), *then* (*ii*) A, B, S *and* T *have a unique common fixed point in* X.

Corollary 9. *Let* A, B, S *and* T *be mappings from a metric space* (X, d) *into itself satisfying:*

(*I*) $A(X) \bigcup B(X) \subset S(X) \bigcap T(X)$,

(*II*) $d(Ax, By) \leq \alpha \max\{d(Sx, Ax)d(Ty, By), d(Sx, By)d(Ty, Ax)\},$
$$+ \beta_1[d(Sx, Ty)] + \beta_2[d(Sx, Ax) + d(Ty, By)] + \beta_3[d(Sx, By) + d(Ty, Ax)]$$

for every $x, y \in X$, *where* $\alpha \geq 0$ *and* $\beta_1, \beta_2, \beta_3 \geq 0$ *and* $\beta_1 + 2\beta_2 + 2\beta_3 < 1$. *If* $S(X) \bigcap T(X)$ *be a closed subspace of* X, *then* (*i*) A, S, B *and* T *have* *a coincidence point. If A, S and B, T are weakly compatible maps of type (A), then (ii) A, B, S and T have a unique common fixed point in X.*

Corollary 10. *Let A, B, S and T be mappings from a metric space (X, d) into itself satisfying:*

(I) $A(X) \bigcup B(X) \subset S(X) \bigcap T(X)$,

$$(II)\ d(Ax, By) \leq \alpha \max\{(Sx, Ax)d(Ty, By), d(Sx, By)d(Ty, Ax)\}, \\ + f(\max\{d(Sx, Ty), d(Sx, Ax), d(Ty, By), \\ d(Sx, By), d(Ty, Ax)\})$$

for every $x, y \in X$, where $\alpha \geq 0$ and $f : \Re^+ \to \Re^+$ is a function satisfying (i) f is non-decreasing; (ii) f is upper semi-continuous, and (iii) $f(t) < t$ for each $t > 0$. If $S(X) \bigcap T(X)$ be a closed subspace of X, then (i) A, S, B and T have a coincidence point. If A, S and B, T are weakly compatible maps of type (A), then (ii) A, B, S and T have a unique common fixed point in X.

Example 1. Let $X = [0, 1)$ and d be the Euclidean metric on X. Define A, B, S and T as follows:

$$A(x) = \begin{cases} 0 & \text{if} \quad x \in [0, \frac{1}{2}), \\ x & \text{if} \quad x \in [\frac{1}{2}, 1), \end{cases}$$

and

$$B(x) = \begin{cases} 0 & \text{if} \quad x \in [0, \frac{1}{2}), \\ x & \text{if} \quad x \in [\frac{1}{2}, 1), \end{cases}$$

$$S(x) = T(x) = x$$

Hence $A(X) \bigcup B(X) \subset S(X) \bigcap T(X)$. Also, (A, S) and (B, T) are the weakly compatible of type (A). It is easy to see that all conditions of Theorem 2 hold and there exists a $x = 0$ such that $Ax = Bx = Sx = Tx = x$.

Example 2. Let $X = \Re$ and d be the Euclidean metric on X. Define A, B, S and T as follows:

$$A(x) = \begin{cases} 0 & \text{if} \quad x \leq 0, \\ x & \text{if} \quad x > 0, \end{cases}$$

and

$$B(x) = \begin{cases} 0 & \text{if} \quad x \leq 0, \\ x & \text{if} \quad x > 0, \end{cases}$$

$$S(x) = T(x) = x$$

for all x in X and let $\gamma : \mathfrak{R}^+ \to \mathfrak{R}^+$ be given by

$$\gamma(t) < t$$

and let $\varphi : \mathfrak{R}^+ \to \mathfrak{R}^+$ be given by $\varphi(t_1, t_2, t_3, t_4, t_5) = \beta \max\{t_i\}$ for some $0 < \beta < 1$, $i = 1, 2, 3, 4, 5$. Hence $A(X) \bigcup B(X) \subset S(X) \bigcap T(X)$. Also, (A, S) and (B, T) are the weakly compatible of type (A). It is easy to see that all conditions of Corollary 3 hold and for $x = 0$ we have $Ax = Bx = Sx = Tx = x$.

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