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THE RADIAL SOLUTION OF THE HEAT EQUATION IN THE CYLINDRICAL RING

Abstract. The subject of the paper is the construction of a solution to the parabolic problem for the cylindrical ring with initial condition of Cauchy type and boundary conditions of Dirichlet type. To construct the radial solution we do not use the Bessel functions but we apply the convenient Green function, heat potentials and Banach fixed point theorem.

Key words: parabolic equation, radial solution, initial-boundary value problems, Green function, heat potentials, nonlinear integral equation, Banach fixed point theorem.

AMS Mathematics Subject Classification: 35K05, 35K15.

1. Introduction

In the present paper we shall construct the radial solution of the heat equation

$$(1)\quad Pu(x,t) = F(x,t), \quad x = (x_1, x_2), \quad P = D^2_{x_1} + D^2_{x_2} - D_t - C(|x|, t),$$

$$|x| = (x_1^2 + x_2^2)^{\frac{1}{2}}$$

in the domain

$$D = \{(x,t) : 0 < a_1 < |x| < a_2, \;\; t \in (0,T), \;\; a_i > 0, \;\; i = 1,2\},$$

satisfying the initial condition

$$(2)\qquad u(x,0) = f(x),$$
$$x \in D_1 = \{(x,0) : 0 < a_1 < |x| < a_2, \;\; a_i > 0, \;\; i = 1,2\}$$

and the boundary conditions

$$(3)\qquad u(x,t) = h_i(x,t),$$
$$(x,t) \in S_i = \{(x,t) : x_1^2 + x_2^2 = a_i^2, \;\; i = 1,2, \;\; t \in (0,T)\}.$$

2. Motivation of the considered problem

In [6] the radial solution for the spherical shell is considered. In [1], [2], [3] and [5] are considered the radial solutions of polyparabolic equations in spherical shell, i.e., $x \in R^3$.

In four dimensional time-spatial space the equation (1) in the radial coordinates has following form

$$(D_r^2 - D_t)w(r,t) = rf(r,t), \quad r \in R, \ \ t \in (0,T),$$

where $w = ru$.

For $x \notin R^3$ the equation (1) in radial coordinates is more complicated and the equation (1) has the form

$$(D_r^2 + \frac{n-1}{r} D_r - D_t)v(r,t) = f(r,t), \quad r \in R, \ \ t \in (0,T).$$

In this paper $x \in R^2$ and the equation (1) has following form

$$(D_r^2 + \frac{1}{r} D_r - D_t)v(r,t) = f(r,t), \quad r \in R, \ \ t \in (0,T).$$

We construct the radial solution of (1) - (3) problem not using of Bessel functions.

3. Radial transformation of the (1) - (3) problem

Let $r =\mid x \mid= (x_1^2 + x_2^2)^{\frac{1}{2}}$ and let $u(r,t)$ will be the radial solution of the homogeneous equation

$$Pu(x,t) = 0, \quad (x,t) \in D. \tag{1a}$$

Let

$$F(x,t) = F_1(\mid x \mid, t) = F_1(r,t), \quad r = |x|.$$

By [6], p.226, we have that the radial solution $U(r,t)$ of the equation (1a) satisfies the equation

$$D_r^2 U(r,t) + r^{-1} D_r U(r,t) - D_t U(r,t) = F_1(r,t), \tag{1b}$$

in the domain

$$D_2 = \{(r,t) : r \in (a_1, a_2), \ \ t \in (0,T), \ \ 0 < a_1 < a_2\}.$$

In the sequel we will give the construction the solution

$$D_r^2 U(r,t) - D_t U(r,t) = F_1(r,t) - r^{-1} D_r U(r,t) \quad \text{in } D_2 \tag{1c}$$

satisfying the initial condition

(2c) $$U(r,0) = f_1(r), \quad f_1(r) = f(|x|), \;\; r \in (a_1, a_2)$$

and the boundary conditions

(3c) $$U(a_i,t) = H_i(t), \quad H_i(t) = H_i(|x|,t)|_{|x|=a_i}, \;\; i = 1,2, \;\; t \in (0,T).$$

4. Green functions

Let

$$r \in (a_1, a_2).$$

Let us consider the points sequence

$$\begin{aligned} r^1_{2n} &= r + 2n(a_2 - a_1), \;\; n \in N, \\ r^2_{2n} &= r - 2n(a_2 - a_1), \;\; n \in N, \\ r^1_{2n+1} &= -r + 2a_1 - 2n(a_2 - a_1), \;\; n \in N, \\ r^2_{2n+1} &= -r + 2a_2 + 2n(a_2 - a_1), \;\; n \in N, \\ r &= r^1_0 = r^2_0. \end{aligned}$$

Let

$$p \in [a_1, a_2]$$

and let

$$\begin{aligned} U_0(r,t;p,s) &= (t-s)^{\frac{-1}{2}} \exp(B(t,s)(r-p)^2), \\ U_{n,i}(r,t;p,s) &= (t-s)^{\frac{-1}{2}} \exp(B(t,s)(r^i_n - p)^2), \;\; i = 1,2, \\ B(t,s) &= (-4(t-s))^{-1}, \;\; t > s \geq 0, \\ H_1(r,t;p,s) &= \sum_{n=1}^{\infty} (-1)^n [U_{n,1}(r,t;p,s) + U_{n,2}(r,t;p,s)]. \end{aligned}$$

Let us consider the function G of the form

$$G(r,t;p,s) = U_0(r,t;p,s) + H_1(r,t;p,s)$$

or

$$G(r,t;p,s) = U_0(r,t;p,s) - U_{1,1}(r,t;p,s) - U_{2,1}(r,t;p,s) + H_2(r,t;p,s),$$

where

$$\begin{aligned} U_{1,1}(r,t;p,s) &= (t-s)^{\frac{-1}{2}} \exp(B(t,s)(-r + 2a_1 - p)^2), \\ U_{2,1}(r,t;p,s) &= (t-s)^{\frac{-1}{2}} \exp(B(t,s)(-r + 2a_2 - p)^2), \\ H_2(r,t;p,s) &= (t-s)^{\frac{-1}{2}} \sum_{n=2}^{\infty} (-1)^n [\exp(B(t,s)(r^1_n - p)^2) \\ &\qquad + \exp(B(t,s)(r^2_n - p)^2)]. \end{aligned}$$

It is known [4], p.475 that function $G(r,t;p,s)$ is the Green function for the equation

$$D_r^2G(r,t;p,s) - D_tG(r,t;p,s) = 0, \quad (r,t;p,s) \in \overline{D_2} \times \overline{D_2} \tag{1a}$$

satisfying the boundary conditions

$$G(a_i,t;p,s) = 0, \quad i = 1,2. \tag{1a, a}$$

5. Some definitions

Definition 1. *Denote by (K_1) the class of all functions f belonging to the class $C^1([a_1,a_2])$ and $D_p^if(a_i) = 0$, $i = 1,2$.*

Definition 2. *Denote by (K_2) the class of all functions $H : [0,T] \to R$, such that $H \in C^1([0,T])$ and $H(0) = 0$.*

Definition 3. *Denote by (K_3) the class of all functions $F : D_2 \to R$, such that $F \in C^{1,0}(\overline{D_2})$ and $D_p^iF(a_j,0) = 0$, $i = 0,1$, $j = 1,2$ and $D_s^iF(p,0) = 0$, $i = 0,1$.*

6. Green potentials

Let us consider the Green potentials of the form

$$\begin{aligned} u_1(r,t) &= A\int_{a_1}^{a_2} f_1(p)G(r,t;p,0)dp, \quad (r,t) \in D_2, \\ u_{i+2}(r,t) &= (-1)^{i+1}2A\int_0^t H_{i+1}(s)D_pG(r,t;a_{i+1},s)ds, \quad i = 0,1, \quad (r,t) \in D_2, \\ u_4(r,t) &= A\int_0^t\int_{a_1}^{a_2} F_1(p,s)G(r,t;p,s)dpds, \\ A &= (2\sqrt{\pi})^{-1}, \end{aligned}$$

7. Properties of the potential u_1

Let $\overline{f} = f$ for $p \in [a_1,a_2]$, $\overline{f} = 0$ for $p \in R\backslash[a_1,a_2]$. Let us consider the function u_1 in the form

$$u_1(r,t) = \sum_{i=0}^{3} u_1^i(r,t),$$

where

$$
\begin{aligned}
u_1^0(r,t) &= A\int_{-\infty}^{+\infty}\overline{f}_1(p)U_0(r,t;p,0)dp,\\
u_1^1(r,t) &= -A\int_{-\infty}^{+\infty}\overline{f}_1(p)U_{1,1}(r,t;p,0)dp,\\
u_1^2(r,t) &= -A\int_{-\infty}^{+\infty}\overline{f}_1(p)U_{2,1}(r,t;p,0)dp,\\
u_1^3(r,t) &= A\int_{a_1}^{a_2}\overline{f}_1(p)H_2(r,t;p,0)dp.
\end{aligned}
$$

Lemma 1. *If* $f_1 \in (K_1)$*, then*

(1^0) $P_{r,t}u_1(r,t) = 0, \ \ (r,t) \in D_2, \quad P_{r,t} = D_r^2 - D_t,$

(2^0) $u_1(r,0) = f_1(r), \ \ r \in [a_1, a_2],$

(3^0) $u_1(a_i,t) = 0, \ \ t \in (0,T), \ \ i = 1,2,$

(4^0) $D_r u_1(r,t) \in C(\overline{D_2}).$

Proof. Ad 1^0. By [4], p. 448, we obtain that

$$P_{r,t}u_1^i(r,t) = 0, \quad i = 1,2,3, \ \ (r,t) \in D_2.$$

The integrals

$$A\int_{a_1}^{a_2}\overline{f_1}(p)D_r^iH_2(r,t;p,0)dp, \ \ i = 0,1,2$$

and

$$A\int_{a_1}^{a_2}\overline{f_1}(p)D_tH_2(r,t;p,0)dp$$

have the common majorant

$$C\int_{a_1}^{a_2}\sum_{n=2}^{\infty}(n-1)^{-2}dp = C_1\sum_{n=2}^{\infty}(n-1)^{-2} < \infty.$$

Consequently we obtain

$$
\begin{aligned}
P_{r,t}u_1(r,t) &= P_{r,t}\int_{a_1}^{a_2} f_1(p)\sum_{n=2}^{\infty}(-1)^n[U_{n,1}(r,t;p,0) + U_{n,2}(r,t;p,0)]dp\\
&= \int_{a_1}^{a_2} f_1(p)\sum_{n=2}^{\infty}(-1)^nP_{r,t}[U_{n,1}(r,t;p,0) + U_{n,2}(r,t;p,0)]dp\\
&= 0,
\end{aligned}
$$

because by [4], p. 448, we have

$$P_{r,t}U_{n,i}(r,t;p,0) = 0, \quad i = 1,2 \ \text{for} \ (r,t) \in D_2.$$

Ad 2^0. By [4], p. 448, we have

$$u_1^0(r,t) = A\int_{-\infty}^{\infty} \overline{f_1}(p)U_0(r,t;p,0)dp \to f_1(r_0) \tag{4}$$

as

$$(r,t) \to (r_0,0), \quad r_0 \in [a_1,a_2]$$

and

$$\begin{aligned} (5)\ \ u_1^1(r,t) &= A\int_{-\infty}^{+\infty} \overline{f_1}(p)U_{1,1}(r,t;p,0)\,dp \\ &= A\int_{-\infty}^{\infty} \overline{f_1}(p)t^{\frac{-1}{2}}\exp(B(t,0)((-r+2a_1)-p)^2)\,dp \\ &\to \overline{f_1}(p)\,|_{p=-r_0+2a_1} = \overline{f_1}(-r_0+2a_1) = 0 \ \text{ as } \ (r,t) \to (r_0,0), \end{aligned}$$

and

$$\begin{aligned} (6)\qquad u_1^2(r,t) &= A\int_{-\infty}^{\infty} \overline{f_1}(p)U_{2,1}(r,t;p,0)\,dp \\ &= A\int_{-\infty}^{\infty} \overline{f_1}(p)t^{\frac{-1}{2}}\exp(B(t,0)((-r_0+2a_2)-p)^2)\,dp \\ &\to \overline{f_1}(-r_0+2a_2) = 0 \quad \text{as} \quad (r,t) \to (r_0,0), \end{aligned}$$

and

$$\begin{aligned} (7)\quad u_1^3(r,t) &= A\sum_{n=2}^{\infty}(-1)^n\int_{-\infty}^{\infty} \overline{f_1}(p)t^{\frac{-1}{2}}\exp(B(t,0)(r_n^1-p)^2)\,dp \\ &\quad + A\sum_{n=2}^{\infty}(-1)^n\int_{-\infty}^{\infty} \overline{f_1}(p)t^{\frac{-1}{2}}\exp(B(t,0)(r_n^2-p)^2)\,dp \\ &\to A\sum_{n=2}^{\infty}(-1)^n\overline{f_1}((r_0)_n^1) + A\sum_{n=2}^{\infty}(-1)^n\overline{f_1}((r_0)_n^2) = 0. \end{aligned}$$

Consequently by (4) - (8) we obtain the assertion 2^0.

Ad 3^0. Since the integral

$$\int_{-\infty}^{\infty} \overline{f_1}(p)G(r,t;p,0)\,dp$$

is locally uniformly convergent at the point (a_i, t), $i = 1, 2$, thus by continuity of the last integral and by (1a,a) we obtain

$$\begin{aligned}\lim_{r\to a_i}\int_{-\infty}^{\infty}\overline{f_1}(p)G(r,t;p,0)dp &= \int_{-\infty}^{\infty}\overline{f_1}\lim_{r\to a_i}G(r,t;p,0)dp\\ &= \int_{-\infty}^{\infty}\overline{f_1}(p)G(a_i,t;p,0)dp = 0, \quad i=1,2.\end{aligned}$$

Ad 4^0. Let us consider the integral

$$I(r,t) = \int_{-\infty}^{\infty}\overline{f_1}(p)D_rG(r,t;p,0)dp.$$

It is easy to verify that

$$D_rG(r,t;p,0) = -D_pG(r,t;p,0)$$

and

$$I(r,t) = \int_{-\infty}^{\infty}\overline{f_1}(p)D_pG(r,t;p,0)\,dp.$$

Integrating by parts we have

$$\begin{aligned}I(r,t) &= -\overline{f_1}(p)G(r,t;p,0)\,|_{-\infty}^{\infty} + \int_{-\infty}^{\infty}D_p(\overline{f_1}(p))G(r,t;p,0)dp\\ &= \int_{-\infty}^{\infty}D_p(\overline{f_1}(p))G(r,t;p,0)\,dp \;\in\; C(\overline{D_2}).\end{aligned}$$

■

8. Properties of the potentials u_i, $i = 2, 3$

By [5], p. 121, we obtain

Lemma 2. *If* $(r,t) \in \overline{D_2}$, $0 \le s < t \le T$, *then*

$$\begin{aligned}(1^0)\qquad u_2(r,t) &= -2A\int_0^t H_1(s)D_pG(r,t;p,s)|_{p=a_1}\,ds\\ &= -2A\int_0^t H_1(s)(a_1-r)(t-s)^{\frac{-3}{2}}\exp(B(t,s)(r-a_1)^2)\\ &\quad + 2A\int_0^t H_1(s)(t-s)^{\frac{-3}{2}}\sum_{n=1}^{\infty}A_n\exp(B(t,s)(A_n)^2)ds\\ &\quad - 2A\int_0^t H_1(s)(t-s)^{\frac{-3}{2}}\sum_{n=1}^{\infty}B_n\exp(B(t,s)(B_n)^2)ds,\end{aligned}$$

where

$$\begin{aligned} A_n &= -r - 2(n+1)(a_2 - a_1) + a_1, \\ B_n &= r - 2(n+1)(a_2 - a_1) - a_1. \end{aligned}$$

$$(2^0) \quad \begin{aligned} u_3(r,t) &= -2A \int_0^t H_1(s) D_p G(r,t;p,s)|_{p=a_2} ds \\ &= -2A \int_0^t H_2(s)(a_2 - r)(t-s)^{\frac{-3}{2}} \exp(B(t,s)(a_2 - r)^2) ds \\ &+ 2A \int_0^t H_2(s)(t-s)^{\frac{-3}{2}} \sum_{n=1}^{\infty} D_n \exp(B(t,s)(D_n)^2) ds \\ &- 2A \int_0^t H_2(s)(t-s)^{\frac{-3}{2}} \sum_{n=1}^{\infty} C_n \exp(B(t,s)(C_n)^2) ds, \end{aligned}$$

where

$$\begin{aligned} C_n &= r + 2(n+1)(a_2 - a_1) - 2a_1 + a_2, \\ D_n &= r - 2(n+1)(a_2 - a_1) - a_2. \end{aligned}$$

Lemma 3. *If* $H_i \in (K_2)$, $i = 1, 2$, *then*

$$(1^0) \quad P_{r,t} u_i(r,t) = 0, \quad (r,t) \in D_2, \quad i = 2, 3,$$

$$(2^0) \quad u_i(r,0) = 0, \quad r \in [a_1, a_2], \quad i = 2, 3,$$

$$(3^0) \quad u_2(a_1,t) = H_1(t), \quad t \in [0,T],$$

$$(4^0) \quad u_2(a_2,t) = 0, \quad t \in [0,T],$$

$$(5^0) \quad u_3(a_1,t) = 0, \quad t \in [0,T],$$

$$(6^0) \quad u_3(a_2,t) = H_2(t), \quad t \in [0,T].$$

Proof. Ad 1^0. We shall give the proof only for u_2 because the proof for u_3 is similar.

Let

$$u_2(r,t) = u_2^1(r,t) + u_2^2(r,t),$$

where

$$\begin{aligned} u_2^1(r,t) &= -2A \int_0^t H_1(s)(t-s)^{\frac{-3}{2}} (a_1 - r) \exp(B(t,s)(a_1 - r)^2) ds, \\ u_2^2(r,t) &= -2A \int_0^t H_1(s)(t-s)^{\frac{-3}{2}} \sum_{n=1}^{\infty} A_n \exp(B(t,s)(A_n)^2) ds \\ & \quad -2A \int_0^t H_1(s)(t-s)^{\frac{-3}{2}} \sum_{n=1}^{\infty} B_n \exp(B(t,s)(B_n)^2) ds. \end{aligned}$$

By [4], p. 483, we have

$$P_{r,t}u_2^1(r,t) = 0, \ \ (r,t) \in D_2$$

and

$$\begin{aligned} P_{r,t}u_2^2(r,t) &= -2A\int_0^t H_1(s)\sum_{n=1}^{\infty} P_{r,t}[(t-s)^{\frac{-3}{2}} A_n \exp(B(t,s)(A_n)^2)]ds \\ &= -2A\int_0^t H_1(s)\sum_{n=1}^{\infty} P_{r,t}[(t-s)^{\frac{-3}{2}} B_n \exp(B(t,s)(B_n)^2)]ds = 0. \end{aligned}$$

Ad 2^0. We have

$$\begin{aligned} |u_2^1| &\le 2A\sup_{[0,t]}|H_1(s)|\int_{-\infty}^t (r-a_1)(t-s)^{\frac{-3}{2}}\exp(B(t,s)(r-a_1)^2)ds \\ &\le \sup_{[0,t]}|H_1(s)| \to H_1(0) \quad \text{as} \quad s \to 0 \end{aligned}$$

and

$$|u_2^2| \le C\sup_{[0,t]}|H_1(s)| \to 0 \quad \text{as} \quad t \to 0.$$

Ad 3^0. By [4], p.483, we have

$$u_2^1(r,t) \to H_1(t_0) \quad \text{as} \quad (r,t) \to (a_1,t_0), \ \ t_0 \in [0,T]$$

and

$$u_2^2(a_1,t_0) = 0 \ \text{ for } \ t_0 \in [0,T].$$

Ad 4^0. By the assertion 1^0 of Lemma 2, we obtain 4^0.

Ad 5^0 and 6^0. By the assertion 2^0 of Lemma 2 we obtain 5^0 and 6^0. ■

9. Properties of the potential u_4

Lemma 4. *If $F_1 \in (K_3)$, then*

(1^0) $$P_{r,t}u_4(r,t) = F_1(r,t), \ \ (r,t) \in D_2,$$

(2^0) $$u_4(r,0) = 0, \ \ r \in [a_1,a_2],$$

(3^0) $$u_4(a_i,t) = 0, \ \ t \in (0,T), \ \ i = 1,2.$$

Proof. Ad 1^0. By Poisson's theorem [4], p. 522, we have

$$\begin{aligned} P_{r,t}u_4(r,t) &= P_{r,t}A\int_0^t\int_{a_1}^{a_2}F_1(p,s)G(r,t;p,s)\,dpds \\ &= F_1(p,s)|_{p=r_{s=t}} = F_1(r,t), \quad (r,t)\in D_2. \end{aligned}$$

Ad 2^0. We have

$$\begin{aligned} |u_4(r,t)| &\le C\sup_{D_2}|F_1|\int_0^t (t-s)^{\frac{-1}{2}}\,ds = C\sup_{D_2}|F_1|2t^{\frac{1}{2}} \\ &= C\sup_{D_2}|F_1|2t^{\frac{1}{2}} \to 0 \quad \text{as} \quad t\to 0. \end{aligned}$$

Ad 3^0. By the assertion 2^0 we have that the integral $u_4(r,t)$ is locally uniformly convergent at every point (a_i,t), $i=1,2$, and by properties of the Green function we obtain

$$u_4(a_i,t) = A\int_0^t\int_{a_1}^{a_2}F_1(y,s)G(a_i,t;y,s)\,dyds = 0, \quad i=1,2, \quad t\in(0,T).$$

■

10. Integro - differential equations

Let us consider the following integro - differential equation of the form

$$v(r,t) = \sum_{i=1}^4 u_i(r,t) + \int_0^t\int_{a_1}^{a_2} G(r,t;p,s)\cdot\left(F_1(p,s)+\frac{1}{p}D_p v(p,s)\right)dpds. \tag{8}$$

Let

$$v(p,s) = w_1(p,s), \qquad D_p v(p,s) = w_2(p,s) \tag{9}$$

By (9) and (10) we have

$$\begin{aligned} D_r w_1(r,t) = \sum_{i=1}^4 D_r u_i(r,t) &+ \int_0^t\int_{a_1}^{a_2} D_r G(r,t;p,s)F_1(p,s)dyds \\ &+ \int_0^t\int_{a_1}^{a_2}\frac{1}{p}(D_r G(r,t;p,s))w_2(p,s)dpds. \end{aligned}$$

By (9), (10) and (11) we obtain the following system integro - differential equations

$$\begin{aligned} w_1(r,t) = \sum_{i=1}^4 u_i(r,t) &+ \int_0^t\int_{a_1}^{a_2} G(r,t;p,s)F_1(p,s)\,dyds \\ &+ \int_0^t\int_{a_1}^{a_2}\frac{1}{p}G(r,t;p,s)w_2(p,s)\,dpds, \end{aligned} \tag{10}$$

$$(11)\quad w_2(r,t) = \sum_{i=1}^{4} D_r u_i(r,t) \; + \int_0^t \int_{a_1}^{a_2} D_r G(r,t;p,s) F_1(p,s) dyds$$
$$+ \int_0^t \int_{a_1}^{a_2} \frac{1}{p}(D_r G(r,t;p,s) w_2(p,s) dpds.$$

11. The solution of the (12) - (13) system

Let

$$\begin{aligned}
\overline{W} &= (w_1, w_2),\\
\overline{Z} &= (z_1, z_2),\\
z_1(r,t) &= \sum_{i=1}^{4} u_i(r,t) + \int_0^t \int_{a_1}^{a_2} G(r,t;p,s) F_1(p,s) dyds,\\
z_2(r,t) &= \sum_{i=1}^{4} D_r u_i(r,t) + \int_0^t \int_{a_1}^{a_2} D_r G(r,t;p,s) F_1(p,s) dyds.
\end{aligned}$$

The system (12) - (13) we write in the form

$$(12)\qquad \overline{W}(r,t) = \overline{Z}(r,t) + \overline{N}(\overline{W}),$$

where

$$\begin{aligned}
\overline{N}(\overline{W}) &= (N_1(\overline{W}), N_2(\overline{W})),\\
N_1(\overline{W}) &= \int_0^t \int_{a_1}^{a_2} G(r,t;p,s) w_2(p,s) dpds,\\
N_2(\overline{W}) &= \int_0^t \int_{a_1}^{a_2} (D_r G(r,t;p,s)) w_2(p,s) dpds.
\end{aligned}$$

We solve the system (14) applying the method of the fixed point.

Let us consider the transformation

$$(13)\qquad \overline{S}(\overline{W}) = \overline{Z} + \overline{N}(\overline{W}).$$

Definition 4. *Denote by (B) the Banach space of all functions $\overline{W}$, such that*

$$(1^0)\qquad w_i \in C(\overline{D_2}),\quad i = 1,2,$$
$$(2^0)\qquad \|\overline{W}\| = \sup_{\overline{D_2}} |w_1(r,t)| + \sup_{\overline{D_2}} |w_2(r,t)| \le L,$$

where L is a positive constant.

Definition 5. *Denote by* (K_4) *the class of all functions* $\overline{Z}$*, such that* $\|Z\| \leq (1-Q)L$*,* $Q \in (0,1)$*.*

Let

$$\frac{1}{2}Q = \max_i \left\{ \sup_{\overline{D_2}} \int_0^t \int_{a_1}^{a_2} |D_r^i G(r,t;p,s)|\, dpds, \quad i = 0,1 \right\}.$$

Lemma 5. *If* $\overline{Z} \in (K_4)$*,* $\overline{W} \in (B)$*, then*
1^0 *the transformation* $\overline{S}$ *is contracting,*
2^0 *the transformation* $\overline{S}$ *is invariant in the set* (B)*.*

Proof.
Ad 1^0. Let the vectors

$$\overline{W} = (w_1^i, w_2^i), \quad i = 1,2$$

belong to (B).

By (15) we have

$$\begin{aligned}
(14) \qquad & \|\overline{S}(\overline{W^1}) - \overline{S}(\overline{W^2})\| = \|\overline{N}(\overline{W^1}) - \overline{N}(\overline{W^2})\| \\
&= \sup_{\overline{D_2}} |N_1(\overline{W^1}) - N_1(\overline{W^2})| \\
&\quad + \sup_{\overline{D_2}} |N_2(\overline{W^1}) - N_2(\overline{W^2})| \\
&= \sup_{\overline{D_2}} \Big| \int_0^t \int_{a_1}^{a_2} G(r,t;p,s) w_2^1(p,s) dpds \\
&\quad - \int_0^t \int_{a_1}^{a_2} G(r,t;p,s) w_2^2(p,s) dpds \Big| \\
&\quad + \sup_{\overline{D_2}} \Big| \int_0^t \int_{a_1}^{a_2} \frac{1}{p} D_r G(r,t;p,s) w_2^1(p,s) dpds \\
&\quad - \int_0^t \int_{a_1}^{a_2} \frac{1}{p} (D_r G(r,t;p,s) w_2^2(p,s) dpds \Big| \\
&= \sup_{\overline{D_2}} \Big| \int_0^t \int_{a_1}^{a_2} G(r,t;p,s)(w_2^1(p,s) - w_2^2(p,s)) dpds \Big| \\
&\quad + \sup_{\overline{D_2}} \Big| \int_0^t \int_{a_1}^{a_2} \frac{1}{p} D_r G(r,t;p,s)(w_2^1(p,s) - w_2^2(p,s)) dpds \Big| \\
&\leq \int_0^t \int_{a_1}^{a_2} | G(r,t;p,s) | \sup_{\overline{D_2}} | w_2^1(p,s) - w_2^2(p,s) | \, dpds \\
&\quad + \int_0^t \int_{a_1}^{a_2} \frac{1}{p} | D_r G(r,t;p,s) | \sup_{\overline{D_2}} | w_2^1(p,s) - w_2^2(p,s) | \, dpds
\end{aligned}$$

$$\le \sup_{\overline{D_2}} |w_2^1(p,s) - w_2^2(p,s)| \int_0^t \int_{a_1}^{a_2} |G(r,t;p,s)| dpds$$

$$+ \sup_{\overline{D_2}} \mid w_2^1(p,s) - w_2^2(p,s) \mid \int_0^t \int_{a_1}^{a_2} \frac{1}{p} \mid D_r G(r,t;p,s) \mid dpds$$

$$\le \Big[\sup_{\overline{D_2}} \mid w_2^1(p,s) - w_2^2(p,s)|$$

$$+ \sup_{\overline{D_2}} \mid w_1^1(p,s) - w_2^2(p,s) \mid \Big] \int_0^t \int_{a_1}^{a_2} \mid G(r,t;p,s) \mid dpds$$

$$+ \Big[\sup_{\overline{D_2}} \mid w_2^1(p,s) - w_2^2(p,s) \mid$$

$$+ \sup_{\overline{D_2}} \mid w_1^2(p,s) - w_2^2(p,s) \mid \Big] \int_0^t \int_{a_1}^{a_2} \frac{1}{p} \mid D_r G(r,t;p,s) \mid dpds$$

$$\le \|\overline{W^1} - \overline{W^2}\| \int_0^t \int_{a_1}^{a_2} \Big[\mid G(r,t;p,s) \mid + \frac{1}{p} \mid D_r G(r,t;p,s) \mid \Big] dpds$$

$$\le Q \|\overline{W^1} - \overline{W^2}\| \quad \text{for} \quad Q \in (0,1).$$

Consequently

$$\|\overline{S}(\overline{W^1}) - \overline{S}(\overline{W^2})\| \le Q \|\overline{W^1} - \overline{W^2}\|, \quad Q \in (0,1).$$

Ad 2^0. Let $\overline{W} \in (B)$. By (15) we have

$$\|\overline{S}(\overline{W})\| = \|\overline{Z} + \overline{N}(\overline{W})\| \le \|\overline{Z}\| + \|\overline{N}(\overline{W})\|.$$

By (16) we obtain

$$\|\overline{S}(\overline{W})\| \le (1-Q)L + QL$$

because

$$\|\overline{N}(\overline{W})\| = \|\overline{N}(\overline{W}) - \overline{N}(\overline{\Theta})\| = \|\overline{W} - \overline{\Theta}\| Q = \|\overline{W}\| Q \le QL,$$

where Θ is zero vector. ■

By Banach theorem on contracting mapping we obtain

Lemma 6. *If $Z \in (K_4)$, $Q \in (0,1)$, then there exists the unique solution w_1, w_2 of the system of the integral equations (12) - (13) and the function w_1 is the solution of the equation (12).*

12. Properties of the function v

Lemma 7. *If* $f_1 \in (K_1)$, $H_i \in (K_2)$, $F_1 \in (K_3)$, *then*

$$(1^0) \qquad P_{r,t}v(r,t) = F_1(r,t) - r^{-1}D_rU(r,t), \quad (r,t) \in D_2,$$

$$(2^0) \qquad v(r,0) = f_1(r,t), \quad r \in [a_1, a_2],$$

$$(3^0) \qquad v(a_i,t) = H_i(t), \quad t \in [0,T], \quad i = 1,2.$$

Proof. Ad 1^0. By Lemmas 1 - 4 and by Poisson's theorem [4], p.522, we have

$$\begin{aligned} P_{r,t}v(r,t) &= P_{r,t}\sum_{i=1}^{4} u_i(r,t) \\ &\quad + P_{r,t}\int_0^t \int_{a_1}^{a_2} G(r,t;p,s)\,\frac{1}{p}D_pv(p,s)dpds \\ &= F_1(r,t) + \frac{1}{r}D_rv(r,t), \quad (r,t) \in D_2. \end{aligned}$$

Ad 2^0. By Lemmas 1 - 4 and by properties of the function G we obtain the assertion 2^0.

Ad 3^0. By Lemmas 1 - 4 and by properties of the function G we obtain the assertion 3^0. ■

13. Existence theorem on the problem (1c) - (3c)

By the foregoing reasoning we obtain the following fundamental theorem

Theorem. *If the assumptions of Lemmas 1-7 are satisfied, then the function*

$$v(r,t) = \sum_{i=1}^{4} u_i(r,t) + \int_0^t \int_{a_1}^{a_2} G(r,t;p,s)D_pv(p,s)dpds$$

is the solution of the radial problem (1c), (2c), (3c) and the function

$$u(x,t) = v(r,t)|_{r=|x|}$$

is the radial solution of the problem (1) - (3).

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