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ON I-LIMIT SUPERIOR AND I-LIMIT INFERIOR OF SEQUENCES OF FUZZY NUMBERS

Abstract. In this article we introduce the notions of I-limit superior and I-limit inferior for sequences of fuzzy real numbers . We prove fuzzy analogue of some results on I-limit superior and I-limit inferior for real sequences.

Key words: fuzzy real numbers, I-convergence, I-limit superior, I-limit inferior.

AMS Mathematics Subject Classification: 40A05, 40D25.

1. Introduction

The notion of I-convergence of real valued sequence was studied at the initial stage by Kostyrko, Šalát and Wilczynski [3]. It generalizes and unifies different notions of convergence of sequences. Generalizing the concepts of limit superior and limit inferior for real sequences, Demirci [1] introduced the concepts of I-limit superior and I-limit inferior for sequences of real numbers.

The concepts of fuzzy sets was first introduced by Zadeh [9]. Bounded and convergent sequences of fuzzy numbers are studied by Matloka [4]. Later on sequences of fuzzy numbers have been discussed by Nanda [5], Nuray and Savas [7], Nuray [6], Fang and Huang [2], Tripathy and Nanda [8] and many others.

2. Definitions and background

Throughout N and R denote the sets of natural and real numbers respectively.

If X is a non empty set, then a non-void class $I \subseteq 2^X$ is called an ideal if I is additive (i.e. $A, B \in I \Rightarrow A \cup B \in I$) and hereditary (i.e. $A \in I and B \subseteq A \Rightarrow B \in I$). An ideal $I \subseteq 2^X$ is said to be non-trivial if $I \neq 2^X$. A non-trivial ideal I is said to be admissible if I contains every finite subset of X. A non-trivial ideal I is said to be maximal if there does not exist any non-trivial ideal $J \neq I$ containing I as a subset.

Example. (a) Let $I = I_f$, class of all finite subsets of N. Then I_f is a non-trivial admissible ideal.

(b) Let $A \subset N$. Put $d(A) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_A(k)$ and $\delta(A) = \lim_{n\to\infty} \frac{1}{S_n} \sum_{k=1}^{n} \frac{\chi_A(k)}{k}$, where $S_n = \sum_{k=1}^{n} \frac{1}{k}$. The class $I_d(I_\delta)$ of all $A \subset N$ with $d(A) = 0(\delta(A) = 0)$ forms a non-trivial admissible ideal.

(c) The uniform density of a set $A \subset N$ is defined as follows. For integers $t \geq 0$ and $s \geq 1$, let $A(t+1, t+s) = \text{card}\ \{n \in A : t+1 \leq n \leq t+s\}$. Put $\beta_s = \liminf_{t\to\infty} A(t+1, t+s)$, $\beta^s = \limsup_{t\to\infty} A(t+1, t+s)$. If $\lim_{s\to\infty} \frac{\beta_s}{s} = \lim_{s\to\infty} \frac{\beta^s}{s}$ (= u(A), say), then $u(A)$ is called the uniform density of A. The class I_u of all $A \subset N$ with $u(A) = 0$ forms a non-trivial ideal.

For any ideal there is a filter $\Im(I)$ corresponding to I, given by

$$\Im(I) = \{K \subseteq N : N \setminus K \in I\}.$$

Let D denote the set of all closed bounded intervals $X = [a_1, a_2]$ on the real line R. For $X = [a_1, a_2] \in D$ and $Y = [b_1, b_2] \in D$, we define $X \leq Y$ if and only if $a_1 \leq b_1$ and $a_2 \leq b_2$.

$$d(X, Y) = \max(|a_1 - b_1|, |a_2 - b_2|)$$

It is known that (D, d) is a complete metric space and $\leq$ is a partial order on D.

A fuzzy real number X is a fuzzy set on R i.e. a mapping $X : R \to L$ $(= [0, 1])$ associating each real number t with its grade of membership $X(t)$. Every real number r can be expressed as a fuzzy real number $\overline{r}$ as follows: $\overline{r}(t) = \begin{cases} 1 & \text{if}\, t = r \\ 0 & otherwise \end{cases}$. A fuzzy real number X is called convex, if $X(t) \geq X(s) \wedge X(r) = \min(X(s), X(r))$, where $s < t < r$. If there exists $t_0 \in R$ such that $X(t_0) = 1$, then the fuzzy real number X is called normal. The α- level set of a fuzzy real number X, $0 < \alpha \leq 1$ denoted by X^α is defined as $X^\alpha = \{t \in R : X(t) \geq \alpha\}$. A fuzzy real number X is said to be upper semi-continuous if for each $\epsilon > 0$, $X^{-1}([0, a + \epsilon))$, for all $a \in L$ is open in the usual topology of R. We denote the set of all upper semi-continuous, normal, convex fuzzy numbers by $L(R)$. A fuzzy real number η is said to be non-negative if $\eta(t) = 0$ for all $t < 0$.

Arithmetic operations $\oplus$ and $\ominus$ on $L(R) \times L(R)$ can be defined as follows:

$$(\eta \oplus \delta)(t) = \sup_{s\in R} \{\eta(s) \wedge \delta(t - s)\}, \quad t \in R$$

$$(\eta \ominus \delta)(t) = \sup_{s\in R} \{\eta(s) \wedge \delta(s - t)\}, \quad t \in R.$$

Let $\bar{d} : L(R) \times L(R) \to R$ be defined by

$$\bar{d}(X,Y) = \sup_{0\leq\alpha\leq1} d(X^\alpha, Y^\alpha) \quad \text{for } X, Y \in L(R).$$

Then $\bar{d}$ defines a metric on $L(R)$.

We define $X \leq Y$ if and only if $X^\alpha \leq Y^\alpha$ for any $\alpha \in L$. The additive identity and multiplicative identity in $L(R)$ are denoted by $\bar{0}$ and $\bar{1}$ respectively.

Definition 1. *A sequence $X = (X_n)$ of fuzzy numbers is said to be I-convergent if there exists a fuzzy numbers X_0 such that for all $\epsilon > 0$, the set $\{n \in N : \bar{d}(X_n, X_0) \geq \epsilon\} \in I$. We write $I - limit X_n = X_0$. Throughout the paper I will be an admissible ideal.*

3. I-limit superior and I-limit inferior of sequences of fuzzy numbers

In this section we introduce the notions of I-limit superior and I-limit inferior for sequences of fuzzy real numbers.

A subset E of $L(R)$ is said to be bounded above if there exists a fuzzy number μ, called an upper bound of E, such that $X \leq \mu$, for all $X \in E$. μ is called the least upper bound (lub or sup) of E if μ is an upper bound and is the smallest of all upper bounds. A lower bound and the greatest lower bound (glb or inf) are defined similarly. E is said to be bounded if it is bounded above and bounded below.

For a fuzzy real valued sequence $X = (X_n)$ let B_X denotes the set:

$$B_X = \{\mu \in L(R) : \{n \in N : X_n > \mu\} \notin I\}.$$

Similarly,

$$A_X = \{\lambda \in L(R) : \{n \in N : X_n < \lambda\} \notin I\}.$$

Definition 2. *For $X = (X_n)$ a fuzzy real valued sequence and I an admissible ideal, the I-limit superior of (X_n) is given by*

$$I - \text{limit}\sup X = \begin{cases} \sup B_X, & if \quad B_X \neq \emptyset \\ -\bar{\infty}, & if \quad B_X = \emptyset \end{cases}$$

Also , the I-limit inferior of X is given by

$$I - \text{limit}\inf X = \begin{cases} \inf A_X, & if \quad A_X \neq \emptyset \\ +\bar{\infty}, & if \quad A_X = \emptyset \end{cases}$$

Remark 1. For $I = I_d$, from the above definition we get the statistical limit superior and statistical limit inferior. For $I = I_d$, we get the notions of logarithmic limit superior and logarithmic limit inferior. Similarly we can get the other notions of I-limit superior and I-limit inferior for different ideals.

From definition we can easily prove the following two theorems.

Theorem 1. *If* $\beta = I - \text{limit}\sup X$ *is finite, then for every positive fuzzy number* η

$$\{n \in N : X_n > \beta \ominus \eta\} \notin I \quad \textit{and} \quad \{n \in N : X_n > \beta \oplus \eta\} \in I \tag{1}$$

Conversely if (1) holds for every positive fuzzy number η*, then* $\beta = I - \textit{limit}\sup X$*.*

Theorem 2. *If* $\gamma = I - \text{limit}\inf X$ *is finite, then for every positive fuzzy number* γ

$$\{n \in N : X_n < \gamma \oplus \eta\} \notin I \quad \textit{and} \quad \{n \in N : X_n < \gamma \ominus \eta\} \in I \tag{2}$$

Conversely if (2) holds for every positive fuzzy number η*, then* $\gamma = I - \text{limit}\inf X$*.*

Theorem 3. *For every fuzzy real valued sequence* X*,*

$$I - \text{limit}\inf X \leq I - \text{limit}\sup X.$$

Proof. First we consider the case in which $I - \textit{limit}\sup X = -\bar{\infty}$. Then $B_X = \emptyset$. So for every μ in $L(R)$, $\{n \in N : X_n > \mu\} \in I\}$. This implies that $\{n \in N : X_n \leq \mu\} \in \Im(I)\}$. Hence for every in $\lambda \in L(R)$,the set $\{n \in N : X_n \leq \lambda\} \notin I$. So I-limit inf $X = -\bar{\infty}$.

If $I - \text{limit}\sup X = \bar{\infty}$, then the result follows immediately.

Now we assume that $\beta = I - \text{limit}\sup X$ be finite and $\gamma = I - \textit{limit}\inf X$. Let $\epsilon > 0$ be real. We show that $\beta \oplus \bar{\epsilon} \in A_X$. Since $\beta = I - \textit{limit}\sup X$, by Theorem 1, the set $\{n \in N : X_n > \beta \oplus \frac{1}{2}\bar{\epsilon}\} \in I$ which implies that $\{n \in N : X_n \leq \beta \oplus \frac{1}{2}\bar{\epsilon}\} \in \Im(I)$. Since $\{n \in N : X_n \leq \beta \oplus \frac{1}{2}\bar{\epsilon}\} \subseteq \{n \in N : X_n < \beta \oplus \bar{\epsilon}\}$ and $\Im(I)$ is a filter on N, so $\{n \in N : X_n < \beta \oplus \bar{\epsilon}\} \in \Im(I)$. Thus $\{n \in N : X_n < \beta \oplus \bar{\epsilon}\} \notin I$. Hence $\beta \oplus \bar{\epsilon} \in A_X$ and so from definition of I- limit inferior $\gamma \leq \beta \oplus \bar{\epsilon}$. Since $\bar{\epsilon}$ is arbitrary, so $\gamma \leq \beta$. From definition and Theorem 3, it can be easily shown that

$$\text{limit}\inf X \leq I - \text{limit}\inf X \leq I - \text{limit}\sup X \leq \text{limit}\sup X.$$

■

Definition 3. *A fuzzy real valued sequence $X = (X_n)$ is said to be I-bounded if there exists a real numbers $B > 0$ such that $\{n \in N : \bar{d}(X_n, \bar{0}) > B\} \in I$.*

Note. Suppose there exists a real number $B > 0$ such that

$$\{n \in N : \bar{d}(X_n, \bar{0}) > B\} \in I \tag{3}$$

Then we can easily check that the set $\{n \in N : X_n > \bar{B}\} \in I$ which implies that I-limit sup $X \leq \bar{B}$. Also from (3) we have $-\bar{B} \leq I$-limit inf X. Hence I- boundedness of a fuzzy real valued sequence $X = (X_n)$ implies $I - \operatorname{limit\,inf} X$ and $I - \operatorname{limit\,sup} X$ are finite and so properties (1) and (2) of Theorem 1 and Theorem 2 hold.

Theorem 4. *The fuzzy real valued I-bounded sequence $X = (X_n)$ is I-convergent if and only if $I - \operatorname{limit\,inf} X = I - \operatorname{limit\,sup} X$.*

Proof. Let $\gamma = I - \operatorname{limit\,inf} X$ and $\beta = I - \operatorname{limit\,sup} X$.

Suppose $I - \lim X_n = X_0$. Then given $\epsilon > 0$, the set $\{n \in N : \bar{d}(X_n, X_0) \geq \epsilon\} \in I$. Thus $\{n \in N : X_n > X_0 \oplus \bar{\epsilon}\} \in I$ and so $\beta \leq X_0$. Also $\{n \in N : X_n < X_0 \ominus \bar{\epsilon}\} \in I$. Hence $X_0 \leq \gamma$. Thus $\beta \leq \gamma$. But from Theorem 3, we get $\gamma \leq \beta$. Hence $\gamma = \beta$.

Next we assume that $\gamma = \beta$. Let $X_0 = \gamma$. Then properties (1) and (2) of Theorem 1 and Theorem 2 imply that the sets $\{n \in N : X_n > X_0 \oplus \frac{1}{2}\bar{\epsilon}\} \in I$ and $\{n \in N : X_n < X_0 \ominus \frac{1}{2}\bar{\epsilon}\} \in I$. Thus the set $\{n \in N : \bar{d}(X_n, X_0) \geq \epsilon\} \in I$. Hence $I - \lim X_n = X_0$. ■

Theorem 5. *If (X_n) is a fuzzy sequence of real numbers, then $I - \operatorname{limit\,inf}(-X_n) = -(I - \operatorname{limit\,sup} X_n)$, $I - \operatorname{limit\,sup}(-X_n) = -(I - \operatorname{limit\,inf} X_n)$.*

Proof. Let $Y_n = -X_n$. Then

$$\begin{aligned} I - \operatorname{limit\,inf}(-X_n) &= I - \operatorname{limit\,inf} Y_n \\ &= \inf\{\lambda \in L(R) : \{n \in N : Y_n < \lambda\} \notin I\} \\ &= \inf\{\lambda \in L(R) : \{n \in N : X_n > -\lambda\} \notin I\} \\ &= \inf\{-\mu \in L(R) : \{n \in N : X_n > \mu\} \notin I\} \\ &= -\sup\{\mu \in L(R) : \{n \in N : X_n > \mu\} \notin I\} \\ &= -(I - \operatorname{limit\,sup}(X_n)). \end{aligned}$$

Similarly, we can show that $I - \operatorname{limit\,sup}(-X_n) = -(I - \operatorname{limit\,inf} X_n)$. ■

Theorem 6. *If (X_n) and (Y_n) are I- bounded sequences, then*
(i) $I - \operatorname{limit\,inf} X_n \oplus I - \operatorname{limit\,inf} Y_n \leq I - \operatorname{limit\,inf}(X_n \oplus Y_n)$
(ii) $I - \operatorname{limit\,inf}(X_n \oplus Y_n) \leq I - \operatorname{limit\,inf} X_n \oplus I - \operatorname{limit\,sup} Y_n$

(*iii*) $I - \text{limit}\inf X_n \oplus I - \text{limit}\sup Y_n \leq I - \text{limit}\sup(X_n \oplus Y_n)$
(*iv*) $I - \text{limit}\sup(X_n \oplus Y_n) \leq I - \text{limit}\sup X_n \oplus I - \text{limit}\sup Y_n$.

Remark 2. For (*ii*) and (*iii*) I should be a maximal ideal.

Proof. We will prove (*i*) and (*iii*). (*ii*) and (*iv*) will follow from (*iii*) and (*i*) respectively taking the sequences $(-X_n)$ and $(-Y_n)$ in places of (X_n) and (Y_n) and using Theorem 5.

(*i*) Let $I - \liminf X_n = \alpha$ and $I - \liminf Y_n = \beta$.

Since (X_n) and (Y_n) are I- bounded, so α and β are finite. Let $\epsilon > 0$ be real. Then by Theorem 2, $A = \{n \in N : X_n < \alpha \ominus \frac{1}{2}\bar{\epsilon}\} \in I$ and $B = \{n \in N : Y_n < \beta \ominus \frac{1}{2}\bar{\epsilon}\} \in I$. Now for each $n \in A \cap B$, $X_n \oplus Y_n < (\alpha \oplus \beta) \ominus \bar{\epsilon}$. But $A \cap B \in I$.

Hence $(\alpha \oplus \beta) \ominus \bar{\epsilon} \leq I - \text{limit}\inf(X_n \oplus Y_n)$. Since $\bar{\epsilon}$ is arbitrary, so $(\alpha \oplus \beta) \leq I - \text{limit}\inf(X_n \oplus Y_n)$.

(*iii*) Let $I - \liminf X_n = \alpha$ and $I - \limsup Y_n = \beta$. Then α and β are finite. Let $\epsilon > 0$ be real. Then by Theorem 1 and Theorem 2, $\{n \in N : X_n < \alpha \ominus \frac{1}{2}\bar{\epsilon}\} \in I \Rightarrow \{n \in N : X_n \geq \alpha \ominus \frac{1}{2}\bar{\epsilon}\} \in \Im(I)$. Also $\{n \in N : Y_n > \beta \ominus \frac{1}{2}\bar{\epsilon}\} \notin I \Rightarrow \{n \in N : Y_n > \beta \ominus \frac{1}{2}\bar{\epsilon}\} \in \Im(I)$, as I is maximal. Hence $\{n \in N : X_n \oplus Y_n > (\alpha \oplus \beta) \ominus \bar{\epsilon}\} \notin I$. Hence $(\alpha \oplus \beta) \ominus \bar{\epsilon} \leq I - \text{limit}\sup(X_n \oplus Y_n)$. Since $\bar{\epsilon}$ is arbitrary, so $(\alpha \oplus \beta) \leq I - \text{limit}\sup(X_n \oplus Y_n)$. Hence the theorem. ■

4. I-limit points and I-cluster points

Definition 4. *A fuzzy real valued number μ is said to be I-limit point of the fuzzy real valued sequence $X = (X_n)$ provided that there exists a set $M = \{m_1 < m_2 < \cdots\cdots\} \subset N$ such that $M \notin I$ and* $\lim_k X_{m_k} = \mu$.

Definition 5. *A fuzzy real valued number μ is said to be I-cluster point of the fuzzy real valued sequence $X = (X_n)$ if and only if for each $\epsilon > 0$, the set $\{n \in N : \bar{d}(X_n, \mu) < \epsilon\} \notin I$.*

Let Λ_X^I and Γ_X^I denotes the set of all I-limit points and I-cluster points of X respectively.

From the definition of I - cluster point of fuzzy real valued sequence and from Theorem 1 and Theorem 2 we can interpret that $I - \text{limit}\sup X$ and $I - \text{limit}\inf X$ are the greatest and the least I- cluster points of X. The following result follows easily from Theorem 4.

Theorem 7. *A necessary and sufficient condition for the I- convergence of a fuzzy real valued sequence is that it is I- bounded and has a unique cluster point.*

Theorem 8. *If I is an admissible ideal, then for each fuzzy real valued sequence $X = (X_n)$ of elements of $L(R)$, we have $\Lambda_X^I \subset \Gamma_X^I$.*

Proof. Let $\mu \in \Lambda_X^I$. Then there exists a set $M = \{m_1 < m_2 < \cdots\cdots\} \notin I$ such that $\lim_k X_{m_k} = \mu$.

So given $\epsilon > 0$, there exists $k_0 \in N$ such that $\bar{d}(X_{m_k}, \mu) < \epsilon$ for all $k \geq k_0$.

But the set $A = \{k \in N : \bar{d}(X_k, \mu) < \epsilon\} \supset M \setminus \{m_1, m_2, \cdots\cdots, m_{k_0}\}$. So $A \notin I$. Hence $\mu \in \Gamma_X^I$. ■

Theorem 9. *The set Γ_X^I is closed in $L(R)$ for each sequence $X = (X_n)$ of elements of $L(R)$.*

Proof. Let Y be a limit point of Γ_X^I. Let $\epsilon > 0$. Then every open ball $\bar{B}(Y, \epsilon)$ with centre at Y and radius ϵ must contain a point of Γ_X^I different from Y.

Let $Y_0 \in \Gamma_X^I \cap \bar{B}(Y, \epsilon)$. Choose $\delta > 0$ such that $\bar{B}(Y_0, \delta) \subset \bar{B}(Y, \epsilon)$. Then we have, $\{n \in N : \bar{d}(X_n, Y) < \epsilon\} \supset \{n \in N : \bar{d}(X_n, Y_0) < \delta\} \notin I$. Hence $Y \in \Gamma_X^I$. ■

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