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THE NÖRLUND SPACE OF DOUBLE ENTIRE SEQUENCES

ABSTRACT. Let Γ^2 denote the spaces of all double entire sequences. Let Λ^2 denote the spaces of all double analytic sequences. This paper is devoted to a study of the general properties of Nörlund double entire sequence space $\eta(\Gamma^2)$, Γ^2 and also study some of the properties of $\eta(\Gamma^2)$ and $\eta(\Lambda^2)$.

KEY WORDS: entire sequence, analytic sequence, double sequence, Nörlund space.

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1. Introduction

Let (x_{mn}) be a double sequence of real or complex numbers. Then the series $\sum_{m,n=1}^{\infty} x_{mn}$ is called a double series. The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is said to be convergent if and only if the double sequence (S_{mn}) is convergent, where

$$S_{mn} = \sum_{i,j=1}^{m,n} x_{ij} \quad (m, n = 1, 2, 3, \dots) \quad (\text{see}[1]).$$

We denote w^2 as the class of all complex double sequences (x_{mn}) . A sequence $x = (x_{mn})$ is said to be double analytic if

$$\sup_{mn} |x_{mn}|^{1/m+n} < \infty.$$

The vector space of all prime sense double analytic sequences are usually denoted by Λ^2 . A sequence $x = (x_{mn})$ is called double entire sequence if

$$|x_{mn}|^{1/m+n} \rightarrow 0 \quad \text{as } m, n \rightarrow \infty.$$

The vector space of all prime sense double entire sequences are usually denoted by Γ^2 . The space Λ^2 as well as Γ^2 is a metric space with the metric

$$(1) \quad d(x, y) = \sup_{mn} \left\{ |x_{mn} - y_{mn}|^{1/m+n} : m, n = 1, 2, 3, \dots \right\},$$

forall $x = \{x_{mn}\}$ and $y = \{y_{mn}\}$ in Γ^2 .

Let $(P_{m,n})_{m,n=0}^\infty$ be a sequence of non-negative real numbers with $p_{00} > 0$. Consider the transformation

$$y_{mn} = \frac{1}{\sum_{i=0}^m \sum_{j=0}^n p_{ij}} \sum_{i=0}^m \sum_{j=0}^n p_{ij} x_{m-i, n-j}$$

for $m, n = 0, 1, 2, \dots$. The set of all (x_{mn}) for which $(y_{mn}) \in \Gamma^2$ is called the Nörlund space of double entire sequence. The Nörlund space of double entire sequence is denoted by $\eta(\Gamma^2)$. Similarly the set of all (x_{mn}) for which $(y_{mn}) \in \Lambda^2$ is called the Nörlund space of double analytic sequence is denoted by $\eta(\Lambda^2)$. We write $P_{mn} = p_{00} + \dots + p_{mn}$, for $m, n = 0, 1, 2, \dots$.

All absolutely convex absorbent closed subset of locally convex Topological Vector Space X is called barrel. X is called barreled space if each barrel is a neighbourhood of zero.

A locally convex Topological Vector Space X is said to be semi reflexive if each bounded closed set in X is $\sigma(X, X')$ -compact.

Consider a double sequence $x = (x_{ij})$. The $(m, n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij} \delta_{ij}$ for all $m, n \in \mathbb{N}$, where

$$\delta_{mn} = \begin{pmatrix} 0, & 0, & \dots, & 0, & \dots \\ 0, & 0, & \dots, & 0, & \dots \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ 0, & 0, & \dots, & 1, & 0, & \dots \\ 0, & 0, & \dots, & 0, & 0, & \dots \end{pmatrix}$$

with 1 in the $(m, n)^{th}$ position and zero other wise. An FK-space(or a metric space) X is said to have AK property if (δ_{mn}) is a Schauder basis for X . Or equivalently $x^{[m,n]} \rightarrow x$. We need the following inequality in the sequel of the paper:

Lemma 1. For $a, b \geq 0$ and $0 < p < 1$, we have

$$(a + b)^p \leq a^p + b^p.$$

2. Preliminaries

Let us define the following sets of double sequences:

$$\mathcal{M}_u(t) := \left\{ (x_{mn}) \in w^2 : \sup_{m,n \in \mathbb{N}} |x_{mn}|^{t_{mn}} < \infty \right\},$$

$$\begin{aligned} \mathcal{C}_p(t) &:= \left\{ (x_{mn}) \in w^2 : p - \lim_{m,n \rightarrow \infty} |x_{mn} - l|^{t_{mn}} = 1 \text{ for some } l \in \mathbb{C} \right\}, \\ \mathcal{C}_{0p}(t) &:= \left\{ (x_{mn}) \in w^2 : p - \lim_{m,n \rightarrow \infty} |x_{mn}|^{t_{mn}} = 1 \right\}, \\ \mathcal{L}_u(t) &:= \left\{ (x_{mn}) \in w^2 : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |x_{mn}|^{t_{mn}} < \infty \right\}, \\ \mathcal{C}_{bp}(t) &:= \mathcal{C}_p(t) \cap \mathcal{M}_u(t) \quad \text{and} \quad \mathcal{C}_{0bp}(t) = \mathcal{C}_{0p}(t) \cap \mathcal{M}_u(t); \end{aligned}$$

where $t = (t_{mn})$ is the sequence of strictly positive reals t_{mn} for all $m, n \in \mathbb{N}$ and $p - \lim_{m,n \rightarrow \infty}$ denotes the limit in the Pringsheim's sense. In the case $t_{mn} = 1$ for all $m, n \in \mathbb{N}$; $\mathcal{M}_u(t)$, $\mathcal{C}_p(t)$, $\mathcal{C}_{0p}(t)$, $\mathcal{L}_u(t)$, $\mathcal{C}_{bp}(t)$ and $\mathcal{C}_{0bp}(t)$ reduce to the sets \mathcal{M}_u , \mathcal{C}_p , \mathcal{C}_{0p} , \mathcal{L}_u , \mathcal{C}_{bp} and \mathcal{C}_{0bp} , respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Göhan and Colak [10, 11] have proved that $\mathcal{M}_u(t)$ and $\mathcal{C}_p(t)$, $\mathcal{C}_{bp}(t)$ are complete paranormed spaces of double sequences and gave the α -, β -, γ - duals of the spaces $\mathcal{M}_u(t)$ and $\mathcal{C}_{bp}(t)$. Quite recently, in her PhD thesis, Zelter [12] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [13] have recently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Nextly, Mursaleen [14] and Mursaleen and Edely [15] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the M -core for double sequences and determined those four dimensional matrices transforming every bounded double sequences $x = (x_{jk})$ into one whose core is a subset of the M -core of x . More recently, Altay and Basar [16] have defined the spaces \mathcal{BS} , $\mathcal{BS}(t)$, \mathcal{CS}_p , \mathcal{CS}_{bp} , \mathcal{CS}_r and \mathcal{BV} of double sequences consisting of all double series whose sequence of partial sums are in the spaces \mathcal{M}_u , $\mathcal{M}_u(t)$, \mathcal{C}_p , \mathcal{C}_{bp} , \mathcal{C}_r and \mathcal{L}_u , respectively, and also examined some properties of those sequence spaces and determined the α - duals of the spaces \mathcal{BS} , \mathcal{BV} , \mathcal{CS}_{bp} and the $\beta(\vartheta)$ - duals of the spaces \mathcal{CS}_{bp} and \mathcal{CS}_r of double series. Quite recently Basar and Sever [17] have introduced the Banach space \mathcal{L}_q of double sequences corresponding to the well-known space ℓ_q of single sequences and examined some properties of the space \mathcal{L}_q . Quite recently Subramanian and Misra [18,19] have studied the space $\chi_M^2(p, q, u)$ of double sequences and gave some inclusion relations and also studied characterization and general properties of gai sequences via double Orlicz space of χ_M^2 of χ^2 establishing some inclusion relations.

Some initial works on double sequence spaces is found in Bromwich[3]. Later on it was investigated by Hardy[5], Moricz[6], Moricz and Rhoades[7],

Basarir and Solankan[2], Tripathy[8], Colak and Turkmenoglu[4], Turkmenoglu[9], and many others.

3. Main results

Proposition 1. $\eta(\Gamma^2) = \Gamma^2$.

Proof. Let $x = (x_{mn}) \in \eta(\Gamma^2)$. Then $y \in \Gamma^2$ so that for every $\epsilon > 0$, we have a positive integer n_0 such that

$$\left| \frac{p_{00}x_{mn} + \cdots + p_{mn}x_{00}}{P_{mn}} \right| < \epsilon^{m+n} \text{ for all } m, n \geq n_0$$

Take $p_{00} = 1$; $p_{11} = \cdots = p_{mn} = 0$. We then have $|x_{mn}| < \epsilon^{m+n}$, $\forall m, n \geq n_0$. Therefore $x = (x_{mn}) \in \Gamma^2$. Hence

$$(2) \quad \eta(\Gamma^2) \subset \Gamma^2$$

On the other hand, let $x = (x_{mn}) \in \Gamma^2$. But for any given $\epsilon > 0$, there exists a positive integer n_0 such that $|x_{mn}| < \epsilon^{m+n}$, $\forall m, n \geq n_0$. We have

$$\begin{aligned} |y_{mn}|! &\leq \left| \frac{p_{00}x_{mn} + \cdots + p_{mn}x_{00}}{P_{mn}} \right| \\ &\leq \frac{1}{P_{mn}} [p_{00}|x_{mn}| + \cdots + p_{mn}|x_{00}|] \\ &\leq \frac{1}{P_{mn}} [p_{00}\epsilon^{m+n} + \cdots + p_{mn}\epsilon^{0+0}] \\ &\leq \frac{\epsilon^{m+n}}{P_{mn}} [p_{00} + \cdots + p_{mn}] \\ &\leq \frac{\epsilon^{m+n}}{P_{mn}} P_{mn} = \epsilon^{m+n} \forall m, n \geq n_0. \end{aligned}$$

Therefore $(y_{mn}) \in \Gamma^2$. Consequently $x \in \eta(\Gamma^2)$. Hence

$$(3) \quad \Gamma^2 \subset \eta(\Gamma^2)$$

From (2) and (3) we obtain $\eta(\Gamma^2) = \Gamma^2$. This completes the proof. ■

Proposition 2. $\eta(\Lambda^2) = \Lambda^2$.

Proof. Let $(x_{mn}) \in \Lambda^2$. Then there exists a positive constant T such that

$$\begin{aligned} |x_{mn}| &\leq T^{m+n} \text{ for } m, n = 0, 1, 2, \dots . \\ |y_{mn}| &\leq \frac{p_{00}T^{m+n} + \dots + p_{mn}T^{0+0}}{P_{mn}} \\ &\leq \frac{T^{m+n}}{P_{mn}} \left[p_{00} + \dots + \frac{p_{mn}}{T^{m+n}} \right] \\ &\leq \frac{T^{m+n}}{P_{mn}} [p_{00} + \dots + p_{mn}] \\ &\leq \frac{T^{m+n}}{P_{mn}} P_{mn} = T^{m+n}, \text{ for } m, n = 0, 1, 2, \dots . \end{aligned}$$

Hence $(y_{mn}) \in \Lambda^2$. But then $x = (x_{mn}) \in \eta(\Gamma^2)$. Consequently

$$(4) \quad \Lambda^2 \subset \eta(\Lambda^2)$$

On the other hand let $(x_{mn}) \in \eta(\Lambda^2)$. Then $(y_{mn}) \in \Lambda^2$. Hence there exists a positive constant T such that $|y_{mn}| < T^{m+n}$ for $m, n = 0, 1, 2, \dots$. This in turn implies that

$$\left| \frac{p_{00}x_{mn} + \dots + P_{mn}x_{00}}{P_{mn}} \right| < T^{m+n}$$

Hence

$$\frac{1}{P_{mn}} (|p_{00}x_{mn} + \dots + p_{mn}x_{00}|) < T^{m+n}$$

and thus

$$|p_{00}x_{mn} + \dots + p_{mn}x_{00}| < P_{mn}T^{m+n}.$$

Take $p_{00} = 1; p_{11} = \dots = p_{mn} = 0$. Then it follows that $P_{mn} = 1$ and so $|x_{mn}| < T^{m+n}$ for all m, n . Consequently $x = (x_{mn}) \in \Lambda^2$. Hence

$$(5) \quad \eta(\Lambda^2) \subset \Lambda^2$$

From (4) and (5) we get $\eta(\Lambda^2) = \Lambda^2$. This completes the proof. ■

Proposition 3. Γ^2 is not a barreled space.

Proof. Let

$$A = \left\{ x \in \Gamma^2 : |x_{mn}|^{\frac{1}{m+n}} \leq \frac{1}{m+n}, \forall m, n \right\}.$$

Then A is an absolutely convex, closed absorbent in Γ^2 . But A is not a neighbourhood of zero. Hence Γ^2 is not barreled. ■

Proposition 4. Γ^2 is not semi reflexive.

Proof. Let $\{\delta^{(mn)}\} \in U$ be the unit closed ball in Γ^2 . Clearly no subsequence of $\{\delta^{(mn)}\}$ can converge weakly to any $y \in \Gamma^2$. Hence Γ^2 is not semi reflexive. ■

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