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ON THE SOLUTIONS OF THE DIFFERENCE EQUATION $x_{n+1} = \max\left\{\frac{1}{x_n}, \frac{x_{n-1}}{x_n}\right\}$

Abstract. We study the solutions of the following difference equation

$$x_{n+1} = \max\left\{\frac{1}{x_n}, \frac{x_{n-1}}{x_n}\right\},$$

where initial conditions $x_{-1}$and x_0 are nonzero real numbers. In most of the cases we determine the solutions in function of the initial conditions x_{-1} and x_0.

Key words: difference equation, max operator, fibonacci number.

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1. Introduction

In this paper we study the solutions of the following difference equation

$$x_{n+1} = \max\left\{\frac{1}{x_n}, \frac{x_{n-1}}{x_n}\right\}, \quad n = 0, 1, ..., \tag{1}$$

where initial conditions $x_{-1}$and x_0 are nonzero real numbers.

Some closely related equations were investigated, for example [1-6]. For example, the investigation of the difference equation

$$x_{n+1} = \max\left\{\frac{A_0}{x_n}, \frac{A_1}{x_{n-1}}, ..., \frac{A_k}{x_{n-k}}\right\}, \quad n = 0, 1, ..., \tag{2}$$

where A_i, $i = 0, 1, ..., k$ are real numbers, such that at least one of them is different from zero and initial conditions $x_0, x_{-1}, ..., x_{-k}$ are different from zero, was proposed in [2] and [3].

A special case the max operator in Eq.(2) arises naturally in certain models in automatic control theory (see [4, 5]).

2. Main results

We consider the solutions of the Eq.(1). The following theorem completely desciribes the solutions of Eq.(1).

Theorem 1. *Consider the Eq.(1), then the general solution of Eq.(1) is* $x_m = (\frac{x_{-1}^p}{x_0^s})^{2^k}$ *where* p, s *are any integers and* m *is a function of* k, *(for* $k = 0, 1, ...$*).*

Proof. (A) Let $1 < x_0$.

Also, $F(j)$ is the smallest fibonacci (for $F(0) = F(1) = 1$) number which is holds (3) or (4) for $x_0 < x_{-1}$.

$$x_{-1}^{F(j-1)} < x_0^{F(j)} \quad \text{and} \quad x_{-1}^{F(j)} < x_0^{F(j+1)}, \tag{3}$$

$$x_0^{F(j)} < x_{-1}^{F(j-1)} \quad \text{and} \quad x_0^{F(j+1)} < x_{-1}^{F(j)}, \tag{4}$$

(i) If $1 < x_0 < x_{-1}$ and $F(j)$ holds (3), then

$x_1 = \max\left\{\frac{1}{x_0}, \frac{x_{-1}}{x_0}\right\} = \frac{x_{-1}}{x_0}, \quad x_2 = \max\left\{\frac{x_0}{x_{-1}}, \frac{x_0^2}{x_{-1}}\right\} = \frac{x_0^2}{x_{-1}},$

$x_3 = \max\left\{\frac{x_{-1}}{x_0^2}, \frac{x_{-1}^2}{x_0^3}\right\} = \frac{x_{-1}^2}{x_0^3}, ..., x_{j+1} = \max\left\{\frac{x_{-1}^{F(j-1)}}{x_0^{F(j)}}, \frac{x_{-1}^{F(j)}}{x_0^{F(j+1)}}\right\} = \frac{x_{-1}^{F(j)}}{x_0^{F(j+1)}},$

$x_{j+2} = \max\left\{\frac{x_0^{F(j+1)}}{x_{-1}^{F(j)}}, \frac{x_0^{F(j+2)}}{x_{-1}^{F(j+1)}}\right\} = \frac{x_0^{F(j+2)}}{x_{-1}^{F(j+1)}},$

$x_{j+3} = \max\left\{\frac{x_{-1}^{F(j+1)}}{x_0^{F(j+2)}}, \frac{x_{-1}^{F(j+2)}}{x_0^{F(j+3)}}\right\} = \frac{x_{-1}^{F(j+1)}}{x_0^{F(j+2)}},$

$x_{j+4} = \max\left\{\frac{x_0^{F(j+2)}}{x_{-1}^{F(j+1)}}, \frac{x_0^{2F(j+2)}}{x_{-1}^{2F(j+1)}}\right\} = \frac{x_0^{2F(j+2)}}{x_{-1}^{2F(j+1)}} = (\frac{x_0^{F(j+2)}}{x_{-1}^{F(j+1)}})^2, ...,$

A simple argument yields that, the general solution of Eq.(1) are

$$x_{j+2+2k} = (\frac{x_0^{F(j+2)}}{x_{-1}^{F(j+1)}})^{2^k}, \quad x_{j+2+2k+1} = (\frac{x_{-1}^{F(j+1)}}{x_0^{F(j+2)}})^{2^k} \text{ for } k = 0, 1, 2, ...,$$

Also, First $(j + 2)$ solutions of Eq.(1) are

$$x_{2l+1} = \frac{x_{-1}^{F(2l)}}{x_0^{F(2l+1)}}, \quad x_{2l+2} = \frac{x_0^{F(2l+2)}}{x_{-1}^{F(2l+1)}} \text{ for } l = 0, 1, 2, ..., \frac{j}{2},$$

If $F(j)$ is the smallest fibonacci number which is holds (4), then

$x_1 = \max\left\{\frac{1}{x_0}, \frac{x_{-1}}{x_0}\right\} = \frac{x_{-1}}{x_0}, \quad x_2 = \max\left\{\frac{x_0}{x_{-1}}, \frac{x_0^2}{x_{-1}}\right\} = \frac{x_0^2}{x_{-1}},$

$x_3 = \max\left\{\frac{x_{-1}}{x_0^2}, \frac{x_{-1}^2}{x_0^3}\right\} = \frac{x_{-1}^2}{x_0^3}, ..., x_{j+1} = \max\left\{\frac{x_0^{F(j)}}{x_{-1}^{F(j-1)}}, \frac{x_0^{F(j+1)}}{x_{-1}^{F(j)}}\right\} = \frac{x_0^{F(j+1)}}{x_{-1}^{F(j)}},$

$$x_{j+2} = \max\left\{\frac{x_{-1}^{F(j)}}{x_0^{F(j+1)}}, \frac{x_{-1}^{F(j+1)}}{x_0^{F(j+2)}}\right\} = \frac{x_{-1}^{F(j+1)}}{x_0^{F(j+2)}},$$
$$x_{j+3} = \max\left\{\frac{x_0^{F(j+2)}}{x_{-1}^{F(j+1)}}, \frac{x_0^{F(j+3)}}{x_{-1}^{F(j+2)}}\right\} = \frac{x_0^{F(j+2)}}{x_{-1}^{F(j+1)}},$$
$$x_{j+4} = \max\left\{\frac{x_{-1}^{F(j+1)}}{x_0^{F(j+2)}}, \frac{x_{-1}^{2F(j+1)}}{x_0^{2F(j+2)}}\right\} = \frac{x_{-1}^{2F(j+1)}}{x_0^{2F(j+2)}} = (\frac{x_{-1}^{F(j+1)}}{x_0^{F(j+2)}})^2, ...,$$

A simple argument yields that, the general solution of Eq.(1) are

$$x_{j+2+2k} = (\frac{x_{-1}^{F(j+1)}}{x_0^{F(j+2)}})^{2^k}, \quad x_{j+2+2k+1} = (\frac{x_0^{F(j+2)}}{x_{-1}^{F(j+1)}})^{2^k} \quad \text{for} k = 0, 1, 2, ...,$$

Also, First $(j+1)$ solutions of Eq.(1) are

$$x_{2l+1} = \frac{x_{-1}^{F(2l)}}{x_0^{F(2l+1)}}, \quad x_{2l} = \frac{x_0^{F(2l)}}{x_{-1}^{F(2l-1)}} \qquad for\ l = 0, 1, 2, ..., \frac{j}{2},$$

(ii) If $1 \le x_{-1} \le x_0$, then similarly

$$x_1 = \max\left\{\frac{1}{x_0}, \frac{x_{-1}}{x_0}\right\} = \frac{x_{-1}}{x_0}, \qquad x_2 = \max\left\{\frac{x_0}{x_{-1}}, \frac{x_0^2}{x_{-1}}\right\} = \frac{x_0^2}{x_{-1}},$$
$$x_3 = \max\left\{\frac{x_{-1}}{x_0^2}, \frac{x_{-1}^2}{x_0^3}\right\} = \frac{x_{-1}}{x_0^2}, ...,$$

A simple argument yields that, the general solution of Eq.(1) are

$$x_{2k+2} = (\frac{x_0^2}{x_{-1}})^{2^k}, \quad x_{2k+3} = (\frac{x_{-1}}{x_0^2})^{2^k} \quad \text{for } k = 0, 1, 2, ...,$$

(iii) If $x_{-1} \le 1 \le x_0$ and $x_{-1} \ne 0$, then similarly

$$x_1 = \max\left\{\frac{1}{x_0}, \frac{x_{-1}}{x_0}\right\} = \frac{1}{x_0}, \qquad x_2 = \max\left\{x_0, x_0^2\right\} = x_0^2,$$
$$x_3 = \max\left\{\frac{1}{x_0^2}, \frac{1}{x_0^3}\right\} = \frac{1}{x_0^2}, ...,$$

A simple argument yields that, the general solution of Eq.(1) are

$$x_{2k} = (x_0)^{2^k}, \quad x_{2k+1} = (\frac{1}{x_0})^{2^k} \quad \text{for } k = 0, 1, 2, ...,$$

(B) Let $0 < x_0 \le 1$.

(i) If $1 \le x_{-1}$, then

$$x_1 = \max\left\{\frac{1}{x_0}, \frac{x_{-1}}{x_0}\right\} = \frac{x_{-1}}{x_0}, \qquad x_2 = \max\left\{\frac{x_0}{x_{-1}}, \frac{x_0^2}{x_{-1}}\right\} = \frac{x_0}{x_{-1}},$$
$$x_3 = \max\left\{\frac{x_{-1}}{x_0}, \frac{x_{-1}^2}{x_0^2}\right\} = \frac{x_{-1}^2}{x_0^2}, ...,$$

A simple argument yields that, the general solution of Eq.(1) are

$$x_{2k+1} = (\frac{x_{-1}}{x_0})^{2^k}, \quad x_{2k+2} = (\frac{x_0}{x_{-1}})^{2^k} \quad \text{for } k = 0, 1, 2, ...,$$

(ii) If $x_{-1} \le 1$, $(x_{-1} \ne 0)$, then

$x_1 = \max\left\{\frac{1}{x_0}, \frac{x_{-1}}{x_0}\right\} = \frac{1}{x_0}, \quad x_2 = \max\left\{x_0, x_0^2\right\} = x_0$
$x_3 = \max\left\{\frac{1}{x_0}, \frac{1}{x_0^2}\right\} = \frac{1}{x_0^2}, ...,$
A simple argument yields that, the general solution of Eq.(1) are

$$x_{2k+1} = (\frac{1}{x_0})^{2^k}, \ \ x_{2k+2} = (x_0)^{2^k} \quad \text{for } \ k = 0, 1, 2, ...,$$

(C) Let $x_0 < 0$.

(i) If $0 < x_{-1} \leq 1$, then
$x_1 = \max\left\{\frac{1}{x_0}, \frac{x_{-1}}{x_0}\right\} = \frac{x_{-1}}{x_0}, \quad x_2 = \max\left\{\frac{x_0}{x_{-1}}, \frac{x_0^2}{x_{-1}}\right\} = \frac{x_0^2}{x_{-1}},$
$x_3 = \max\left\{\frac{x_{-1}}{x_0^2}, \frac{x_{-1}^2}{x_0^3}\right\} = \frac{x_{-1}}{x_0^2}.$
If $x_0^2 < x_{-1}$, then
$x_4 = \max\left\{\frac{x_0^2}{x_{-1}}, \frac{x_0^4}{x_{-1}^2}\right\} = \frac{x_0^2}{x_{-1}}, \quad x_5 = \max\left\{\frac{x_{-1}}{x_0^2}, \frac{x_{-1}^2}{x_0^4}\right\} = \frac{x_{-1}^2}{x_0^4}, ...,$
A simple argument yields that, the general solution of Eq.(1) are

$$x_{2k+3} = (\frac{x_{-1}}{x_0^2})^{2^k}, \ \ x_{2k+4} = (\frac{x_0^2}{x_{-1}})^{2^k} \ \ \text{for } \ k = 0, 1, 2, ...,$$

If $x_{-1} < x_0^2$, then
$x_4 = \max\left\{\frac{x_0^2}{x_{-1}}, \frac{x_0^4}{x_{-1}^2}\right\} = \frac{x_0^4}{x_{-1}^2}, \quad x_5 = \max\left\{\frac{x_{-1}^2}{x_0^4}, \frac{x_{-1}^3}{x_0^6}\right\} = \frac{x_{-1}^2}{x_0^4}, ...$
A simple argument yields that, the general solution of Eq.(1) are

$$x_{2k+2} = (\frac{x_0^2}{x_{-1}})^{2^k}, \quad x_{2k+3} = (\frac{x_{-1}}{x_0^2})^{2^k} \ \ \text{for } \ k = 0, 1, 2, ...,$$

(ii) If $-1 \leq x_0 < 0$ and $1 \leq x_{-1}$, then
$x_1 = \max\left\{\frac{1}{x_0}, \frac{x_{-1}}{x_0}\right\} = \frac{1}{x_0}, \quad x_2 = \max\left\{x_0, x_0^2\right\} = x_0^2,$
$x_3 = \max\left\{\frac{1}{x_0^2}, \frac{1}{x_0^3}\right\} = \frac{1}{x_0^2}, ...$
A simple argument yields that, the general solution of Eq.(1) are

$$x_{2k+1} = (\frac{1}{x_0})^{2^k}, \quad x_{2k+4} = (x_0^2)^{2^k} \ \ \text{for } \ k = 0, 1, 2, ...,$$

(iii) $x_0 \leq -1$ and $1 \leq x_{-1}$, then
$x_1 = \max\left\{\frac{1}{x_0}, \frac{x_{-1}}{x_0}\right\} = \frac{1}{x_0}, \quad x_2 = \max\left\{x_0, x_0^2\right\} = x_0^2, ...,$
A simple argument yields that, the general solution of Eq.(1) are

$$x_{2k} = (x_0)^{2^k}, \ \ x_{2k+1} = (\frac{1}{x_0})^{2^k} \ \ \text{for } \ k = 0, 1, 2, ...,$$

(iv) If $x_{-1} \leq x_0 < 0$, then

$x_1 = \max\left\{\frac{1}{x_0}, \frac{x_{-1}}{x_0}\right\} = \frac{x_{-1}}{x_0}, \quad x_2 = \max\left\{\frac{x_0}{x_{-1}}, \frac{x_0^2}{x_{-1}}\right\} = \frac{x_0}{x_{-1}},$
$x_3 = \max\left\{\frac{x_{-1}}{x_0}, \frac{x_{-1}^2}{x_0^2}\right\} = \frac{x_{-1}^2}{x_0^2}, ...,$
A simple argument yields that, the general solution of Eq.(1) are

$$x_{2k+1} = (\frac{x_{-1}}{x_0})^{2^k}, \quad x_{2k+2} = (\frac{x_0}{x_{-1}})^{2^k} \text{ for } k = 0, 1, 2, ...,$$

(v) If $x_0 \leq x_{-1} < 0$, then
$x_1 = \max\left\{\frac{1}{x_0}, \frac{x_{-1}}{x_0}\right\} = \frac{x_{-1}}{x_0}, \quad x_2 = \max\left\{\frac{x_0}{x_{-1}}, \frac{x_0^2}{x_{-1}}\right\} = \frac{x_0}{x_{-1}},$
$x_3 = \max\left\{\frac{x_{-1}}{x_0}, \frac{x_{-1}^2}{x_0^2}\right\} = \frac{x_{-1}}{x_0}, ...,$
A simple argument yields that, the general solution of Eq.(1) are

$$x_{2k+2} = (\frac{x_0}{x_{-1}})^{2^k}, \quad x_{2k+3} = (\frac{x_{-1}}{x_0})^{2^k} \text{ for } k = 0, 1, 2, ...,$$

Then, the proof is completed. ■

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