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GLOBAL ASYMPTOTIC STABILITY OF A SYSTEM OF TWO NONLINEAR DIFFERENCE EQUATIONS

Abstract. In this paper a sufficient condition is obtained for the global asmptotic stability of the following system of difference equations

$$z_{n+1} = \frac{t_n + z_{n-1}}{t_n z_{n-1} + a}, \quad t_{n+1} = \frac{z_n + t_{n-1}}{z_n t_{n-1} + a}, \ \ n = 0, 1, 2, ...$$

where the parameter $a \epsilon (0, \infty)$ and the initial values $(z_k, t_k) \epsilon (0, \infty)$ (for $k = -1, 0$).

Key words: rational difference equation, system, global asymptotic stability, semicycle, equilibrium point.

AMS Mathematics Subject Classification: 39A10.

1. Introduction

Recently there has been an increasing interest in the study of qualitative analysis of rational difference equations and systems of difference equations. Difference equations appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations having applications in biology, ecology, economy, physics, etc [5].Although difference equations are very simple in form, it is extremely difficult to understand thoroughly the global behaviors of their solutions. (see [1-8] and the references cited therein).

In [2] De Vault et al. proved that the unique equilibrium of the difference equation

$$x_{n+1} = A + \frac{x_n}{x_{n-1}}, \quad n = 0, 1, 2, \ldots$$

where $A \epsilon (0, \infty)$, is globally asymptotically stable and proved the oscillatory behavior of the positive solutions of this difference equation.

From on, Papaschinopoluos and Schinas [5] extended the results obtained in [2] to the following system of difference equations:

$$x_{n+1} = A + \frac{y_n}{y_{n-p}}, \qquad y_{n+1} = A + \frac{x_n}{x_{n-q}}, \quad n = 0, 1, 2, \ldots$$

where $A \in (0,\infty)$, p, q are positive integers and $x_{-q}, x_{-q+1}, \dots x_0$, y_{-p}, $y_{-p+1}, \dots y_0$, are positive initial values.

Li and Zhu [4] proved that the unique positive equilibrium of the difference equation

$$x_{n+1} = \frac{x_n x_{n-1} + a}{x_n + x_{n-1}}, \quad n = 0, 1, 2, \dots$$

where $a \epsilon [0,\infty)$ and x_{-1}, x_0 are positive, is globally asymptotically stable.

From on, we [1] extended the results obtained in [4] to the following difference equation

$$x_{n+1} = \frac{x_n x_{n-k} + a}{x_n + x_{n-k}}, \quad n = 0, 1, 2, \dots$$

where k is nonnegative integer, $a \in [0,\infty)$ and $x_{-k}, ..., x_0$ are positive, is globally asymptotically stable.

Also, we [8] extended the results obtained in [4] to the following system of difference equations

$$z_{n+1} = \frac{z_n t_{n-1} + a}{z_n + t_{n-1}}, \quad t_{n+1} = \frac{t_n z_{n-1} + a}{t_n + z_{n-1}}, \quad n = 0, 1, 2, \dots$$

where $a \in (0,\infty)$ and the initial values $(z_k, t_k) \in (0,\infty)$ (for $k = -1, 0$), is globally asymptotically stable.

In this paper, we consider the following system of difference equations

$$(1) \qquad z_{n+1} = \frac{t_n + z_{n-1}}{t_n z_{n-1} + a}, \qquad t_{n+1} = \frac{z_n + t_{n-1}}{z_n t_{n-1} + a}, \quad n = 0, 1, 2, \dots$$

where $a \in (0,\infty)$ and the initial values $(z_k, t_k) \in (0,\infty)$ (for $k = -1, 0$). Our main aim is to investigate the global asymptotic behavior of its solutions.

It is clear that the change of variables

$$(z_n, t_n) = (\sqrt{a} x_n, \sqrt{a} y_n)$$

reduces the system (1) to the system

$$(2) \qquad x_{n+1} = \frac{y_n + x_{n-1}}{y_n x_{n-1} + 1}, \qquad y_{n+1} = \frac{x_n + y_{n-1}}{x_n y_{n-1} + 1}, \quad n = 0, 1, 2, \dots$$

where the initial values $(x_k, y_k) \in (0,\infty)$ (for $k = -1, 0$).

We need the following definitions and theorem [3]:

Let I be some interval of real numbers and let

$$f, g : I \times I \to I$$

be continuously differentiable functions. Then for all initial values $(x_k, y_k) \in I$, $k = -1, 0$, the system of difference equations

$$(3) \qquad x_{n+1} = f(x_n, y_{n-1}), \qquad y_{n+1} = g(y_n, x_{n-1}), \quad n = 0, 1, 2, \dots$$

has a unique solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$.

Definition 1. *A point* $(\overline{x}, \overline{y})$ *called an equilibrium point of the system (3) if*

$$\overline{x} = f(\overline{x}, \overline{y}) \text{ and } \overline{y} = g(\overline{x}, \overline{y}).$$

It is easy to see that the system (2) has the unique positive equilibrium $(\overline{x}, \overline{y}) = (1, 1)$.

Definition 2. *Let* $(\overline{x}, \overline{y})$ *be an equilibrium point of the system (3).*

(a) *An equilibrium point* $(\overline{x}, \overline{y})$ *is said to be stable if for any* $\varepsilon > 0$ *there is* $\delta > 0$ *such that for every initial points* (x_{-1}, y_{-1}) *and* (x_0, y_0) *for which* $\|(x_{-1}, y_{-1}) - (\overline{x}, \overline{y})\| + \|(x_0, y_0) - (\overline{x}, \overline{y})\| < \delta$ *, the iterates* (x_n, y_n) *of* (x_{-1}, y_{-1}) *and* (x_0, y_0) *satisfy* $\|(x_n, y_n) - (\overline{x}, \overline{y})\| < \varepsilon$ *for all* $n > 0$. *An equilibrium point* $(\overline{x}, \overline{y})$ *is said to be unstable if it is not stable. (By* $\|.\|$ *we denote the Euclidean norm in* $\mathbb{R}^2$ *given by* $\|(x, y)\| = \sqrt{x^2 + y^2}$*)*

(b) *An equilibriumpoint* $(\overline{x}, \overline{y})$ *is said to be asymptotically stable if there exists* $r > 0$ *such that* $(x_n, y_n) \to (\overline{x}, \overline{y})$ *as* $n \to \infty$ *for all* (x_{-1}, y_{-1}) *and* (x_0, y_0) *that satisfy* $\|(x_{-1}, y_{-1}) - (\overline{x}, \overline{y})\| + \|(x_0, y_0) - (\overline{x}, \overline{y})\| < r$.

Definition 3. *Let* $(\overline{x}, \overline{y})$ *be an equilibrium point of a map* $F = (f, g)$, *where* f *and* g *are continuously differentiable functions at* $(\overline{x}, \overline{y})$. *The Jacobian matrix of* F *at* $(\overline{x}, \overline{y})$ *is the matrix*

$$J_F(\overline{x}, \overline{y}) = \begin{bmatrix} \frac{\partial f}{\partial x}(\overline{x}, \overline{y}) & \frac{\partial f}{\partial y}(\overline{x}, \overline{y}) \\ \frac{\partial g}{\partial x}(\overline{x}, \overline{y}) & \frac{\partial g}{\partial y}(\overline{x}, \overline{y}) \end{bmatrix}.$$

The linear map $J_F(\overline{x}, \overline{y}) : \mathbb{R}^2 \to \mathbb{R}^2$ *given by*

$$J_F(\overline{x}, \overline{y}) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x}(\overline{x}, \overline{y})x + \frac{\partial f}{\partial y}(\overline{x}, \overline{y})y \\ \frac{\partial g}{\partial x}(\overline{x}, \overline{y})x + \frac{\partial g}{\partial y}(\overline{x}, \overline{y})y \end{bmatrix} \tag{4}$$

is called the linearization of the map F *at* $(\overline{x}, \overline{y})$.

Theorem 1 (Linearized Stability Theorem). *Let* $F = (f, g)$ *be a continuously differentiable function defined on an open set* I *in* $\mathbb{R}^2$, *and let* $(\overline{x}, \overline{y})$ *in* I *be an equilibrium point of the map* $F = (f, g)$.

(a) *If all the eigenvalues of the Jacobian matrix* $J_F(\overline{x}, \overline{y})$ *have modulus less than one, then the equilibrium point* $(\overline{x}, \overline{y})$ *is asymptotically stable.*

(b) *If at least one of the eigenvalues of the Jacobian matrix* $J_F(\overline{x}, \overline{y})$ *has modulus greater than one, then the equilibrium point* $(\overline{x}, \overline{y})$ *is unstable.*

(c) *An equilibrium point* $(\overline{x}, \overline{y})$ *of the map* $F = (f, g)$ *is locally asymptotically stable if and only if every solution of the characteristic equation*

$$\lambda^2 - trJ_F(\overline{x}, \overline{y})\lambda + \det J_F(\overline{x}, \overline{y}) = 0$$

lies inside the unit circle, that is, if and only if

$$|trJ_F(\overline{x}, \overline{y})| < 1 + \det J_F(\overline{x}, \overline{y}) < 2.$$

Definition 4. *Let* $(\overline{x}, \overline{y})$ *be positive equilibrium point of the system (3). [see 6]*

A "string" of consecutive terms $\{x_s, ..., x_m\}$ *(resp.* $\{y_s, ..., y_m\}$*),* $s \geq -1$*,* $m \leq \infty$ *is said to be a positive semicycle if* $x_i \geqslant \overline{x}$ *(resp.* $y_i \geqslant \overline{y}$*),* $i \in \{s, ..., m\}$*,* $x_{s-1} < \overline{x}$ *(resp.* $y_{s-1} < \overline{y}$*), and* $x_{m+1} < \overline{x}$ *(resp.* $y_{m+1} < \overline{y}$*).*

A "string" of consecutive terms $\{x_s, ..., x_m\}$ *(resp.* $\{y_s, ..., y_m\}$*),* $s \geq -1$*,* $m \leq \infty$ *is said to be a negative semicycle if* $x_i < \overline{x}$ *(resp.* $y_i < \overline{y}$*),* $i \in \{s, ..., m\}$*,* $x_{s-1} \geqslant \overline{x}$ *(resp.* $y_{s-1} \geqslant \overline{y}$*), and* $x_{m+1} \geqslant \overline{x}$ *(resp.* $y_{m+1} \geqslant \overline{y}$*).*

A "string" of consecutive terms $\{(x_s, y_s), ..., (x_m, y_m)\}$ *is said to be a positive (resp. negative) semicycle if* $\{x_s, ..., x_m\}$*,* $\{y_s, ..., y_m\}$ *are positive (resp. negative) semicycles. Finally a "string" of consecutive terms* $\{(x_s, y_s), ..., (x_m, y_m)\}$ *is said to be a semicycle positive (resp. negative) with respect to* x_n *and negative (resp. positive) with respect to* y_n *if* $\{x_s, ..., x_m\}$ *is a positive (resp. negative) semicycle and* $\{y_s, ..., y_m\}$ *is a negative (resp. positive) semicycle.*

We now give new definitions. These definitions can be used for different subsequences of $\{x_n\}$ (resp. $\{y_n\}$).

Definition 5. *Let* $(\overline{x}, \overline{y})$ *be positive equilibrium point of the system (3).*

A "string" of consecutive terms $\{x_{2s}, x_{2s+2}, ..., x_{2m}\}$ *(resp.* $\{y_{2s}, ..., y_{2m}\}$*),* $s \geq 1$*,* $m \leq \infty$ *is said to be a positive sub-semicycle associated with* $\{x_{2n}\}$ *(resp.* $\{y_{2n}\}$*) if* $x_i \geqslant \overline{x}$ *(resp.* $y_i \geqslant \overline{y}$*),* $i \epsilon \{2s, 2s+2, ..., 2m\}$*,* $x_{2s-2} < \overline{x}$ *(resp.* $y_{2s-2} < \overline{y}$*), and* $x_{2m+2} < \overline{x}$ *(resp.* $y_{2m+2} < \overline{y}$*).*

A "string" of consecutive terms $\{x_{2s}, x_{2s+2}, ..., x_{2m}\}$ *(resp.* $\{y_{2s}, ..., y_{2m}\}$*),* $s \geq 1$*,* $m \leq \infty$ *is said to be a negative sub-semicycle associated with* $\{x_{2n}\}$ *(resp.* $\{y_{2n}\}$*) if* $x_i < \overline{x}$ *(resp.* $y_i < \overline{y}$*),* $i \epsilon \{2s, 2s+2, ..., 2m\}$*,* $x_{2s-2} \geqslant \overline{x}$ *(resp.* $y_{2s-2} \geqslant \overline{y}$*), and* $x_{2m+2} \geqslant \overline{x}$ *(resp.* $y_{2m+2} \geqslant \overline{y}$*).*

A "string" of consecutive terms $\{(x_{2s}, y_{2s}), (x_{2s+2}, y_{2s+2}), ..., (x_{2m}, y_{2m})\}$ *is said to be a positive (resp. negative) sub-semicycle if* $\{x_{2s}, x_{2s+2}, ..., x_{2m}\}$*,* $\{y_{2s}, ..., y_{2m}\}$ *are positive (resp. negative) sub-semicycles. Finally a "string" of consecutive terms* $\{(x_{2s}, y_{2s}), (x_{2s+2}, y_{2s+2}), ..., (x_{2m}, y_{2m})\}$ *is said to be a sub-semicycle positive (resp. negative) with respect to* x_{2n} *and negative (resp. positive) with respect to* y_{2n} *if* $\{x_{2s}, x_{2s+2}, ..., x_{2m}\}$ *is a positive (resp. negative) sub-semicycle and* $\{y_{2s}, ..., y_{2m}\}$ *is a negative (resp. positive) sub-semicycle.*

Definition 6. *Let* $(\overline{x}, \overline{y})$ *be positive equilibrium point of the system (3).*

A "string" of consecutive terms $\{x_{2s-1}, x_{2s+1}, ..., x_{2m+1}\}$ *(resp.* $\{y_{2s-1}, ..., y_{2m+1}\}$*),* $s \geq 1$*,* $m \leq \infty$ *is said to be a positive sub-semicycle associated with* $\{x_{2n-1}\}$ *(resp.* $\{y_{2n-1}\}$*) if* $x_i \geqslant \overline{x}$ *(resp.* $y_i \geqslant \overline{y}$*),* $i \in \{2s+1, 2s+1, ..., 2m+1\}$*,* $x_{2s-3} < \overline{x}$ *(resp.* $y_{2s-3} < \overline{y}$*), and* $x_{2m+3} < \overline{x}$ *(resp.* $y_{2m+3} < \overline{y}$*).*

A "string" of consecutive terms $\{x_{2s-1}, x_{2s+1}, ..., x_{2m+1}\}$ *(resp.* $\{y_{2s-1}, ..., y_{2m+1}\}$*),* $s \geq 1$*,* $m \leq \infty$ *is said to be a negative sub-semicycle associated with* $\{x_{2n-1}\}$ *(resp.* $\{y_{2n-1}\}$*) if* $x_i < \overline{x}$ *(resp.* $y_i < \overline{y}$*),* $i \in \{2s-1, 2s+1, ..., 2m+1\}$*,* $x_{2s-3} \geqslant \overline{x}$ *(resp.* $y_{2s-3} \geqslant \overline{y}$*), and* $x_{2m+3} \geqslant \overline{x}$ *(resp.* $y_{2m+3} \geqslant \overline{y}$*).*

A "string" of consecutive terms $\{(x_{2s-1}, y_{2s-1}), (x_{2s+1}, y_{2s+1}), ..., (x_{2m+1}, y_{2m+1})\}$ *is said to be a positive (resp. negative) sub-semicycle if* $\{x_{2s-1}, x_{2s+1}, ..., x_{2m+1}\}$*,* $\{y_{2s-1}, ..., y_{2m+1}\}$ *are positive (resp. negative) sub-semicycles. Finally a "string" of consecutive terms* $\{(x_{2s-1}, y_{2s-1}), (x_{2s+1}, y_{2s+1}), \dots, (x_{2m+1}, y_{2m+1})\}$ *is said to be a sub-semicycle positive (resp. negative) with respect to* x_{2n-1} *and negative (resp. positive) with respect to* y_{2n-1} *if* $\{x_{2s-1}, x_{2s+1}, ..., x_{2m+1}\}$ *is a positive (resp. negative) sub-semicycle and* $\{y_{2s-1}, \dots, y_{2m+1}\}$ *is a negative (resp. positive) sub-semicycle.*

2. Some auxiliary results

In this section, we give the following lemmas which show us the behavior of semicycles of positive solutions of system (2). Proofs of Lemmas 1 and 2 are clear from (2), So, they will be omitted.

Lemma 1. *Assume that* $\{(x_n, y_n)\}_{n=-1}^{\infty}$ *is a solution of the system (2) and consider the cases:*

$Case\ a : x_0 = x_{-1} = 1,$ $Case\ b : y_0 = y_{-1} = 1,$

$Case\ c : x_0 = y_0 = 1,$ $Case\ d : x_{-1} = y_{-1} = 1.$

If one of the above cases occurs, then every positive solution of system (2) equal to $(1, 1)$*.*

Lemma 2. *Assume that* $\{(x_n, y_n)\}_{n=-1}^{\infty}$ *is a positive solution of the system (2) which is not eventually equal to* $(1, 1)$*. Then the following statements are true:*

(i) $(x_{n+1} - x_{n-1})(1 - x_{n-1}) > 0$ *and* $(y_{n+1} - y_{n-1})(1 - y_{n-1}) > 0$ *for all* $n \geq 0$*,*

(ii) $(x_{n+1} - 1)(x_{n-1} - 1)(1 - y_n) > 0$ *and* $(y_{n+1} - 1)(y_{n-1} - 1)(1 - x_n) > 0$ *for all* $n \geq 0$*.*

Proof of Lemma 3 is clear from Lemma 2 $(i) - (ii)$ and proofs of Lemmas 4, 5 and 6 can easily be obtained from Lemma 2 (ii). So, they will be omitted.

Lemma 3. *Assume that* $\{(x_n, y_n)\}_{n=-1}^{\infty}$ *is a solution of system (2) and suppose that the case, Case1:* $x_k, y_k < 1$ *(for* $k = -1, 0$*), holds. Then* (x_{2n-1}, y_{2n-1}) *and* (x_{2n}, y_{2n}) *are negative sub-semicycles of system(2) with an infinite number of terms and they monotonically tend to the positive equilibrium* $(\overline{x}, \overline{y}) = (1, 1)$*.*

Lemma 4. *Assume that* $\{(x_n, y_n)\}_{n=-1}^{\infty}$ *is a solution of system (2), and consider the cases:*

Case 2 : $x_{-1}, x_0 > 1$ *and* $y_{-1}, y_0 < 1$;
Case 3 : $x_{-1}, x_0 < 1$ *and* $y_{-1}, y_0 > 1$;
Case 4 : $x_{-1}, y_{-1}, y_0 > 1$ *and* $x_0 < 1$;
Case 5 : $x_0, y_{-1}, y_0 > 1$ *and* $x_{-1} < 1$;
Case 6 : $x_{-1}, x_0, y_{-1} > 1$ *and* $y_0 < 1$;
Case 7 : $x_{-1}, x_0, y_0 > 1$ *and* $y_{-1} < 1$;
Case 8 : $x_{-1}, y_{-1} > 1$ *and* $x_0, y_0 < 1$;
Case 9 : $x_{-1}, y_{-1} < 1$ *and* $x_0, y_0 > 1$;
Case 10 : $x_{-1}, x_0, y_{-1}, y_0 > 1$.

If one of the above cases occurs, then

(*i*) *Every positive sub-semicycle associated with* $\{x_{2n-1}\}$ *and* $\{x_{2n}\}$ (*resp.* $\{y_{2n-1}\}$ *and* $\{y_{2n}\}$) *of system (2) consists of two terms;*

(*ii*) *Every negative sub-semicycle associated with* $\{x_{2n-1}\}$ *and* $\{x_{2n}\}$ (*resp.* $\{y_{2n-1}\}$ *and* $\{y_{2n}\}$) *of system (2) consists of one term;*

(*iii*) *Every positive sub-semicycle of length two is followed by a negative sub-semicycle of length one;*

(*iv*) *Every negative sub-semicycle of length one is followed by a positive sub-semicycle of length two.*

Lemma 5. *Assume that* $\{(x_n, y_n)\}_{n=-1}^{\infty}$ *be a solution of system(2) and consider the cases:*

Case 11 : $x_{-1}, y_{-1}, y_0 < 1$ *and* $x_0 > 1$;
Case 12 : $x_{-1}, x_0, y_0 < 1$ *and* $y_{-1} > 1$;
Case 13 : $x_{-1}, y_0 < 1$ *and* $x_0, y_{-1} > 1$.

If one of the above cases occurs, then

(*i*) $\{x_{2n-1}\}$ *and* $\{y_{2n}\}$ *are negative sub-semicycles of system (2) with an infinite number of terms* (*monotonically tend to the positive equilibrium* $(\overline{x}, \overline{y}) = (1, 1)$)*;*

(*ii*) *Every positive sub-semicycle associated with* $\{x_{2n}\}$ *and* $\{y_{2n-1}\}$ *of system (2) consists of two terms;*

(*iii*) *Every negative sub-semicycle associated with* $\{x_{2n}\}$ *and* $\{y_{2n-1}\}$ *of system (2) consists of one term;*

(*iv*) *Every positive sub-semicycle of length two is followed by a negative sub-semicycle of length one;*

(*v*) *Every negative sub-semicycle of length one is followed by a positive sub-semicycle of length two.*

Lemma 6. *Assume that* $\{(x_n, y_n)\}_{n=-1}^{\infty}$ *be a solution of system (2) and consider the cases:*

Case 14 : $x_{-1}, x_0, y_{-1} < 1$ *and* $y_0 > 1$;
Case 15 : $x_0, y_{-1}, y_0 < 1$ *and* $x_{-1} > 1$;

Case 16 : $x_0, y_{-1} < 1$ *and* $x_{-1}, y_0 > 1$.

If one of the above cases occurs, then

(i) $\{x_{2n}\}$ *and* $\{y_{2n-1}\}$ *are negative sub-semicycles of system(2) with an infinite number of terms (monotonically tend to the positive equilibrium* $(\overline{x}, \overline{y}) = (1, 1)$*);*

(ii) *Every positive sub-semicycle associated with* $\{x_{2n-1}\}$ *and* $\{y_{2n}\}$ *of system (2) consists of two terms;*

(iii) *Every negative sub-semicycle associated with* $\{x_{2n-1}\}$ *and* $\{y_{2n}\}$ *of system (2) consists of one term;*

(iv) *Every positive sub-semicycle of length two is followed by a negative sub-semicycle of length one;*

(v) *Every negative sub-semicycle of length one is followed by a positive sub-semicycle of length two.*

3. Main result

Theorem 2. *The positive equilibrium point* $(\overline{x}, \overline{y}) = (1, 1)$ *of the system (2) is globally asymptotically stable.*

Proof. We must show that the positive equilibrium point $(\overline{x}, \overline{y}) = (1, 1)$ of the system (2) is both locally asymptotically stable and $(x_n, y_n) \to (\overline{x}, \overline{y})$ as $n \to \infty$ (or equivalently $(x_{2n-1}, y_{2n-1}) \to (\overline{x}, \overline{y})$ and $(x_{2n}, y_{2n}) \to (\overline{x}, \overline{y})$ as $n \to \infty$). The characteristic equation of the system (2) about the positive equilibrium point $(\overline{x}, \overline{y}) = (1, 1)$ is

$$\lambda^2 - 0.\lambda + 0 = 0$$

and so it is clear from Theorem 1 that positive equilibrium point $(\overline{x}, \overline{y}) = (1, 1)$ of the system (2) is locally asymptotically stable. It remains to verify that every positive solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ of the system (2) converges to $(\overline{x}, \overline{y}) = (1, 1)$ as $n \to \infty$. Namely, we want to prove

$$\begin{aligned} \lim_{n\to\infty} x_{2n} &= \lim_{n\to\infty} x_{2n-1} = \overline{x} = 1 \\ \lim_{n\to\infty} y_{2n} &= \lim_{n\to\infty} y_{2n-1} = \overline{y} = 1 \end{aligned} \tag{5}$$

If the solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ of equation (2) is nonoscillatory about the positive equilibrium point $(\overline{x}, \overline{y}) = (1, 1)$ of the system (2), then according to Lemmas 1 and 3 respectively, we know that the solution is either eventually equal to $(1, 1)$ or an eventually negative one which has an infinite number of terms and monotonically tends the positive equilibrium point $(\overline{x}, \overline{y}) = (1, 1)$ of the system (2) and so equation (5) holds. Therefore, it suffices to prove that equation (5) holds for strictly oscillatory solutions. Now let

$\{(x_n, y_n)\}_{n=-1}^{\infty}$ be strictly oscillatory about the positive equilibrium point $(\overline{x}, \overline{y}) = (1, 1)$ of the system (2). By virtue of Lemmas 2 (ii) and 4 one can see that every positive sub-semicycle associated with $\{x_{2n-1}\}$ (resp. $\{x_{2n}\}$, $\{y_{2n-1}\}$, $\{y_{2n}\}$) of this solution has two terms and every negative sub-semicycle associated with $\{x_{2n-1}\}$ (resp. $\{x_{2n}\}$, $\{y_{2n-1}\}$, $\{y_{2n}\}$) except perhaps for the first has exactly one term. Every positive sub-semicycle of length two is followed by a negative sub-semicycle of length one.

We consider the sub-semicycles associated with $\{x_{2n}\}$ and $\{y_{2n}\}$.

For the convenience of statement, without loss of generality, we use the following notation. We denote by x_{2p}, x_{2p+2} (resp. y_{2p}, y_{2p+2}) the terms of a positive sub-semicycle of length two, followed by x_{2p+4} (resp. y_{2p+4}) which is the term of a negative sub-semicycle of length one. Afterwards, there is the positive sub-semicycles x_{2p+6}, x_{2p+8} (resp. y_{2p+6}, y_{2p+8}) in turn followed by the negative sub-semicycles so on.

Therefore, we have the following sequences consisting of positive and negative sub-semicycles (for $n = 0, 1, ...$): $\{x_{2p+6n}, x_{2p+6n+2}\}_{n=0}^{\infty}$, $\{x_{p+6n+4}\}_{n=0}^{\infty}$ and $\{y_{2p+6n}, y_{2p+6n+2}\}_{n=0}^{\infty}$, $\{y_{2p+6n+4}\}_{n=0}^{\infty}$.

We have the following assertions:

(*i*) $x_{2p+6n} > x_{2p+6n+2}$ and $y_{2p+6n} > y_{2p+6n+2}$;

(*ii*) $x_{2p+6n+2}x_{2p+6n+4} > 1$ and $y_{2p+6n+2}y_{2p+6n+4} > 1$;

(*iii*) $x_{2p+6n+4}x_{2p+6n+6} < 1$ and $y_{2p+6n+4}y_{2p+6n+6} < 1$.

Combining the above inequalities, we derive

$$\begin{aligned} &\frac{1}{x_{2p+6n}} < \frac{1}{x_{2p+6n+2}} < x_{2p+6n+4} < \frac{1}{x_{2p+6n+6}} \\ &\frac{1}{y_{2p+6n}} < \frac{1}{y_{2p+6n+2}} < y_{2p+6n+4} < \frac{1}{y_{2p+6n+6}} \end{aligned} \tag{6}$$

From equation (6), one can see that $\{x_{2p+6n}\}_{n=0}^{\infty}$ and $\{y_{2p+6n}\}_{n=0}^{\infty}$ are decreasing with upper bound 1. So the limits

$$\lim_{n\to\infty} x_{2p+6n} = L_1 \quad \text{and} \quad \lim_{n\to\infty} y_{2p+6n} = L_2$$

exist and are finite. Accordingly, in view of equation (6), we obtain

$$\lim_{n\to\infty} x_{2p+6n+2} = \lim_{n\to\infty} x_{2p+6n+6} = L_1 \quad \text{and} \quad \lim_{n\to\infty} x_{2p+6n+4} = 1/L_1$$

and

$$\lim_{n\to\infty} y_{2p+6n+2} = \lim_{n\to\infty} y_{2p+6n+6} = L_2 \quad \text{and} \quad \lim_{n\to\infty} y_{2p+6n+4} = 1/L_2.$$

Now, we consider the sub-semicycles associated with $\{x_{2n-1}\}$ and $\{y_{2n-1}\}$.

Similarly, for the convenience of statement, without loss of generality, we use the following notation. We denote by x_{2p+1}, x_{2p+3} (resp. y_{2p+1},

y_{2p+3}) the terms of a positive sub-semicycle of length two, followed by x_{2p+5} (resp. y_{2p+5}) which is the term of a negative sub-semicycle of length one. Afterwards, there are the positive sub-semicycles x_{2p+7}, x_{2p+9} (resp. y_{2p+7}, y_{2p+9}) in turn followed by the negative sub-semicycles so on.

Therefore, we have the following sequences consisting of positive and negative sub-semicycles (for $n = 0, 1, ...$): $\{x_{2p+6n+1}, x_{2p+6n+3}\}_{n=0}^{\infty}$, $\{x_{p+6n+5}\}_{n=0}^{\infty}$ and $\{y_{2p+6n+1}, y_{2p+6n+3}\}_{n=0}^{\infty}$, $\{y_{2p+6n+5}\}_{n=0}^{\infty}$.

We have the following assertions:

(i) $x_{2p+6n+1} > x_{2p+6n+3}$ and $y_{2p+6n+1} > y_{2p+6n+3}$;

(ii) $x_{2p+6n+3}x_{2p+6n+5} > 1$ and $y_{2p+6n+3}y_{2p+6n+5} > 1$;

(iii) $x_{2p+6n+5}x_{2p+6n+7} < 1$ and $y_{2p+6n+5}y_{2p+6n+7} < 1$.

Combining the above inequalities, we derive

$$\begin{aligned} &\frac{1}{x_{2p+6n+1}} < \frac{1}{x_{2p+6n+3}} < x_{2p+6n+5} < \frac{1}{x_{2p+6n+7}} \\ &\frac{1}{y_{2p+6n+1}} < \frac{1}{y_{2p+6n+3}} < y_{2p+6n+5} < \frac{1}{y_{2p+6n+7}} \end{aligned} \tag{7}$$

From equation (7), one can see that $\{x_{2p+6n+1}\}_{n=0}^{\infty}$ and $\{y_{2p+6n+1}\}_{n=0}^{\infty}$ are decreasing with upper bound 1. So the limits

$$\lim_{n\to\infty} x_{2p+6n+1} = L_3 \quad \text{and} \quad \lim_{n\to\infty} y_{2p+6n+1} = L_4$$

exist and are finite. Accordingly, in view of equation (7), we obtain

$$\lim_{n\to\infty} x_{2p+6n+3} = \lim_{n\to\infty} x_{2p+6n+7} = L_3 \quad \text{and} \quad \lim_{n\to\infty} x_{2p+6n+5} = 1/L_3$$

and

$$\lim_{n\to\infty} y_{2p+6n+3} = \lim_{n\to\infty} y_{2p+6n+7} = L_4 \quad \text{and} \quad \lim_{n\to\infty} y_{2p+6n+5} = 1/L_4.$$

It suffices to verify that

$$L_1 = L_2 = L_3 = L_4 = 1.$$

To this end, note that

$$x_{2p+6n+6} = \frac{y_{2p+6n+5} + x_{2p+6n+4}}{y_{2p+6n+5}x_{2p+6n+4} + 1} \quad \text{and} \quad y_{2p+6n+6} = \frac{x_{2p+6n+5} + y_{2p+6n+4}}{x_{2p+6n+5}y_{2p+6n+4} + 1}$$

Take the limits on both sides of the above equality and obtain

$$L_1 = \frac{1/L_4 + 1/L_1}{1/L_4 \cdot 1/L_1 + 1} \quad \text{and} \quad L_2 = \frac{1/L_3 + 1/L_2}{1/L_3 \cdot 1/L_2 + 1}$$

which imply that $L_1 = L_2 = 1$. Similarly, one can see that $L_3 = L_4 = 1$.

Moreover, by virtue of Lemmas 2 (ii) and 5 (resp. 6) one can see that equation (5) holds. Therefore, the proof is complete. ■

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