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# ON A LINEAR DIFFERENCE EQUATION WITH SEVERAL INFINITE LAGS*

Abstract. This paper deals with asymptotic properties of the solutions of a variable order linear difference equation. As the main result, we derive the effective asymptotic estimate valid for all solutions of this equation. Moreover, we are going to discuss some consequences of this theoretical result, especially with respect to the numerical analysis of the multi-pantograph differential equation.

Key words: delay difference equation, multi-pantograph equation, stability, asymptotic behaviour.

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## 1. Introduction and preliminaries

We consider the linear difference equation

$$y(n+1) = ay(n) + \sum_{j=1}^{m} b_j y(\lfloor \lambda_j n \rfloor), \quad n = 0, 1, \dots, \tag{1}$$

where $-1 < a < 1$, $b_j \neq 0$ and $0 < \lambda_j < 1$, $j = 1, \dots, m$ are real scalars and the symbol $\lfloor \ \rfloor$ means an integer part. Our principal interest in this paper is the discussion on the asymptotic behaviour of the solutions $y(n)$ of (1) as $n \to \infty$.

The problem of asymptotic behaviour of linear difference equations has a long tradition. Among most frequent topics belong especially asymptotic expansions for Poincaré difference equations (see, e.g. [1], [8] and [9]) or the study of asymptotic properties for delay difference equations (see, e.g. [3], [7], [10] and [16]). However, contrary to (1), the difference equations considered in the frame of these investigations are usually of a fixed constant order. This is not the case of the equation (1) which is a linear difference

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equation of a variable order which becomes even unbounded as $n \to \infty$. This makes its asymptotic (and more generally qualitative) analysis more difficult.

The equation (1) can be viewed as the discrete analogue of the multi-pantograph equation

$$u'(t) = pu(t) + \sum_{j=1}^{m} q_j u(\lambda_j t), \quad t \geq 0\,. \tag{2}$$

More precisely, if we employ a suitable discretization of the equation (2) with a constant stepsize, then we arrive either directly at the form (1) or at the form very close to (1). To obtain the discretization of (2) of a fixed order, the non-constant stepsize method has to be used (see [14]). We note that a general reference for the investigation of various discretizations of the pantograph equations provide books [2] and [12]. Among numerous papers on this topic we can mention, e.g. [4], [11], [15] or [6]. In particular, it follows from the analysis performed in these papers that some problems remain open especially in the framework of the constant stepsize discretizations, i.e. discretizations leading to the difference equation of the form (1) or its modifications.

As we have declared above, we wish to give the asymptotic description of the solutions of the difference equation (1) with $m$ infinite lags. In Section 2, we present the main result of this paper, namely the asymptotic estimate of the solutions of (1). This result shows that any solution of (1) can be estimated via a power sequence whose exponent can be determined as the (unique) real root of an auxiliary nonlinear equation. Some consequences of the main result, especially with respect to the numerical analysis of the differential equation (2), are the topic of Section 3.

## 2. The main result

We start with some preliminary considerations which enable us to formulate the asymptotic description of the solutions of (1). For its given real coefficients $a$, $b_j$ and $\lambda_j$, where $|a| < 1$, $b_j \neq 0$ and $0 < \lambda_j < 1$, we introduce the function

$$F(\gamma) := \sum_{j=1}^{m} |b_j| (\lambda_j)^{\gamma} + |a| - 1$$

of a real variable $\gamma$. Since $F$ is decreasing and

$$\lim_{\gamma \to -\infty} F(\gamma) = \infty, \qquad \lim_{\gamma \to \infty} F(\gamma) = |a| - 1 < 0\,,$$

there exists a unique real root $\gamma^*$ of the equation

$$F(\gamma) = 0\,.$$

Moreover, $\gamma^* \geq 0$ if and only if $\sum_{j=1}^{m} |b_j| + |a| \geq 1$. Further, we put

$$\lambda := \max(\lambda_1, \dots, \lambda_m)$$

and for $n = 1, 2, \dots$ we define

$$\varphi(n) = \begin{cases} n^{\gamma^*}, & \text{if } \sum_{j=1}^{m} |b_j| + |a| \geq 1, \\ \left(n + \frac{1}{1-\lambda}\right)^{\gamma^*}, & \text{if } \sum_{j=1}^{m} |b_j| + |a| < 1\,. \end{cases} \tag{3}$$

Using this notation we can formulate and prove the main result of this paper.

**Theorem 1.** *Consider the equation (1), where* $|a| < 1$, $b_j \neq 0$ *and* $0 < \lambda_j < 1$, $j = 1, \dots, m$. *Then*

$$y(n) = O\left(n^{\gamma^*}\right) \qquad \text{as} \quad n \to \infty \tag{4}$$

*for any solution* $y(n)$ *of (1).*

**Proof.** First we note that the upper bound sequence $(n^{\gamma *})$ is asymptotically equivalent to the sequence $(\varphi(n))$. Hence, we can replace (4) by the property $y(n) = O(\varphi(n))$ and that is what we are going to prove.

Put $n_0 = 0$ and $n_i = \lfloor \lambda^{-i} \rfloor$, $i = 1, 2, \dots$. Assuming $a \neq 0$ we divide (1) by $a^{n+1}$ to get

$$a^{-n-1} y(n+1) = a^{-n} y(n) + a^{-n-1} \sum_{j=1}^{m} b_j y(\lfloor \lambda_j n \rfloor)\,.$$

Since $a^{-n-1} y(n+1) - a^{-n} y(n) = \Delta(a^{-n} y(n))$, we can write

$$\Delta(a^{-n} y(n)) = a^{-n-1} \sum_{j=1}^{m} b_j y(\lfloor \lambda_j n \rfloor)\,. \tag{5}$$

Now consider arbitrary $n \in \mathbb{Z}^+$, $n \geq \lambda^{-1}$ and let $k = \lfloor -\log_\lambda n \rfloor$. Then $n_k \leq n < n_{k+1}$. Further, we rewrite (5) with $n$ replaced by a different running index and sum from $n_k$ to $n - 1$. Then using the convention $\sum_{l=n_k}^{n_k - 1} c_l = 0$ we have

$$y(n) = a^{n-n_k} y(n_k) + \sum_{l=n_k}^{n-1} a^{n-l-1} \sum_{j=1}^{m} b_j y(\lfloor \lambda_j l \rfloor)\,.$$

If we set $z(n) = y(n)/\varphi(n)$, then $z(n)$ satisfies the relation

$$z(n) = \frac{a^{n-n_k}\varphi(n_k)}{\varphi(n)} z(n_k) + \sum_{l=n_k}^{n-1} \frac{a^{n-l-1}}{\varphi(n)} \sum_{j=1}^{m} b_j \varphi(\lfloor \lambda_j l \rfloor) z(\lfloor \lambda_j l \rfloor)$$

and can be estimated as

$$|z(n)| \le M_k \frac{|a|^{n-n_k}\varphi(n_k)}{\varphi(n)} + M_k \sum_{l=n_k}^{n-1} \frac{|a|^{n-l-1}}{\varphi(n)} \sum_{j=1}^{m} |b_j| \varphi(\lfloor \lambda_j l \rfloor)\,,$$

where we denoted $M_k := \max(|z(s)|,\, n_0 \le s \le n_k)$. Further, it holds

$$\sum_{j=1}^{m} |b_j| \varphi(\lfloor \lambda_j l \rfloor) \le (1 - |a|)\varphi(l)\,. \tag{6}$$

To verify (6) we first assume that $\sum_{j=1}^{m} |b_j| + |a| \ge 1$, i.e. $\gamma^* \ge 0$. Then

$$\begin{aligned} \sum_{j=1}^{m} |b_j| \varphi(\lfloor \lambda_j l \rfloor) &= \sum_{j=1}^{m} |b_j| (\lfloor \lambda_j l \rfloor)^{\gamma^*} \le \sum_{j=1}^{m} |b_j| (\lambda_j)^{\gamma^*} l^{\gamma^*} \\ &= \varphi(l) \sum_{j=1}^{m} |b_j| (\lambda_j)^{\gamma^*} = (1 - |a|)\varphi(l)\,. \end{aligned}$$

If $\sum_{j=1}^{m} |b_j| + |a| < 1$, then $\gamma^* < 0$ and

$$\begin{aligned} \sum_{j=1}^{m} |b_j| \varphi(\lfloor \lambda_j l \rfloor) &= \sum_{j=1}^{m} |b_j| (\lfloor \lambda_j l \rfloor + \frac{1}{1-\lambda})^{\gamma^*} \le \sum_{j=1}^{m} |b_j| (\lambda_j l - 1 + \frac{1}{1-\lambda})^{\gamma^*} \\ &= \sum_{j=1}^{m} |b_j| (\lambda_j l + \frac{\lambda_j}{1-\lambda})^{\gamma^*} \\ &= \sum_{j=1}^{m} |b_j| \lambda_j^{\gamma^*} \big(l + \frac{1}{1-\lambda}\big)^{\gamma^*} = (1 - |a|)\varphi(l)\,. \end{aligned}$$

Using (6) we can write

$$\begin{aligned} |z(n)| &\le M_k \left( \frac{|a|^{n-n_k}\varphi(n_k)}{\varphi(n)} + \sum_{l=n_k}^{n-1} \frac{|a|^{n-l-1}}{\varphi(n)} (1 - |a|)\varphi(l) \right) \\ &= M_k \left( \frac{|a|^{n-n_k}\varphi(n_k)}{\varphi(n)} + \sum_{l=n_k}^{n-1} \frac{\varphi(l)}{\varphi(n)} \Delta |a|^{n-l} \right)\,. \end{aligned} \tag{7}$$

The sum on the right-hand side of (7) can be processed via the summation by parts:

$$\sum_{l=n_k}^{n-1} \frac{\varphi(l)}{\varphi(n)} \Delta |a|^{n-l} = 1 - \frac{|a|^{n-n_k}\varphi(n_k)}{\varphi(n)} - \sum_{l=n_k}^{n-1} |a|^{n-l-1} \frac{\Delta\varphi(l)}{\varphi(n)} .$$

Substituting back into (7) we have

$$|z(n)| \le M_k \left( 1 - \sum_{l=n_k}^{n-1} |a|^{n-l-1} \frac{\Delta\varphi(l)}{\varphi(n)} \right) .$$

Now, if $\sum_{j=1}^{m} |b_j| + |a| \ge 1$, then $\Delta\varphi(l) \ge 0$, hence

$$|z(n)| \le M_k . \tag{8}$$

If $\sum_{j=1}^{m} |b_j| + |a| < 1$, then $\Delta\varphi(l) < 0$ and, moreover, $\Delta\varphi(l) \le \Delta\varphi(l+1)$ for all $l$. On this account we get

$$|z(n)| \le M_k \left( 1 - \frac{\Delta\varphi(n_k)}{\varphi(n)} \sum_{l=n_k}^{n-1} |a|^{n-l-1} \right) \le M_k \left( 1 - \frac{\Delta\varphi(n_k)}{(1-|a|)\varphi(n_{k+1})} \right) .$$

By the mean value theorem,

$$\begin{aligned} \frac{-\Delta\varphi(n_k)}{\varphi(n_{k+1})} &= \frac{\left(\lfloor\lambda^{-k}\rfloor + \frac{1}{1-\lambda}\right)^{\gamma^*} - \left(\lfloor\lambda^{-k}\rfloor + \frac{1}{1-\lambda} + 1\right)^{\gamma^*}}{\left(\lfloor\lambda^{-k-1}\rfloor + \frac{1}{1-\lambda}\right)^{\gamma^*}} \\ &\le -\gamma^* \frac{\left(\lfloor\lambda^{-k}\rfloor + \frac{1}{1-\lambda}\right)^{\gamma^*-1}}{\left(\lfloor\lambda^{-k-1}\rfloor + \frac{1}{1-\lambda}\right)^{\gamma^*}} \le -\gamma^* \frac{\left(\lambda^{-k} - 1 + \frac{1}{1-\lambda}\right)^{\gamma^*-1}}{\left(\lambda^{-k-1} + \frac{1}{1-\lambda}\right)^{\gamma^*}} = O(\lambda^k) , \end{aligned}$$

hence

$$|z(n)| \le M_k(1 + O(\lambda^k)) . \tag{9}$$

Now considering (8) and (9) we can conclude that

$$M_{k+1} \le M_k(1 + O(\lambda^k)) ,$$

hence $z(n)$ is bounded. Then $y(n) = O(\varphi(n))$ and the property (4) is proved.

Recall that previous procedures have been performed under the assumption $a \ne 0$. The proof in the case $a = 0$ is much more simple. Utilizing our previous notation we have

$$z(n) = \frac{1}{\varphi(n)} \sum_{j=1}^{m} b_j \varphi(\lfloor\lambda_j(n-1)\rfloor) z(\lfloor\lambda_j(n-1)\rfloor) ,$$

i.e.

$$|z(n)| \le \frac{M_k}{\varphi(n)} \sum_{j=1}^{m} |b_j| \varphi(\lfloor \lambda_j (n-1) \rfloor) \le M_k \frac{\varphi(n-1)}{\varphi(n)} \,,$$

where we utilized the inequality (6). If $\sum_{j=1}^m |b_j| + |a| \ge 1$, then $\varphi(n-1) \le \varphi(n)$ and we get (8). If $\sum_{j=1}^m |b_j| + |a| < 1$, then

$$\frac{\varphi(n-1)}{\varphi(n)} = \frac{\left(n-1+\frac{1}{1-\lambda}\right)^{\gamma^*}}{\left(n+\frac{1}{1-\lambda}\right)^{\gamma^*}} = 1 + O(\lambda^k)$$

which implies (9). The derivation of the property (4) is now quite analogous as in the case $a \ne 0$. ■

**Remark 1.** The technique employed in the proof of Theorem 1 admits some extensions of this asymptotic result. E.g., we can assume that coefficients $a$, $b_j$ are complex scalars or even that they are depending on $n$ (in such a case some additional restrictions on $a = a(n)$ and $b_j = b_j(n)$ turn out to be necessary). Similarly, we can consider the equation (1) in a slightly modified form such as

$$y(n+1) = ay(n) + \sum_{j=1}^{m} \sum_{k=1}^{r} b_{jk} y(\lfloor \lambda_j n \rfloor + k), \quad n = 0, 1, \dots \tag{10}$$

which appears as a result of various discretizations of the multi-pantograph equation (2). The reformulation of the asymptotic property (4) for the equation (10) requires only some simple technical modifications. In particular, if $y(n)$ is a solution of (10), where $|a| < 1$, $b_{jk} \ne 0$ for some $k = 1, \dots, r$ and $0 < \lambda_j < 1$, $j = 1, \dots, m$, then the asymptotic estimate (4) holds, where $\gamma^*$ is a (unique) real root of

$$\sum_{j=1}^{m} \sum_{k=1}^{r} |b_{jk}| (\lambda_j)^{\gamma} + |a| - 1 = 0 \,.$$

## 3. Some consequences

In this section, we give the application of Theorem 1 to the stability and asymptotic theory of numerical methods for the differential equation (2). We start with the following condition on the asymptotic stability of (2) which is taken from [17].

**Lemma 1.** *Assume that the coefficients $p$ and $q_j$, $j = 1, \dots, m$ in (2) satisfy*

$$\sum_{j=1}^{m} |q_j| < -p \,. \tag{11}$$

*Then any solution $u(t)$ of (2) satisfies*

$$\lim_{t\to\infty} u(t) = 0 \,.$$

Now we investigate the asymptotic stability condition for a numerical discretization of the equation (2). As the illustration, we consider the backward Euler method utilizing a piecewise linear interpolation of the delayed term (for more details and some necessary calculations we refer, e.g. to [12] or [13]). Applying this method to (2) we arrive at

$$y(n+1) = ay(n) + \sum_{j=1}^{m} b_{j1} y(\lfloor \lambda_j (n+1) \rfloor) + b_{j2} y(\lfloor \lambda_j (n+1) \rfloor + 1), \tag{12}$$

$n = 0, 1, \dots,$ where

$$a = \frac{1}{1-ph}, \qquad b_{j1} = \frac{q_j h}{1-ph} - b_{j2}, \tag{13}$$
$$b_{j2} = \frac{q_j h}{1-ph} (\lambda_j (n+1) - \lfloor \lambda_j (n+1) \rfloor) \,,$$

$j = 1, \dots, m$. Here $y(n)$ means an approximation of $u(t_n)$, where $t_n = nh$ and $h > 0$ is the stepsize. The difference equation (12) is of the type (10) with the coefficients $b_{j1}$, $b_{j2}$ formally slightly different from those in(10) (in particular, $b_{j1}$, $b_{j2}$ are now depending on $n$). However, because of the property

$$b_{j1}(n) + b_{j2}(n) = \frac{q_j h}{1-ph}, \quad j = 1, \dots, m$$

it requires only a trivial modification to extend the corresponding assertion concerning the asymptotics of the difference equation (10) (mentioned in Remark 1) also to this case. In particular, if

$$\sum_{j=1}^{m} |b_{j1}(n)| + |b_{j2}(n)| + |a| - 1 < 0 \,, \tag{14}$$

then any solution $y(n)$ of (12) tends to zero as $n \to \infty$. Substituting (13) into (14) one gets

$$\sum_{j=1}^{m} \frac{|q_j| h}{|1-ph|} + \frac{1}{|1-ph|} - 1 < 0$$

which is equivalent to (11). In other words, the asymptotic stability condition for the numerical solution of (2) is equivalent to the asymptotic stability condition for the exact solution. These observations are closely related to the notion of asymptotic stability of a numerical method.

**Definition 1.** *A numerical method for solving the multi-pantograph equations is called asymptotic stable if and only if when it is applied to the test equation (2) with its coefficients satisfying (11), its approximate solution* $y(n)$ *satisfies*

$$\lim_{n\to\infty} y(n) = 0\,,$$

*where* $y(n)$ *is an approximation of* $u(t_n)$*,* $t_n = nh$ *and* $h > 0$ *is arbitrary stepsize.*

An immediate consequence of our previous ideas is the following

**Corollary 1.** *The backward Euler method for the multi-pantograph equation is asymptotic stable.*

Note that Corollary 1 generalizes the result on the asymptotic stability of the backward Euler method for the pantograph equation with one proportional delay (see Corollary 9.2.1 of [12]). Furthermore, we can deduce a stronger result from Theorem 1. Considering the multi-pantograph equation (2) and assuming that $p < 0$, $q_j \neq 0$ and $0 < \lambda_j < 1$, it holds

$$u(t) = O\big(t^{\alpha^*}\big) \quad \text{as} \quad t \to \infty \tag{15}$$

for any solution $u(t)$ of (2), where $\alpha^*$ is a unique real root of the equation

$$\sum_{j=1}^{m} |q_j|(\lambda_j)^\gamma + p = 0 \tag{16}$$

(see [5]). If we take the discretization (12), then Theorem 1 and Remark 1 imply that

$$y(n) = O\big(n^{\gamma^*}\big) \quad \text{as} \quad n \to \infty \tag{17}$$

for any solution $y(n)$ of (12), where $\gamma^*$ is a root of

$$\sum_{j=1}^{m} (|b_{j1}(n)| + |b_{j2}(n)|)(\lambda_j)^\gamma + |a| - 1 = 0\,. \tag{18}$$

Now it is simple to verify that substituting (13) into (18) we can observe the equivalence between (16) and (18). Consequently, the asymptotic relations (15) and (17) present the same estimate for both exact and numerical solution of the multi-pantograph equation (2).

Finally we note that the extension of the previous results and ideas to other discretizations of the multi-pantograph equation (e.g., to the linear $\theta$-methods) is not obvious and some related problems in this area remain open.

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