F A S C I C U L I M A T H E M A T I C I

Nr 44 2010

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THE DELTA-NABLA CALCULUS OF VARIATIONS

Abstract. The discrete-time, the quantum, and the continuous calculus of variations have been recently unified and extended. Two approaches are followed in the literature: one dealing with minimization of delta integrals; the other dealing with minimization of nabla integrals. Here we propose a more general approach to the calculus of variations on time scales that allows to obtain both delta and nabla results as particular cases.

Key words: calculus of variations, Euler-Lagrange equations, time scales.

AMS Mathematics Subject Classification: 49K05, 39A12, 34N05.

1. Introduction

The calculus of variations on time scales was introduced by M. Bohner using the delta derivative and integral [7]: to extremize a functional of the form

$$\mathcal{J}_\Delta(y) = \int_a^b L\left(t, y^\sigma(t), y^\Delta(t)\right) \Delta t \,. \tag{1}$$

Motivated by applications in economics [2, 5], a different formulation for the problems of the calculus of variations on time scales has been considered, which involve a functional with a nabla derivative and a nabla integral [1, 4, 11]:

$$\mathcal{J}_\nabla(y) = \int_a^b L\left(t, y^\rho(t), y^\nabla(t)\right) \nabla t \,. \tag{2}$$

Formulations (1) and (2) are consistent in the sense that results obtained *via* delta and nabla approaches are similar among them and similar to the classical results of the calculus of variations. An example of this is given by the time scale versions of the Euler-Lagrange equations: if $y \in C^2_{\mathrm{rd}}$ is an extremizer of (1), then y satisfies the delta-differential equation

$$\frac{\Delta}{\Delta t}\partial_3 L\left(t, y^\sigma(t), y^\Delta(t)\right) = \partial_2 L\left(t, y^\sigma(t), y^\Delta(t)\right) \tag{3}$$

for all $t \in [a,b]^{\kappa^2}$ [7]; if $y \in C^2_{\mathrm{ld}}$ is an extremizer of (2), then y satisfies the nabla-differential equation

$$\frac{\nabla}{\nabla t}\partial_3 L\left(t, y^\rho(t), y^\nabla(t)\right) = \partial_2 L\left(t, y^\rho(t), y^\nabla(t)\right) \tag{4}$$

for all $t \in [a,b]_{\kappa^2}$ [11], where we use $\partial_i L$ to denote the standard partial derivative of $L(\cdot,\cdot,\cdot)$ with respect to its ith variable, $i = 1,2,3$. In the classical context $\mathbb{T} = \mathbb{R}$ one has

$$\mathcal{J}_\Delta(y) = \mathcal{J}_\nabla(y) = \int_a^b L\left(t, y(t), y'(t)\right) dt \tag{5}$$

and both (3) and (4) coincide with the standard Euler-Lagrange equation: if $y \in C^2$ is an extremizer of the integral functional (5), then

$$\frac{d}{dt}\partial_3 L\left(t, y(t), y'(t)\right) = \partial_2 L\left(t, y(t), y'(t)\right)$$

for all $t \in [a,b]$. However, the problems of extremizing (1) and (2) are intrinsically different, in the sense that is not possible to obtain the nabla results as corollaries of the delta ones and *vice versa*. Indeed, if admissible functions y are of class C^2 then (cf. [10])

$$\mathcal{J}_\Delta(y) = \int_a^b L\left(t, y^\sigma(t), y^\Delta(t)\right) \Delta t = \int_a^b L\left(\rho(t), (y^\sigma)^\rho(t), y^\nabla(t)\right) \nabla t$$

while

$$\mathcal{J}_\nabla(y) = \int_a^b L\left(t, y^\rho(t), y^\nabla(t)\right) \nabla t = \int_a^b L\left(\sigma(t), (y^\rho)^\sigma(t), y^\Delta(t)\right) \Delta t$$

and one easily see that functionals (1) and (2) have a different nature and are not compatible with each other. In this paper we introduce a more general formulation of the calculus of variations that includes, as trivial examples, the problems with functionals $\mathcal{J}_\Delta(y)$ and $\mathcal{J}_\nabla(y)$ that have been previously studied in the literature. Our main result provides an Euler-Lagrange necessary optimality type condition (cf. Theorem 1).

2. Our goal

Let $\mathbb{T}$ be a given time scale with $a, b \in \mathbb{T}$, $a < b$, and $(\mathbb{T} \setminus \{a,b\}) \cap [a,b] \neq \emptyset$; $L_\Delta(\cdot,\cdot,\cdot)$ and $L_\nabla(\cdot,\cdot,\cdot)$ be two given smooth functions from $\mathbb{T} \times \mathbb{R}^2$ to $\mathbb{R}$. The results here discussed are trivially generalized for admissible functions

$y : \mathbb{T} \to \mathbb{R}^n$ but for simplicity of presentation we restrict ourselves to the scalar case $n = 1$. We consider the delta-nabla integral functional

$$(6)\quad \begin{aligned} \mathcal{J}(y) &= \left(\int_a^b L_\Delta\left(t, y^\sigma(t), y^\Delta(t)\right)\Delta t\right)\left(\int_a^b L_\nabla\left(t, y^\rho(t), y^\nabla(t)\right)\nabla t\right) \\ &= \int_a^b \int_a^b \left[L_\Delta\left(t, y^\sigma(t), y^\Delta(t)\right) L_\nabla\left(\tau, y^\rho(\tau), y^\nabla(\tau)\right)\right]\Delta t \nabla\tau\,. \end{aligned}$$

Remark 1. In the particular case $L_\nabla \equiv \frac{1}{b-a}$ functional (6) reduces to (1) (i.e., $\mathcal{J}(y) = \mathcal{J}_\Delta(y)$); in the particular case $L_\Delta \equiv \frac{1}{b-a}$ functional (6) reduces to (2) (i.e., $\mathcal{J}(y) = \mathcal{J}_\nabla(y)$).

Our main goal is to answer the following question: *What is the Euler-Lagrange equation for $\mathcal{J}(y)$ defined by* (6)*?*

For simplicity of notation we introduce the operators $[y]$ and $\{y\}$ defined by $[y](t) = \left(t, y^\sigma(t), y^\Delta(t)\right)$ and $\{y\}(t) = \left(t, y^\rho(t), y^\nabla(t)\right)$. Then,

$$\mathcal{J}_\Delta(y) = \int_a^b L_\Delta[y](t)\Delta t\,, \qquad \mathcal{J}_\nabla(y) = \int_a^b L_\nabla\{y\}(t)\nabla t\,,$$
$$\mathcal{J}(y) = \mathcal{J}_\Delta(y)\mathcal{J}_\nabla(y) = \int_a^b \int_a^b L_\Delta[y](t) L_\nabla\{y\}(\tau)\Delta t \nabla\tau\,.$$

3. Preliminaries to the calculus of variations

Similar to the classical calculus of variations, integration by parts will play an important role in our delta-nabla calculus of variations. If functions $f, g : \mathbb{T} \to \mathbb{R}$ are delta and nabla differentiable with continuous derivatives, then the following formulas of integration by parts hold [8]:

$$(7)\quad \begin{aligned} \int_a^b f^\sigma(t) g^\Delta(t)\Delta t &= (fg)(t)|_{t=a}^{t=b} - \int_a^b f^\Delta(t) g(t)\Delta t\,, \\ \int_a^b f(t) g^\Delta(t)\Delta t &= (fg)(t)|_{t=a}^{t=b} - \int_a^b f^\Delta(t) g^\sigma(t)\Delta t\,, \\ \int_a^b f^\rho(t) g^\nabla(t)\nabla t &= (fg)(t)|_{t=a}^{t=b} - \int_a^b f^\nabla(t) g(t)\nabla t\,, \\ \int_a^b f(t) g^\nabla(t)\nabla t &= (fg)(t)|_{t=a}^{t=b} - \int_a^b f^\nabla(t) g^\rho(t)\nabla t\,. \end{aligned}$$

The following fundamental lemma of the calculus of variations on time scales involving a nabla derivative and a nabla integral has been proved in [11].

Lemma 1. (The nabla Dubois-Reymond lemma [11, Lemma 14]) *Let* $f \in C_{ld}([a,b],\mathbb{R})$. *If*

$$\int_a^b f(t)\eta^{\nabla}(t)\nabla t = 0 \quad \textit{for all } \eta \in C^1_{ld}([a,b],\mathbb{R}) \textit{ with } \eta(a)=\eta(b)=0\,,$$

then $f(t)=c$ *on* $t \in [a,b]_\kappa$ *for some constant* c.

Lemma 2 is the analogous delta version of Lemma 1:

Lemma 2. (The delta Dubois-Reymond lemma [7]) *Let* $g \in C_{rd}([a,b],\mathbb{R})$. *If*

$$\int_a^b g(t)\eta^{\Delta}(t)\Delta t = 0 \quad \textit{for all } \eta \in C^1_{rd} \textit{ with } \eta(a)=\eta(b)=0\,,$$

then $g(t)=c$ *on* $[a,b]^\kappa$ *for some* $c \in \mathbb{R}$.

Proposition 1 gives a relationship between delta and nabla derivatives.

Proposition 1. (Theorems 2.5 and 2.6 of [3]) (*i*) *If* $f : \mathbb{T} \to \mathbb{R}$ *is delta differentiable on* $\mathbb{T}^\kappa$ *and* f^Δ *is continuous on* $\mathbb{T}^\kappa$, *then* f *is nabla differentiable on* $\mathbb{T}_\kappa$ *and*

$$f^{\nabla}(t) = \left(f^{\Delta}\right)^{\rho}(t) \quad \textit{for all } t \in \mathbb{T}_\kappa\,. \tag{8}$$

(*ii*) *If* $f : \mathbb{T} \to \mathbb{R}$ *is nabla differentiable on* $\mathbb{T}_\kappa$ *and* f^∇ *is continuous on* $\mathbb{T}_\kappa$, *then* f *is delta differentiable on* $\mathbb{T}^\kappa$ *and*

$$f^{\Delta}(t) = \left(f^{\nabla}\right)^{\sigma}(t) \quad \textit{for all } t \in \mathbb{T}^\kappa\,. \tag{9}$$

Remark 2. Note that, in general, $f^\nabla(t) \neq f^\Delta(\rho(t))$ and $f^\Delta(t) \neq f^\nabla(\sigma(t))$. In Proposition 1 the assumptions on the continuity of f^Δ and f^∇ are crucial.

Proposition 2. ([3, Theorem 2.8]) *Let* $a, b \in \mathbb{T}$ *with* $a \le b$ *and let* f *be a continuous function on* $[a,b]$. *Then,*

$$\begin{aligned}
\int_a^b f(t)\Delta t &= \int_a^{\rho(b)} f(t)\Delta t + (b-\rho(b))f^{\rho}(b)\,,\\
\int_a^b f(t)\Delta t &= (\sigma(a)-a)f(a) + \int_{\sigma(a)}^b f(t)\Delta t\,,\\
\int_a^b f(t)\nabla t &= \int_a^{\rho(b)} f(t)\nabla t + (b-\rho(b))f(b)\,,\\
\int_a^b f(t)\nabla t &= (\sigma(a)-a)f^{\sigma}(a) + \int_{\sigma(a)}^b f(t)\nabla t\,.
\end{aligned}$$

We end our brief review of the calculus on time scales with a relationship between the delta and nabla integrals.

Proposition 3. ([10, Proposition 7]) *If function $f : \mathbb{T} \to \mathbb{R}$ is continuous, then for all $a, b \in \mathbb{T}$ with $a < b$ we have*

$$\int_a^b f(t)\Delta t = \int_a^b f^\rho(t)\nabla t\,, \tag{10}$$

$$\int_a^b f(t)\nabla t = \int_a^b f^\sigma(t)\Delta t\,. \tag{11}$$

4. Main result

We consider the problem of extremizing the variational functional (6) subject to given boundary conditions $y(a) = \alpha$ and $y(b) = \beta$:

$$\mathcal{J}(y) = \left(\int_a^b L_\Delta[y](t)\Delta t\right)\left(\int_a^b L_\nabla\{y\}(t)\nabla t\right) \longrightarrow \text{extr} \tag{12}$$
$$y(\cdot) \in C^1_\diamond$$
$$y(a) = \alpha\,, \quad y(b) = \beta\,,$$

where $C^1_\diamond$ denote the class of functions $y : [a, b] \to \mathbb{R}$ with y^Δ continuous on $[a, b]^\kappa$ and y^∇ continuous on $[a, b]_\kappa$. Before presenting the Euler-Lagrange equations for problem (12) we introduce the definition of weak local extremum.

Definition 1. *We say that $\hat{y} \in C^1_\diamond([a, b], \mathbb{R})$ is a weak local minimizer (respectively weak local maximizer) to problem (12) if there exists $\delta > 0$ such that $\mathcal{J}(\hat{y}) \le \mathcal{J}(y)$ (respectively $\mathcal{J}(\hat{y}) \ge \mathcal{J}(y)$) for all $y \in C^1_\diamond([a, b], \mathbb{R})$ satisfying the boundary conditions $y(a) = \alpha$, $y(b) = \beta$, and $\parallel y - \hat{y} \parallel_{1,\infty} < \delta$, where*

$$\parallel y \parallel_{1,\infty} := \parallel y^\sigma \parallel_\infty + \parallel y^\rho \parallel_\infty + \parallel y^\Delta \parallel_\infty + \parallel y^\nabla \parallel_\infty$$

and $\parallel y \parallel_\infty := \sup_{t\in[a,b]^\kappa_\kappa} \mid y(t) \mid$.

Theorem 1 gives two different forms for the Euler-Lagrange equation on time scales associated with the variational problem (12).

Theorem 1. (The general Euler-Lagrange equations on time scales.) *If $\hat{y} \in C^1_\diamond$ is a weak local extremizer to problem (12), then $\hat{y}$ satisfies the following delta-nabla integral equations:*

$$\begin{aligned}\mathcal{J}_\nabla(\hat{y}) &\left(\partial_3 L_\Delta[\hat{y}](\rho(t)) - \int_a^{\rho(t)} \partial_2 L_\Delta[\hat{y}](\tau)\Delta\tau\right) \\ &+ \mathcal{J}_\Delta(\hat{y})\left(\partial_3 L_\nabla\{\hat{y}\}(t) - \int_a^t \partial_2 L_\nabla\{\hat{y}\}(\tau)\nabla\tau\right) = const \quad \forall t \in [a, b]_\kappa\,;\end{aligned} \tag{13}$$

$$\begin{aligned}(14)\quad & \mathcal{J}_\nabla(\hat{y})\left(\partial_3 L_\Delta[\hat{y}](t) - \int_a^t \partial_2 L_\Delta[\hat{y}](\tau)\Delta\tau\right) + \mathcal{J}_\Delta(\hat{y}) \\ & \times\left(\partial_3 L_\nabla\{\hat{y}\}(\sigma(t)) - \int_a^{\sigma(t)} \partial_2 L_\nabla\{\hat{y}\}(\tau)\nabla\tau\right) = const \quad \forall t \in [a,b]^\kappa .\end{aligned}$$

Remark 3. In the classical context (i.e., when $\mathbb{T} = \mathbb{R}$) the necessary conditions (13) and (14) coincide with the Euler-Lagrange equations recently given in [9].

Proof. Suppose that $\mathcal{J}$ has a weak local extremum at $\hat{y}$. We consider the value of $\mathcal{J}$ at nearby functions $\hat{y}+\varepsilon\eta$, where $\varepsilon \in \mathbb{R}$ is a small parameter, $\eta \in C^1_\diamond([a,b],\mathbb{R})$ with $\eta(a) = \eta(b) = 0$. Thus, function $\phi(\varepsilon) = \mathcal{J}(\hat{y}+\varepsilon\eta)$ has an extremum at $\varepsilon = 0$. Using the first-order necessary optimality condition $\phi'(\varepsilon)|_{\varepsilon=0} = 0$,

$$\begin{aligned}(15)\quad & \mathcal{J}_\nabla(\hat{y})\int_a^b \left(\partial_2 L_\Delta[\hat{y}](t)\eta^\sigma(t) + \partial_3 L_\Delta[\hat{y}](t)\eta^\Delta(t)\right)\Delta t \\ & + \mathcal{J}_\Delta(\hat{y})\int_a^b \left(\partial_2 L_\nabla\{\hat{y}\}(t)\eta^\rho(t) + \partial_3 L_\nabla\{\hat{y}\}(t)\eta^\nabla(t)\right)\nabla t = 0 .\end{aligned}$$

Let $A(t) = \int_a^t \partial_2 L_\Delta[\hat{y}](\tau)\Delta\tau$ and $B(t) = \int_a^t \partial_2 L_\nabla\{\hat{y}\}(\tau)\nabla\tau$. Then, $A^\Delta(t) = \partial_2 L_\Delta[\hat{y}](t)$, $B^\nabla(t) = \partial_2 L_\nabla\{\hat{y}\}(t)$, and the first and third integration by parts formula in (7) tell us, respectively, that

$$\begin{aligned}\int_a^b \partial_2 L_\Delta[\hat{y}](t)\eta^\sigma(t)\Delta t &= \int_a^b A^\Delta(t)\eta^\sigma(t)\Delta t = A(t)\eta(t)|_{t=a}^{t=b} - \int_a^b A(t)\eta^\Delta(t)\Delta t \\ &= -\int_a^b A(t)\eta^\Delta(t)\Delta t\end{aligned}$$

and

$$\begin{aligned}\int_a^b \partial_2 L_\nabla\{\hat{y}\}(t)\eta^\rho(t)\nabla t &= \int_a^b B^\nabla(t)\eta^\rho(t)\nabla t = B(t)\eta(t)|_{t=a}^{t=b} - \int_a^b B(t)\eta^\nabla(t)\nabla t \\ &= -\int_a^b B(t)\eta^\nabla(t)\nabla t .\end{aligned}$$

If we denote $f(t) = \partial_3 L_\Delta[\hat{y}](t) - A(t)$ and $g(t) = \partial_3 L_\nabla\{\hat{y}\}(t) - B(t)$, then we can write the necessary optimality condition (15) in the form

$$(16)\qquad \mathcal{J}_\nabla(\hat{y})\int_a^b f(t)\eta^\Delta(t)\Delta t + \mathcal{J}_\Delta(\hat{y})\int_a^b g(t)\eta^\nabla(t)\nabla t = 0 .$$

We now split the proof in two parts: we prove (13) transforming the delta integral in (16) to a nabla integral by means of (10); we prove (14) transforming the nabla integral in (16) to a delta integral by means of (11). By (10) the necessary optimality condition (16) is equivalent to

$$\int_a^b \left(\mathcal{J}_\nabla(\hat{y}) f^\rho(t) (\eta^\Delta)^\rho(t) + \mathcal{J}_\Delta(\hat{y}) g(t) \eta^\nabla(t)\right) \nabla t = 0$$

and by (8) to

$$\int_a^b \left(\mathcal{J}_\nabla(\hat{y}) f^\rho(t) + \mathcal{J}_\Delta(\hat{y}) g(t)\right) \eta^\nabla(t) \nabla t = 0 \,. \tag{17}$$

Applying Lemma 1 to (17) we prove (13):

$$\mathcal{J}_\nabla(\hat{y}) f^\rho(t) + \mathcal{J}_\Delta(\hat{y}) g(t) = c \quad \forall t \in [a,b]_\kappa \,,$$

where c is a constant. By (11) the necessary optimality condition (16) is equivalent to $\int_a^b \left(\mathcal{J}_\nabla(\hat{y}) f(t) \eta^\Delta(t) + \mathcal{J}_\Delta(\hat{y}) g^\sigma(t) \left(\eta^\nabla\right)^\sigma(t)\right) \Delta t = 0$ and by (9) to

$$\int_a^b \left(\mathcal{J}_\nabla(\hat{y}) f(t) + \mathcal{J}_\Delta(\hat{y}) g^\sigma(t)\right) \eta^\Delta(t) \Delta t = 0 \,. \tag{18}$$

Applying Lemma 2 to (18) we prove (14):

$$\mathcal{J}_\nabla(\hat{y}) f(t) + \mathcal{J}_\Delta(\hat{y}) g^\sigma(t) = c \quad \forall t \in [a,b]^\kappa \,,$$

where c is a constant. ■

Corollary 1. *Let $L_\Delta\left(t, y^\sigma, y^\Delta\right) = L_\Delta(t)$ and $\mathcal{J}_\Delta(\hat{y}) \neq 0$ (this is true, e.g., for $L_\Delta \equiv \frac{1}{b-a}$ for which $\mathcal{J}_\Delta = 1$; cf. Remark 1). Then, $\partial_2 L_\Delta = \partial_3 L_\Delta = 0$ and the Euler-Lagrange equation (13) takes the form*

$$\partial_3 L_\nabla\{\hat{y}\}(t) - \int_a^t \partial_2 L_\nabla\{\hat{y}\}(\tau) \nabla\tau = const \quad \forall t \in [a,b]_\kappa \,. \tag{19}$$

Remark 4. If $\hat{y} \in C^2_{\mathrm{ld}}$, then nabla-differentiating (19) we obtain the Euler-Lagrange differential equation (4) as proved in [11]:

$$\frac{\nabla}{\nabla t} \partial_3 L_\nabla\{\hat{y}\}(t) - \partial_2 L_\nabla\{\hat{y}\}(t) = 0 \quad \forall t \in [a,b]_{\kappa^2} \,.$$

Corollary 2. *Let $L_\nabla\left(t, y^\rho, y^\nabla\right) = L_\nabla(t)$ and $\mathcal{J}_\nabla(\hat{y}) \neq 0$ (this is true, e.g., for $L_\nabla \equiv \frac{1}{b-a}$ for which $\mathcal{J}_\nabla = 1$; cf. Remark 1). Then, $\partial_2 L_\nabla = \partial_3 L_\nabla = 0$ and the Euler-Lagrange equation (14) takes the form*

$$\partial_3 L_\Delta[\hat{y}](t) - \int_a^t \partial_2 L_\Delta[\hat{y}](\tau) \Delta\tau = const \quad \forall t \in [a,b]^\kappa \,. \tag{20}$$

Remark 5. If $\hat{y} \in C^2_{\mathrm{rd}}$, then delta-differentiating (20) we obtain the Euler-Lagrange differential equation (3) as proved in [7]:

$$\frac{\Delta}{\Delta t}\partial_3 L_\Delta[\hat{y}](t) - \partial_2 L_\Delta[\hat{y}](t) = 0 \quad \forall t \in [a,b]^{\kappa^2} .$$

Example 1. Let $\mathbb{T}$ be a time scale with $0, \xi \in \mathbb{T}$, $0 < \xi$, and $(\mathbb{T} \setminus \{0, \xi\}) \cap [0, \xi] \neq \emptyset$. Consider the problem

$$\text{minimize} \quad \mathcal{J}(\mathrm{y}) = \left(\int_0^\xi (\mathrm{y}^\Delta(\mathrm{t}))^2 \Delta\mathrm{t}\right)\left(\int_0^\xi \left(\mathrm{y}^\nabla(\mathrm{t}))^2\right) \nabla\mathrm{t}\right), \tag{21}$$
$$y(0) = 0, \quad y(\xi) = \xi .$$

Since $L_\Delta = (y^\Delta)^2$ and $L_\nabla = (y^\nabla)^2$, we have $\partial_2 L_\Delta = 0$, $\partial_3 L_\Delta = 2y^\Delta$, $\partial_2 L_\nabla = 0$, and $\partial_3 L_\nabla = 2y^\nabla$. Using equation (14) of Theorem 1 we get the following delta-nabla differential equation:

$$2Ay^\Delta(t) + 2By^\nabla(\sigma(t)) = C, \tag{22}$$

where $C \in \mathbb{R}$ and A, B are the values of functionals $\mathcal{J}_\nabla$ and $\mathcal{J}_\Delta$ in a solution to problem (21), respectively. From (9) we can write equation (22) in the form

$$2Ay^\Delta(t) + 2By^\Delta = C. \tag{23}$$

Observe that $A + B$ cannot be equal to 0. Thus, solving equation (23) subject to the boundary conditions $y(0) = 0$ and $y(\xi) = \xi$ we get $y(t) = t$ as a candidate local minimizer to problem (21).

5. Conclusion

A general necessary optimality condition for problems of the calculus of variations on time scales has been given. The proposed calculus of variations extends the problems with delta derivatives considered in [6, 7] and analogous nabla problems [1, 11] to more general cases described by the product of a delta and a nabla integral. Minimization of functionals given by the product of two integrals were considered by Euler himself, and are now receiving an increasing interest because of their nonlocal properties and their applications in economics [9].

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Received on 27.08.2009 and, in revised form, on 02.12.2009.