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BOUNDEDNESS OF SOLUTIONS OF NONLINEAR THREE-DIMENSIONAL DIFFERENCE SYSTEMS WITH DELAYS

Abstract. In this paper three-dimensional nonlinear difference system with delays

$$\begin{cases} \Delta x_n = a_n f(y_{n-l}), \\ \Delta y_n = b_n g(z_{n-m}), \\ \Delta z_n = \delta c_n h(x_{n-k}), \end{cases}$$

is investigated. The classification of nonoscillatory solutions of the considered system are presented. Next, the sufficient conditions under which nonoscillatory solution of considered system is bounded or is unbounded are given.

Key words: difference equation, nonlinear system, nonoscillatory, bounded, unbounded solution.

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1. Introduction

We consider a nonlinear three-dimensional difference system of the form

$$\begin{cases} \Delta x_n = a_n f(y_{n-l}), \\ \Delta y_n = b_n g(z_{n-m}), \quad n \in N(n_0) = \{n_0, n_0+1, \dots\}, \\ \Delta z_n = \delta c_n h(x_{n-k}), \end{cases} \tag{1}$$

where $n_0 \in \mathbb{N} = \{1, 2, \dots\}$, l, m, k are given positive integer and $\delta = \pm 1$. Here $a, b : N(n_0) \to \mathbb{R}_+ \cup \{0\}$, $c : N(n_0) \to \mathbb{R}_+$, where $\mathbb{R}$, $\mathbb{R}_+$ denote the set of real numbers and the set of positive real numbers respectively. Moreover

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n = \infty. \tag{2}$$

Assume that $f, g, h : \mathbb{R} \to \mathbb{R}$ are functions such that

$$uf(u) > 0, \;\; ug(u) > 0, \;\; uh(u) > 0 \;\text{ for }\; u \neq 0, \tag{3}$$

and there exists a positive constants M^*, M^{**} and M^{***} such that

$$\frac{f(u)}{u} \geq M^*, \quad \frac{g(u)}{u} \geq M^{**} \text{ and } \frac{h(u)}{u} \geq M^{***} \text{ for } u \neq 0. \tag{4}$$

Set $M = min\{M^*, M^{**}, M^{***}\}$.

We don't assume that functions f, g and h are continuous nor monotonic.

We note that for given initial condition $x(n_0), y(n_0), z(n_0)$ there exists the unique solution $(\{x_n\}, \{y_n\}, \{z_n\}) = (x, y, z)$ of system (1).

A solution (x, y, z) of system (1) is called nonoscillatory if all its components are nonoscillatory (that is either eventually positive or eventually negative). A solution (x, y, z) of system (1) is called bounded if all its components are bounded. Otherwise it is called unbounded.

The background for difference systems can be found in the well known monograph [1] by Agarwal and Kocić and Ladas [2].

The oscillatory theory is considered usually for two-dimensional difference systems (see, for example, [3], [4], [6] and [7] and the references cited therein).

Oscillatory results for three-dimensional system are investigated by Thandapani and Ponnammal in [5]. Results which are presented in this paper partially answered the open problems stated in the paper mentioned above.

2. Some basic lemmas

We begin with some lemmas which will be useful in the sequel.

Lemma 1. *Assume that condition (3) holds. Let (x, y, z) be a solution of system (1) and let sequence x be nonoscillatory. Then (x, y, z) is nonoscillatory and sequences x, y, z are monotonic for sufficiently large n.*

Proof. Because sequence x is nonoscillatory then it is of the constant sign for large n. From the third equation of the system (1) and condition (3) we get that sequence z is eventually monotonic. This implies that z is of the constant sign for large n. Analogously we obtain that sequences x and y are monotonic, and y is nonoscillatory. ∎

Corollary 1. *Assume that condition (3) holds. Let (x, y, z) be a solution of system (1) and let sequence y (or z) be nonoscillatory. Then (x, y, z) is nonoscillatory and sequences x, y, z are monotonic for sufficiently large n.*

Lemma 2. *Assume that conditions (2) and (4) hold. Let (x, y, z) be a nonoscillatory solution of system (1). If*

$$\lim_{n\to\infty} x_n \text{ is finite} \tag{5}$$

then

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n = 0.$$

Proof. We note that condition (4) implies usual signed condition (3). Because (x, y, z) is a nonoscillatory solution of (1) then, by Lemma 1, sequence y is monotonic. Hence limit of this sequence exists. Set

$$\lim_{n\to\infty} y_n = L^*.$$

For the sake of contradiction suppose that $L^* > 0$. (In the case $L^* < 0$ the proof is similar and hence omitted.) Since that $y_n > 0$ for large n. Then there exists an integer $n_1 \geq n_0$ such that $y_{n-l} \geq \frac{L^*}{2}$, for $n \geq n_1$. By (4), there exists a positive constant M such that $f(y_{n-l}) \geq M y_{n-l} > 0$. Thus, from the first equation of system (1), we have

$$\Delta x_n \geq M a_n y_{n-l} \geq M a_n \frac{L^*}{2} > 0.$$

Summing the above inequality from n_1 to $n-1$, we get

$$x_n \geq x_{n_1} + M\frac{L^*}{2}\sum_{i=n_1}^{n-1} a_i.$$

Letting $n \to \infty$, by (2), the right hand side of the above inequality tends to infinity, so the left side too. This contradicts (5). Therefore we get $\lim\limits_{n\to\infty} y_n = 0$.

Analogously, using the second equation of system (1), we obtain that

$$\lim_{n\to\infty} z_n = 0.$$

This complete the proof. ■

Lemma 3. *Assume that conditions (2) and (4) hold and (x, y, z) is a nonoscillatory solution of system (1). Then one of the following three cases holds:*

(I) $\quad sgn\ x_n = sgn\ y_n = sgn\ z_n,$
(II) $\quad sgn\ x_n = sgn\ z_n \neq sgn\ y_n,$
(III) $\quad sgn\ x_n = sgn\ y_n \neq sgn\ z_n,$

for large n.

Moreover, if $\delta = -1$ in system (1) then every nonoscillatory solution of (1) fulfills condition (I) or (II), if $\delta = 1$ then every nonoscillatory solution of (1) fulfills condition (I) or (III).

Proof. Let (x, y, z) be a nonoscillatory solution of system (1). Without loss of the generality assume that $x_n > 0$.

First, we assume that $\delta = -1$ in this system. From Lemma 1 sequence y is monotonic for large n. Hence $y_n < 0$ or $y_n > 0$ for large n. By the same arguments $z_n < 0$ or $z_n > 0$ for large n.

For contrary, suppose that $z_n < 0$ for large n. Then there exists n_2 such that $z_{n-m} < 0$ for $n \geq n_2$. From the third equation of system (1) we get that sequence z is decreasing, so $z_{n-m} < z_{n_2-m} < 0$ for $n \geq n_2$. By (4), we get $g(z_{n-m}) \leq Mz_{n-m}$ for $n \geq n_2$. From this and the second equation of system (1), we have

$$\Delta y_n \leq b_n M z_{n-m} < b_n M z_{n_2-m}$$

for $n \geq n_2$. Summing the above inequality from n_2 to $n-1$, we obtain

$$y_n < y_{n_2} + M z_{n_2-m} \sum_{i=n_2}^{n-1} b_i.$$

Letting n to infinity, by (2) and negativity of z_{n_2-m}, the right hand side of the above inequality tends to $-\infty$. So, the left side too. Hence $\lim_{n\to\infty} y_n = -\infty$. Then there exists an integer $n_3 \geq n_2$ such that $y_{n-l} < 0$ for $n \geq n_3$. From (4), we get $f(y_{n-l}) \leq My_{n-l}$ for $n \geq n_3$. From the first equation of system (1), we have

$$\Delta x_n \leq a_n M y_{n-l} < a_n M y_{n_3-l}$$

for $n \geq n_3$. Summing the above inequality from n_3 to $n-1$ and letting n to infinity, we get that $\lim_{n\to\infty} x_n = -\infty$. This contradicts the fact that $x_n > 0$ for large n. On the virtue of this contradiction we exclude that $z_n < 0$. Hence we obtain that $z_n > 0$ for large n. Therefore if $\delta = -1$ the thesis of Lemma 3 holds.

Next, we assume that $\delta = 1$ in system (1). From the third equation of system (1) we get that sequence z is eventually increasing. Therefore $z_n < 0$ or $z_n > 0$ for large n.

Let $z_n > 0$. From the second equation of system (1) we have that sequence y is eventually of one sign. Hence $y_n < 0$ or $y_n > 0$ for large n. Suppose that $y_n < 0$ for large n. Thus sequence x is eventually nonincreasing. Then there exists $\lim_{n\to\infty} x_n = c < \infty$. By Lemma 2, we have

$$\lim_{n\to\infty} z_n = 0.$$

This contradicts the fact that z is an eventually positive increasing sequence and exclude the case that $y_n < 0$ for large n.

Let $z_n < 0$. By the analogous arguments as above, we exclude case $y_n < 0$. Therefore also if $\delta = 1$ the thesis of Lemma 3 holds.

This completes the proof. ■

3. Main results

Theorem 1. *Assume that conditions (2) and (4) hold. Then every solution* (x, y, z) *of system (1) fulfilling condition* (I) *is unbounded.*

Proof. Let (x, y, z) be nonoscillatory solution of system (1) for which condition (I) holds. Without loss of generality $x_n > 0$, $y_n > 0$ and $z_n > 0$ for large n, say $n \geq n_4$. Hence sequence y is eventually nondecreasing. Summing the first equation of system (1) from $n_5 = n_4 + l$ to $n-1$ we have

$$x_n = x_{n_5} + \sum_{i=n_5}^{n-1} a_i f(y_{i-l}) \quad \text{for} \quad n \geq n_5.$$

Therefore, by positivity of sequences x and y and (4) we get

$$x_n \geq M \sum_{i=n_5}^{n-1} a_i y_{i-l}.$$

Since y is nondecreasing then

$$x_n \geq M y_{n_5 - l} \sum_{i=n_5}^{n-1} a_i.$$

Thus, using (2), we obtain that $\lim_{n\to\infty} x_n = \infty$. Then every solution of system (1) which fulfills (I) is unbounded. ■

Example 1. Let as consider the following system of difference equations, where $\delta = -1$,

$$\text{(6)} \qquad \begin{cases} \Delta x_n = 2y_{n-1}, \\ \Delta y_n = 2^{2n-2} z_{n-2}, \qquad n \in \mathbb{N}, \\ \Delta z_n = -2^{-2n+1} x_{n-2}. \end{cases}$$

All assumptions of the Theorem 1 hold. Hence this system has unbounded solution which satisfies condition (I). It easy to see that $(2^n, 2^n, 2^{-n})$ is such solution.

Example 2. Let as consider the following system of difference equations, where $\delta = 1$,

$$\text{(7)} \qquad \begin{cases} \Delta x_n = y_{n-1}, \\ \Delta y_n = 8z_{n-2}, \qquad n \in \mathbb{N}, \\ \Delta z_n = 8x_{n-3}. \end{cases}$$

All assumptions of the Theorem 1 hold. Hence this system has unbounded solution which satisfies condition (I). It easy to see that $(2^n, 2^{n+1}, 2^n)$ is such solution.

Theorem 2. *Assume that conditions (2) and (4) hold. Then every solution (x, y, z) of system (1) fulfilling condition (II) is bounded.*

Proof. Assume that (x, y, z) is nonoscillatory solution of system (1) which satisfied condition (II). Notice that, by Lemma 3 this system has such solution if and only if $\delta = -1$ in the third equation of the system. Without loss of the generality $x_n > 0$, $y_n < 0$ and $z_n > 0$ for large n. Then sequence x is nonincreasing, y is nondecreasing and z is decreasing. Hence sequences x, y and z have finite limits. So, the thesis holds. ■

Theorem 3. *Assume that conditions (2) and (4) hold, and*

$$\sum_{n=1}^{\infty} c_n = \infty. \tag{8}$$

Then system (1) has not solution (x, y, z) which fulfilled condition (III).

Proof. Assume that (x, y, z) is nonoscillatory solution of system (1) which satisfied condition (III). Notice that, by Lemma 3, this system has such solution if and only if that $\delta = 1$ in the third equation of the system. Without loss of the generality $x_n > 0$ for large n, say $n \geq n_6$. Therefore, by Lemma 3, $y_n > 0$ and $z_n < 0$ for large n. Since z is increasing sequence we have that $\lim_{n\to\infty} z_n = L^{**} \leq 0$. For the sake of contradiction suppose that $\lim_{n\to\infty} z_n = L^{**} < 0$. Then there exists $n_7 \in \mathbb{N}$ such that $z_n \leq L^{**}$ for $n \geq n_7$. Summing the second equation of system (1) from $n_8 = max\{n_6 + k, n_7 + m\}$ to $n - 1$ we obtain

$$y_n = y_{n_8} + \sum_{i=n_8}^{n-1} b_i g(z_{i-m}) \quad \text{for} \quad n \geq n_8.$$

Hence, by negativity of sequences z, (3) and (4), we get

$$y_n \leq y_{n_8} + M \sum_{i=n_8}^{n-1} b_i z_{i-m} < y_{n_8} + ML^{**} \sum_{i=n_8}^{n-1} b_i,$$

for $n \geq n_8$. Letting n to infinity and using (2) we obtain $\lim_{n\to\infty} y_n = -\infty$. This contradicts positivity of sequence y. So, $\lim_{n\to\infty} z_n = 0$.

Summing the third equation of system (1) from $n_9 = n_8 + k$ to $n - 1$ we have

$$z_n = z_{n_9} + \sum_{i=n_9}^{n-1} c_i h(x_{i-k}) \quad \text{for} \quad n \geq n_9.$$

Then, by (4), we obtain

$$z_n \geq z_{n_9} + M \sum_{i=n_9}^{n-1} c_i x_{i-k}.$$

Hence, using the fact that sequence x is positive and nondecreasing, we get

$$z_n \geq z_{n_9} + M x_{n_9-k} \sum_{i=n_9}^{n-1} c_i.$$

The left side of the above inequality tends to zero whereas the right hand side, by (8), tends to infinity. This contradiction ends the proof. ■

Corollary 2. *Assume that conditions (2), (4) and (8) hold, and $\delta = 1$. Then every nonoscillatory solution of system (1) is unbounded.*

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