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A NOTE ON JUNGCK'S FIXED POINT THEOREM

Abstract. This paper is twofold. First we prove a common fixed point theorem for a pair of weakly compatible mappings along with E.A. property. This theorem improves and generalizes a result of Jungck [5] without any continuity requirement besides relaxing the containment of the range of one map into the range of other map. Moreover, we prove some results under different variants of R-weakly commuting mappings.

Key words: weakly compatible maps, E.A property, Variants of R-weakly commuting mappings (R-weakly commuting mapping of type (A_g), R-weakly commuting mapping of type (A_f) and R-weakly commuting mapping of type (P)).

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1. Introduction

Pfeffer [14] showed that any involution r of a circle S has a fixed point if and only if there exists a free involution of S which commutes with r. This observation leads to interdependence between commutative pairs and existence of common fixed points. In 1922, the Polish mathematician, Banach [2] proved a common fixed-point theorem, which ensures under appropriate conditions, the existence and uniqueness of a fixed-point. This result of Banach is known as Banach's fixed point theorem (Banach contraction principle states "let (X, d) be a complete metric space. If T satisfies $d(Tx, Ty) \leq kd(x, y)$ for each x, y in X where $0 \leq k < 1$, then T has a unique fixed point in X"). This theorem provides a technique for solving a variety of applied problems in mathematical sciences and engineering. Many authors have extended, generalized and improved Banach fixed point theorem in different ways. For the last quarter of the 20th century, there has been a considerable interest to study common fixed point theorems for a pair (or family) of mappings satisfying contractive conditions in metric spaces. Several interesting and elegant results were obtained in this direction by various authors. In particular, we have to look first why we need such types of maps in the context of common fixed point theorems in metric spaces.

Start with the following contraction conditions:

Let T be a mapping from a complete metric space (X, d) into itself and consider the following conditions:

$$(1) \qquad d(Tx, Ty) \leq \alpha d(x, y) \quad \text{for all } x, y \in X, \quad \text{where } 0 \leq \alpha < 1,$$

$$(2) \qquad d(Tx, Ty) \leq \beta[d(x, Tx) + d(y, Ty)] \quad \text{for all } x, y \in X, \quad \text{where } 0 \leq \beta < 1/2.$$

It is clear that every self map T satisfying condition (1) is continuous but it may fail to be continuous if it satisfies condition (2). In late 70's many generalizations of the condition (1) and (2) appeared. To focus such a pioneer problem mathematician generalize the condition (1) for a pair of self maps S and T in the following ways:

$$(3) \qquad d(Sx, Ty) \leq \alpha d(x, y) \quad \text{for all } x, y \in X, \quad \text{where } 0 \leq \alpha < 1,$$

$$(4) \qquad d(Sx, Sy) \leq \alpha d(Tx, Ty) \quad \text{for all } x, y \in X, \quad \text{where } 0 \leq \alpha < 1.$$

To prove the existence of common fixed points for the condition (3) one can choose an arbitrary point x_0 in X and define a sequence $\{x_n\}$ of X by $x_{2n+1} = Sx_{2n}$, $x_{2n+2} = Tx_{2n+1}$, $n \in N_0$. To find the common fixed points for the condition (4), it is necessary to add some additional assumptions and one has to follow the following pattern:

(*i*) construction of the sequence $\{x_n\}$ (*ii*) some mechanism to obtain common fixed point. Jungck [5] resolved this problem by imposing additional hypothesis of commutative pair of maps. Most of the papers satisfying condition (4) followed the following criteria:

(*i*) contraction (*ii*) continuity of functions (either one or both) and (*iii*) commutativity of maps. In some cases condition (*ii*) can be relaxed but condition (*i*) and (*iii*) are unavoidable. The answer of the Global problem, How to develop extensively this theory? was affirmatively answered when mathematicians diverted their research in the direction of conditions (*i*) and (*iii*).

Now we give preliminaries and basic definitions which are used throughout the paper.

Sessa [15] introduced the concept of weak commutativity and researchers started utilizing weak conditions of commutativity to improve common fixed point theorems.

Two self-mappings f and g be of a metric space (X, d) are said to be weakly commuting if $d(fgx, gfx) \leq d(gx, fx)$ for all x in X.

It was the turning point in the "fixed point arena" when the notion of compatibility of mapping was introduced by Jungck [6] as a sharper tool to obtain common fixed point. This concept has been very useful for obtaining fixed point theorems for pairs of mappings, satisfying a contractive type condition and assuming continuity of at least one of mappings.

Definition 1. *Two self-mappings f and g of a metric space (X, d) are said to be compatible if* $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, *whenever* $\{x_n\}_{n=1}^{\infty}$ *is a sequence in X such that* $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ *for some t in X.*

Consequently, the recent literature of fixed point theory has witnessed the evolution of several weak conditions of commutativity such as: Compatible mappings of type (A), Compatible mappings of type (B), Compatible mappings of type (P), Compatible mappings of type (C), R-weakly commuting mappings and several others whose lucid survey and illustration are available in the paper entitled "Important tools and possible applications of metric fixed point theory, *Nonlinear Analysis*, 47(2001), 3479–3490".

It has been known from the paper of Kannan [10] that there exists maps that have a discontinuity in the domain but which have fixed points, moreover, the maps involved in every case were continuous at the fixed point.

The study of common fixed point's theorems for non compatible mappings is initiated by Pant [12] with the introduction of the notion of R-weakly commuting mappings in metric spaces.

Definition 2. *A pair of self-mappings (f, g) of a metric space (X, d) is said to be R-weakly commuting if there exists some $R > 0$ such that*

$$d(fgx, gfx) \leq Rd(fx, gx,) \quad \textit{for all } x \in X.$$

Now, there arises a natural question: "How fixed point theorems can be improved to the setting of non-complete metric spaces and without continuity of f and g over the whole space X?" Pant [12] gives the partial answer. It seems that fixed point theorems can be improved in two ways: either imposing certain restrictions on the space X or by replacing the notion of R-weakly commutativity of mappings with certain improved notion.

In 1997, Pathak, Cho and Kang [11] improved the notion of R-weakly commuting mappings to the notion of R-weakly commuting mappings of type (A_g) and R-weakly commuting mappings of type (A_f).

Definition 3. *A pair of self-mappings (f, g) of a metric space (X, d) is said to be*

(*i*) *R-weakly commuting mappings of type (Ag) if there exists some $R > 0$ such that*

$$d(gfx, ffx) \leq Rd(fx, gx) \quad \textit{for all} \quad x \in X.$$

(*ii*) *R-weakly commuting mappings of type* (A_f) *if there exists some* $R > 0$ *such that*

$$d(fgx, ggx) \leq Rd(fx, gx) \quad \textit{for all} \quad x \in X.$$

In 1998, Jungck and Rhoades [7] introduced the notion of weakly compatible as follow:

Definition 4. *Two maps* f *and* g *are said to be weakly compatible if they commute at coincidence points.*

Example 1. Weakly compatible maps need not be compatible. Let $X = [2, 20]$ and d be the usual metric on X. Define mappings $B, T : X \to X$ by $Bx = x$ if $x = 2$ or $> 5, Bx = 6$ if $2 < x \leq 5$, $Tx = x$ if $x = 2$, $Tx = 12$ if $2 < x \leq 5$, $Tx = x - 3$ if $x > 5$. The mappings B and T are non-compatible since sequence $\{x_n\}$ defined by $x_n = (5 + (1/n), n \geq 1)$. Then $Tx_n \to 2$, $Bx_n \to 2$, $TBx_n \to 2$ and $BTx_n \to 6$. But they are weakly compatible since they commute at coincidence point at $x = 2$.

Now we introduce the notion of R-weakly commuting mappings of type (P), which seem to be unreported in the metric fixed point theory literature.

Definition 5. *A pair of self-mappings* (f, g) *of a metric space* (X, d) *is said to be R-weakly commuting mappings of type (P) if there exists some* $R > 0$ *such that*

$$d(ffx, ggx) \leq Rd(fx, gx) \quad \textit{for all} \quad x \in X.$$

Remark 1. We have some suitable examples to show that variants of R-weakly commuting mappings (R-weakly commuting of type (A_g), R-weakly commuting of type (A_f) and R-weakly commuting of type (P)) are distinct.

Example 2. Let $X = [-1, 1]$ be equipped with usual metric d defined by $d(x, y) = |x - y|$ for all x, y in X.

Define $fx = |x|$ and $gx = |x| - 1$. Then by a straightforward calculation, one can show that $d(fx, gx) = 1$, $d(fgx, gfx) = 2(1 - |x|)$, $d(fgx, ggx) = 1$, $d(gfx, ffx) = 1$, $d(ffx, ggx) = 2|x|$ for all x in X.

Now we conclude the following:

(*i*) pair (f, g) is not weakly commuting,

(*ii*) for $R = 2$, pair (f, g) is R-weakly commuting, R-weakly commuting of type (P), R-weakly commuting of type (A_g) and R-weakly commuting of type (A_f),

(*iii*) for $R = \frac{3}{2}$, pair (f, g) is R-weakly commuting of type (A_f) but neither R-weakly commuting of type (P) nor R-weakly commuting.

Example 3. Let $X = [0, 1]$ be equipped with usual metric d defined by

$$d(x, y) = |x - y| \quad \text{for all } \; x, y \in X.$$

Define $fx = x$ and $gx = x^2$. Then by a straightforward calculation, one can show that $ffx = x$, $gfx = x^2$, $fgx = x^2$, $ggx = x^4$ and $d(fgx, gfx) = 0$, $d(fgx, ggx) = |x^2(x - 1)(x + 1)|$, $d(gfx, ffx) = |x(x - 1)|$, $d(ffx, ggx) = |x(x - 1)(x^2 + x + 1)|$ and $d(fx, gx) = |x(x - 1)|$ for all x in X. Therefore, we conclude that

(i) pair (f, g) is R-weakly commuting for all positive real values of R,

(ii) for $R = 3$, pair (f, g) is R-weakly commuting of the type (A_f), R-weakly commuting of the type (A_g) and R-weakly commuting of the type (P),

(iii) for $R = 2$, pair (f, g) is R-weakly commuting of type (A_f) and R-weakly commuting of type (A_g) but not R-weakly commuting of type (P) (for this take $x = \frac{3}{4}$).

Example 4. Consider $X = \left[\frac{1}{2}, 2\right]$. Let us define self maps f and g by $fx = \frac{x+1}{3}$, $gx = \frac{x+2}{5}$. We calculate the following:

$$d(fx, gx) = \frac{2x-1}{15}, \quad d(fgx, gfx) = 0, \quad d(fgx, ggx) = \frac{2x-1}{75},$$
$$d(gfx, ffx) = \frac{2x-1}{45} \quad \text{and} \quad d(ffx, ggx) = \frac{8}{225}(2x - 1) \quad \text{for all } x \text{ in } X.$$

Now we conclude the following:

(i) the pair (f, g) is R-weakly commuting for all positive real numbers,

(ii) for $R \geq \frac{8}{15}$, it is R-weakly commuting of type (A_f), R-weakly commuting of type (A_g) and R-weakly commuting of type (P),

(iii) for $\frac{1}{3} \leq R < \frac{8}{15}$, it is R-weakly commuting of type (A_g) and R-weakly commuting of type (A_f) but not R-weakly commuting of type (P),

(iv) for $\frac{1}{5} \leq R < \frac{1}{3}$, it is R-weakly commuting of type (A_f) but neither R-weakly commuting of type (A_g) nor R-weakly commuting of type (P).

Moreover, such mappings commute at their coincidence points.

Recently, Amari and Moutawakil [1] introduced a generalization of non compatible maps as E.A. property.

Definition 6. *Let A and S be two self-maps of a metric space (X,d). The pair (A,S) is said to satisfy E.A. property, if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.*

Example 5. Let $X = [0,+\infty)$. Define $S,T : X \to X$ by $Tx = \frac{x}{4}$ and $Sx = \frac{3x}{4}$, for all x in X. Consider the sequence $x_n = \frac{1}{n}$. Clearly $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = 0$. Then S and T satisfy E.A. property.

Example 6. Let $X = [2,+\infty)$. Define $S,T : X \to X$ by $Tx = x+1$ and $Sx = 2x+1$, for all $x \in X$. Suppose that the E.A. property holds. Then, there exists in a sequence $\{x_n\}$ in X satisfying $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$. Therefore, $\lim_{n\to\infty} x_n = z-1$ and $\lim_{n\to\infty} x_n = \frac{z-1}{2}$. Thus, $z = 1$, which is a contradiction, since 1 is not contained in X. Hence S and T do not satisfy E.A. property.

Notice that weakly compatible and E.A. property are independent of each other (for detail see, H.K. Pathak, Rosana Rodriguez-Lopez and R.K.Verma, A common fixed point theorem using implicit relation and E.A property in metric spaces, *Filomat* 21 (2)(2007), 211-234).

Example 7. Let $X = R^+$ and d be the usual metric on X. Define $f,g : X \to X$ by $Fx = 0$ if $0 < x \leq 1$ and $fx = 1$, if $x > 1$ or $x = 0$; and $gx = [x]$, the greatest integer that is less than or equal to x, for all $x \in X$. Consider a sequence $\{x_n\} = \left\{1 + \frac{1}{n}\right\}$ $n \geq 2$ in $(1,\ 2)$, then we have $\lim_{n\to\infty} fx_n = 1 = \lim_{n\to\infty} gx_n$. Similarly for the sequence $\{y_n\} = \left\{1 - \frac{1}{n}\right\}$ $n \geq 2$ in $(0,1)$, we have $\lim_{n\to\infty} fy_n = 0 = \lim_{n\to\infty} gy_n$. Thus the pair (f,g) satisfies E.A. property. However, f and g are not weakly compatible as each $u_1 \in (0,1)$ and $u_2 \in (1,2)$ are coincidence points of f and g, where they do not commute. Moreover, they commute at $x = 0,1,2,\ldots$ but none of these points are coincidence points of f and g. Thus we can conclude that, E.A. property does not imply weak compatibility.

Following example will revel that a pair of weakly compatible maps need not be compatible.

Example 8. Let $X = [2,20]$ and define mappings $S,T : X \to X$ by

$$S(x) = \begin{cases} x, & \text{if } x = \{2,5\} \\ 6, & \text{if } 2 < x \leq 5 \end{cases} \qquad T(x) = \begin{cases} x, & \text{if } x = 2 \\ 12, & \text{if } 2 < x \leq 5 \\ x-3, & \text{if } x > 5 \end{cases}$$

Now consider a sequence $\{x_n\}$ defined by $x_n = 5 + \frac{1}{n}$, $n \geq 1$. Then $Tx_n \to 2$, $Sx_n \to 2$, $TSx_n \to 2$ and $STx_n \to 6$. The mappings S and T are non-compatible, however, the maps S and T are weakly compatible since they commute at coincidence point at $x = 2$.

2. Fixed point theorems for a pair of mappings

In 1976, Jungck [5] proved the following interesting generalization of Banach contraction principle by replacing identity map with a continuous map.

Theorem 1. *Let f be a continuous mapping of a complete metric space (X, d) into itself and let $g : X \to X$ be a map that satisfy the following conditions:*
(a) $g(X) \subseteq f(X)$
(b) *g commutes with f*
(c) $d(gx, gy) \leq kd(fx, fy)$ *for all* $x, y \in X$ *and for some* $0 \leq k < 1$.

Then f and g have a unique common fixed point provided f and g commute.

Now we prove our main result for a pair of weakly compatible mappings along with E.A property.

Our improvement in this paper is four-fold:
(i) to relax the continuity requirement of maps completely,
(ii) to minimize the commutativity requirement of the maps to the point of coincidence,
(iii) to weaken the completeness requirement of the space
(iv) E.A property buys containment of ranges without any continuity requirement to the points of coincidence.

Theorem 2. *Let (X, d) be a metric space. Let f and g be self maps of X satisfying the following conditions:*
(i) *f and g satisfy E.A. property,*
(ii) *there exists a number $0 \leq q < 1$ such that*

$$d(gx, gy) \leq qd(fx, fy) \text{ for all } x, y \in X,$$

(iii) *$f(X)$ is a closed subspace of X.*

Then f and g have a unique common fixed point in X provided f and g are weakly compatible maps.

Proof. Since f and g satisfy the E.A. property, there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u \in X$. Since $f(X)$ is a closed subspace of X, points in $f(X)$ converges to a point in $f(X)$. Hence

$\lim_{n\to\infty} fx_n = u = fa = \lim_{n\to\infty} gx_n$ for some $a \in X$. This implies $u = fa \in f(X)$. Now we show that $u = fa = ga$.

From (ii), $d(ga, gx_n) \leq qd(fa, fx_n)$.

Proceeding limit as $n \to \infty$, we have $u = ga = fa$. Thus a is the coincidence point of f and g. Since f and g are weakly compatible, $fu = fga = gfa = gu$.

From (ii), we have

$$d(gu, ga) \leq qd(fu, fa),$$

which in turns implies that $fu = u$. Hence u is the unique common fixed point of f and g. Uniqueness follows easily from (ii). ■

Example 9. *Consider $X = [0, 2]$ with usual metric d. Define the self maps f and g on X as follows:*

$$g(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } 1 \leq x \leq 2 \end{cases} \qquad f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 2 & \text{if } 1 \leq x \leq 2 \end{cases}.$$

Consider the sequence $x_n = \frac{1}{n}$. Clearly $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 0$.

Then f and g satisfy E. A. property. Also $g(X) = \{0, 1\}$ and $f(X) = \{0, 2\}$. Here we note that neither $f(X)$ is contained in $g(X)$ nor $g(X)$ is contained in $f(X)$. Theorem 2 holds for $\frac{1}{2} \leq q < 1$.

Remark 2. *Here one needs to note that Jungck's theorem [5] requires $g(X) \subseteq f(X)$, which is not met in the Example 9.*

Remark 3. *We note that requirement of the completeness of the subspace is also essential in the Theorem 2 and cannot be relaxed even if the space is complete.*

Example 10 ([9])**.** *Let $X = \left\{0, 1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3} \dots\right\}$ be a complete metric space with usual metric d for all x, y in X. Define mappings $f, g : X \to X$ by $f(0) = \frac{1}{2^2}$, $f\left(\frac{1}{2^n}\right) = \frac{1}{2^{n+2}} \dots$ and $g(0) = \frac{1}{2^2}$, $g\left(\frac{1}{2^n}\right) = \frac{1}{2^{n+1}} \dots$. Here we note that f and g enjoys the E.A. property. We also notice that f and g have no point of coincidence even though the metric space is complete. Thus we note that the completeness of the subspace is essential requirement for the existence of the unique common fixed point.*

Theorem 3. *Theorem 2 remains true if weakly compatible property is replaced by any one (retaining the rest of hypothesis) of the following:*

(a) R-weakly commuting property,

(*b*) *R-weakly commuting property of type* (A_g),
(*c*) *R-weakly commuting property of type* (A_f),
(*d*) *R-weakly commuting property of type* (P),
(*e*) *weakly commuting property.*

Proof. Since all the conditions of Theorem 2 are satisfied, then the existence of coincidence points for both the pairs is insured. Let x be an arbitrary point of coincidence for the pair (f, g), then using R-weak commutativity one gets

$$d(fgx, gfx) \leq Rd(fx, gx) = 0,$$

which amounts to say that $fgx = gfx$. Thus the pair (f, g) is weakly compatible. Now applying Theorem 2, one concludes that f and g have a unique common fixed point.

In case (f, g) is an R-weakly commuting pair of type (Ag), then $d(gfx, f^2x) \leq d(fx, gx) = 0$, which amounts to say that $gfx = f^2x$. Now

$$d(fgx, gfx) \leq d(fgx, f^2x) + d(f^2x, gfx) = 0 + 0 = 0.$$

In case (f, g) is an R-weakly commuting pair of type (A_f), then $d(fgx, g^2x) = d(fx, gx) = 0$, which amounts to say that $fgx = g^2x$. Now $d(fgx, gfx) \leq d(fgx, g^2x) + d(g^2x, gfx) = 0 + 0 = 0$, yielding thereby $fgx = gfx$.

Similarly, if pair is R-weakly commuting mappings of type (P) or weakly commuting, then (f, g) also commutes at their points of coincidence. Now in view of Theorem 2, in all the cases f and g have a unique common fixed point. This completes the proof. ■

As an application of Theorem 2, we prove a common fixed point theorem for two finite families of mappings which runs as follows:

Theorem 4. *Let* $\{f_1, f_2 \ldots f_m\}$ *and* $\{g_1, g_2, \ldots, g_n\}$ *be two finite families of self-mappings of a metric space* (X, d) *such that* $f = f_1 f_2 \ldots f_m$, $g = g_1 g_2 \ldots g_n$, *satisfy conditions* (i), (ii) *and the following:*

$$g(X) \ \text{ is a complete subspace of } \ X.$$

Then f and g have a point of coincidence.

Moreover, if $f_i f_j = f_j f_i$ and $g_k g_l = g_l g_k$ for all $i, j \in I_1 = \{1, 2, \ldots, m\}$, $k, l \in I_2 = \{1, 2, \ldots, n\}$, then (for all $i \in I_1$, $k \in I_2$) f_i and g_k have a common fixed point.

Proof. From the component wise commutativity of various pairs, one can conclude that $fg = gf$. Therefore, maps f and g are obviously weakly compatible. Note that all the conditions of Theorem 2 are satisfied, therefore maps f and g have a unique common fixed point say z. Now one need to show that z remains the fixed point of all the component maps. For this consider

$$\begin{aligned} f(f_i z) &= ((f_1 f_2 \dots f_m) f_i) z = (f_1 f_2 \dots f_{m-1})((f_m f_i) z) \\ &= (f_1 \dots f_{m-1})(f_i f_m z) = (f_1 \dots f_{m-2})(f_{m-1} f_i (f_m z)) \\ &= (f_1 \dots f_{m-2})(f_i f_{m-1}(f_m z)) = \dots \\ &= f_1 f_i (f_2 f_3 f_4 \dots f_m z) = f_i f_1 (f_2 f_3 \dots f_m z) = f_i(fz) = f_i z. \end{aligned}$$

Similarly, one can show that $f(g_k z) = g_k(fz) = g_k z, g(g_k z) = g_k(gz) = g_k z$ and $g(f_i z) = f_i(gz) = f_i z$, which show that (for all i and k) $f_i z$ and $g_k z$ are other fixed points of the pair (f, g). Now appealing to the uniqueness of common fixed points of both pairs separately, we get $z = f_i z = g_k z$, which shows that z is a common fixed point of f_i, g_k for all i and k. ■

Theorem 5. *Let f and g be self maps of a metric spaces (X, d) satisfying the following conditions:*

(*i*) $f(X) \subset g(X)$,

(*ii*) $d(fx, fy) \leq k \max\{d(gx, gy), d(fx, gx), d(fy, gy), d(gx, fy), d(fx, gy)\}$, *for all x, y in X, where $k \in (0, 1)$,*

(*iii*) *the pair (f, g) satisfies E.A. property,*

(*iv*) *the pair (f, g) is weakly compatible.*

If the range of one of f and g is a closed subset of X, then f and g have a unique common fixed point in X.

Proof. Since (f, g) satisfies the E.A. property, there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = p$ for some $p \in X$.

As $p \in g(X)$ there exists a $u \in X$ such $p = gu$.

Therefore, from (*ii*) we have

$$\begin{aligned} d(fx_n, fu) \leq\ & k \max\{d(gx_n, gu), d(fx_n, gx_n), \\ & d(fu, gu), d(gx_n, fu), d(fx_n, gu)\} \quad \text{for all } n \in N. \end{aligned}$$

Proceeding to the limit as $n \to \infty$, we have $fu = gu$. Let us denote $fu = gu = z$.

Since the pair (f, g) is weak compatible, $fgu = gfu$ i.e, $fz = gz$.

Now we show that $fz = z$.

$$d(fz, fu) \leq k \max\{d(gz, gu), d(fz, gz), d(fu, gu), d(fu, gz), d(fz, gu)\},$$

implies

$$fz = z.$$

Hence $fz = gz = z$ and z is a common fixed point of f and g. Uniqueness follows easily from (*ii*). ■

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