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Arun K. Tripathy

OSCILLATION PROPERTIES OF A CLASS OF NEUTRAL DIFFERENTIAL EQUATIONS WITH POSITIVE AND NEGATIVE COEFFICIENTS

Abstract. In this paper, oscillatory and asymptotic property of solutions of a class of nonlinear neutral delay differential equations of the form

$$(E) \qquad \frac{d}{dt}(r(t)\frac{d}{dt}(y(t)+p(t)y(t-\tau))) + f_1(t)G_1(y(t-\sigma_1)) - f_2(t)G_2(y(t-\sigma_2)) = g(t)$$

and

$$\frac{d}{dt}(r(t)\frac{d}{dt}(y(t)+p(t)y(t-\tau))) + f_1(t)G_1(y(t-\sigma_1)) - f_2(t)G_2(y(t-\sigma_2)) = 0$$

are studied under the assumptions

$$\int_0^\infty \frac{dt}{r(t)} < \infty \qquad \text{and} \qquad \int_0^\infty \frac{dt}{r(t)} = \infty$$

for various ranges of $p(t)$. Sufficient conditions are obtained for existence of bounded positive solutions of (E).

Key words: oscillation, non-oscillation, neutral differential equation, existence of positive solutions, asymptotic behaviour.

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1. Introduction

Recently, there has been many investigations into the nonoscillation of nonlinear neutral delay differential equations with positive and negative coefficients. See [1, 2, 4, 5, 7] for reviews of this theory. However, the study of oscillatory and asymptotic behaviour of solutions of such equations has received much less attention, which is due mainly to the technical difficulties arising in its analysis.

In [6], authors have made an attempt to study the oscillation properties of a nonlinear differential equations of type

$$\frac{d}{dt}(y(t)+p(t)y(t-\tau))+f_1(t)G_1(y(t-\sigma_1))-f_2(t)G_2(y(t-\sigma_2))=0$$

and

$$\frac{d}{dt}(y(t)+p(t)y(t-\tau))+f_1(t)G_1(y(t-\sigma_1))-f_2(t)G_2(y(t-\sigma_2))=g(t)$$

with a suitable transformation. Keeping in view a similar transformation the author has discussed the oscillation properties of a class of nonlinear functional differential equation of the form

$$\frac{d}{dt}\left(r(t)\frac{d}{dt}\left(y(t)+p(t)y(t-\tau)\right)\right)+f_1(t)G_1(y(t-\sigma_1)) \\ -f_2(t)G_2(y(t-\sigma_2))=0, \tag{1}$$

where $\tau>0$, $\sigma_1,\sigma_2\geq 0$, $f_1,f_2,r\in C([0,\infty),[0,\infty))$ and $G_i\in C(R,R)$ such that $xG_i(x)>0$, $x\neq 0$ for $i=1,2$ under the assumptions

$$\int_{t_0}^{\infty}\frac{1}{r(s)}\left(\int_s^{\infty}f_2(t)dt\right)ds<\infty \tag{H_0}$$

$$\int_0^{\infty}\frac{dt}{r(t)}<\infty \tag{H_1}$$

and

$$\int_0^{\infty}\frac{dt}{r(t)}=\infty. \tag{H_2}$$

The associated forced equation

$$(r(t)(y(t)+p(t)y(t-\tau))')'+f_1(t)G_1(y(t-\sigma_1)) \\ -f_2(t)G_2(y(t-\sigma_2))=g(t), \tag{2}$$

where $g\in C([0,\infty),R)$ is also studied under the assumptions (H_0), (H_1) and (H_2). Different ranges of $p(t)$ and a particular type of forcing function is considered.

Equation (1) is considered by the authors Yu and Wang, where the whole text deals with the existence of positive solutions only. It seems that almost there is no work concerning the oscillation properties of solutions of (1) and (2) under the hypotheses (H_1) and (H_2).

The object of this paper is to establish the necessary and sufficient conditions for oscillation of (1) and (2). An extension work of [6] for Equs.(1) and (2) provides own purpose due to the work in [7].

By a solution of (1)/(2), we understand a function $y\in C([-\rho,\infty),R)$ such that $(y(t)+p(t)y(t-\tau))$ is continuously differentiable, $(r(t)(y(t)+$

$p(t)y(t-\tau)')$ is continuously differentiable and equation (1)/(2) is satisfied for $t \geq 0$, where $\rho = \max\{\tau, \sigma_1, \sigma_2\}$ and $\sup\{|y(t)| : t \geq t_0\} > 0$ for every $t_0 \geq 0$. A solution of (1)/(2) is said to be oscillatory if it has arbitrarily large zeros; otherwise, it is called nonoscillatory.

2. Some preliminary results

This section deals with some lemmas which play an important role in establishing the present work.

Lemma 1. *Assume that* (H_1) *hold. Let u(t) be an eventually positive continuously differentiable function such that* $r(t)u'(t)$ *is continuously differentiable and* $(r(t)u'(t))' \leq 0$ *but* $\not\equiv 0$ *for large* t, *where* $r \in C([0,\infty),(0,\infty))$.

(i) *If* $u'(t) > 0$, *then there exists a constant* $C > 0$ *such that* $u(t) \geq CR(t)$, *for large* t.

(ii) *If* $u'(t) < 0$, *then* $u(t) > -r(t)u'(t)R(t)$, *where,* $R(t) = \int_t^\infty \frac{ds}{r(s)}$.

Proof. (i) Since $R(t) < \infty$, $R(t) \to 0$ as $t \to \infty$ and $u(t)$ is nondecreasing, we can find a constant $C > 0$ such that $u(t) \geq CR(t)$ for all large t.

(ii) For $s \geq t$, $r(s)u'(s) \leq r(t)u'(t)$ and hence

$$u(s) \leq u(t) + \int_t^s \frac{r(t)u'(t)}{r(\theta)} d\theta = u(t) + r(t)u'(t) \int_t^s \frac{d\theta}{r(\theta)}.$$

Thus $0 < u(s) \leq u(t) + r(t)u'(t) \int_t^s \frac{d\theta}{r(\theta)}$ implies that $u(t) \geq -r(t)u'(t)R(t)$. ■

Lemma 2. *Assume that* (H_2) *hold. Let* $u(t))$ *and* $u'(t)$ *be positive continuously differentiable functions with* $u''(t) \leq 0$ *for* $t \geq T \geq 0$. *Then* $u(t) \geq (t-T)u'(t) = \beta(t)r(t)u'(t)$ *for* $t \geq T \geq 0$, *where* $\beta(t) = \frac{t-T}{r(t)}$.

Proof. The proof is simple and hence the details are omitted. ■

Lemma 3 ([3). *]Let* $p, y, z \in C([0,\infty), R)$ *be such that* $z(t) = y(t) + p(t)y(t-\tau)$, $t \geq \tau \geq 0$, $y(t) > 0$ *for* $t \geq t_1 > \tau$, $\liminf_{t\to\infty} y(t) = 0$ *and* $\lim_{t\to\infty} z(t) = L$ *exists.*

Let $p(t)$ *be satisfy one of the following conditions:*

(i) $0 \leq p(t) \leq p_1 < 1$

(ii) $1 < p_2 \leq p(t) \leq p_3$,

(iii) $p_4 \leq p(t) \leq 0$,

where p_i *is a constant,* $1 \leq i \leq 4$. *Then* $L = 0$.

3. Oscillation properties of Eq.(1)

This section provides the sufficient conditions for oscillation and asymptotic behaviour of solutions of Eq.(1) under the assumptions (H_1) and (H_2). We need following conditons for our work in the sequel.

(H_3) For $u > 0$ and $v > 0$, there exists $\lambda > 0$ such that

$$G_1(u) + G_1(v) \geq \lambda G_1(u+v)$$

(H_4) $$G_1(uv) = G_1(u)G_1(v) \quad \text{for} \quad u, v \in R$$

(H_5) $$G_1(-u) = -G_1(u), \quad u \in R$$

(H_6) $$\int_0^{\pm C} \frac{dx}{G_1(x)} < \infty \ .$$

Remark 1. The prototype of G_1 satisfying (H_3) and (H_4) is

$$G_1(u) = (a + b|u|^\lambda)|u|^\mu sgn\ u,$$

where $a \geq 1$, $b \geq 1$, $\lambda \geq 0$ and $\mu \geq 0$.

Remark 2. (H_4) implies (H_5), indeed, $G_1(1)G_1(1) = G_1(1)$, so that $G_1(1) = 1$. Further, $G_1(-1)G_1(-1) = G_1(1) = 1$ gives $(G_1(-1))^2 = 1$. Because $G_1(-1) < 0$, then $G_1(-1) = -1$. Consequently, $G_1(-u) = G_1(-1)G_1(u) = -G_1(u)$. On the otherhand $G_1(uv) = G_1(u)G_1(v)$ for $u > 0$, $v > 0$ and $G_1(-u) = -G_1(u)$ imply that $G_1(uv) = G_1(u)G_1(v)$ for every $u, v \in R$.

Remark 3. We may note that if $y(t)$ is a solution of (1), then $x(t) = -y(t)$ is also a solution of (1) provided that G_1 satisfies (H_4) or (H_5).

Theorem 1. *Let $0 \leq p(t) \leq d < \infty$. Suppose that ($H_0$), ($H_1$), ($H_3$) - ($H_5$) hold. If*

(H_7) $$\int_0^\infty Q(t)G_1(R(t-\sigma_1))dt = \infty,$$

where $Q(t) = \min\{f_1(t),\ f_1(t-\tau)\}$, $t \geq \tau$, then every solution of (1) either oscillates or tends to zero as $t \to \infty$.

Proof. Suppose on the contrary that $y(t)$ is a non-oscillatory solution of (1) such that $y(t) > 0$ for $t \geq t_0 \geq 0$. Setting

$$K(t) = \int_t^\infty \frac{1}{r(s)} \int_s^\infty f_2(\theta)G_2(y(\theta - \sigma_2))d\theta\ ds,$$

and

$$w(t) = y(t) + p(t)y(t-\tau) - K(t) = z(t) - K(t) \tag{3}$$

for $t \geq t_0 + \rho$, Eq.(1) can be written as

$$(r(t)w'(t))' + f_1(t)G_1(y(t-\sigma_1)) = 0,$$

that is,

$$(r(t)w'(t))' = -f_1(t)G_1(y(t-\sigma_1)) \leq 0, \tag{4}$$

for $t \geq t_1 > t_0 + \rho$. Hence $r(t)w'(t)$ is a monotonic function on $[t_1, \infty)$. Let $w'(t) < 0$ for $t \geq t_1$. If $w(t) < 0$, then $y(t) \leq z(t) \leq K(t)$, $t \geq t_1$. We note that $K(t)$ is bounded with $\lim_{t\to\infty} K(t) = 0$ and hence there exists a constant $\gamma > 0$ such that $y(t) \leq \gamma$ for $t \geq t_2 > t_1$. Ultimately, $w(t)$ is bounded and $\lim_{t\to\infty} w(t)$ exists. This is a contradiction to the fact that $\lim_{t\to\infty} w(t) = \lim_{t\to\infty} z(t) \neq 0$ implies that $z(t) < 0$ for $t \geq t_3 > t_2$. Assume that $w(t) > 0$ for $t \geq t_1$. Successive integration of the inequality $(r(t)w'(t))' \leq 0$ from t_1 to t, we can find a constant $\eta > 0$ such that $w(t) \leq \eta$ for $t \geq t_2 > t_1$. Using Lemma 1 (ii) with $u(t)$ replaced by $w(t)$, we get $w(t) \geq -r(t)w'(t)R(t)$ and hence $z(t) \geq -r(t)w'(t)R(t)$ for $t \geq t_2$. Indeed, $w(t)$ is bounded, $R(t)$ is bounded and $r(t)w'(t)$ is monotonic imply that $\lim_{t\to\infty}(r(t)w'(t))$ exist. Repeated application of Eq.(1) and use of (H_3) and (H_4) yields

$$\begin{aligned} 0 = {} & (r(t)w'(t))' + G_1(d)(r(t-\tau)w'(t-\tau))' \\ & + f_1(t)G_1(y(t-\sigma_1)) \\ & + G_1(d)f_1(t-\tau)G_1(y(t-\sigma_1-\tau)) \end{aligned} \tag{5}$$

that is,

$$\begin{aligned} 0 \geq {} & (r(t)w'(t))' + G_1(d)(r(t-\tau)w'(t-\tau))'\lambda Q(t)G_1(z(t-\sigma_1)) \\ \geq {} & (r(t)w'(t))' + G_1(d)(r(t-\tau)w'(t-\tau))' \\ & + \lambda Q(t)G_1(-r(t-\sigma_1)w'(t-\sigma_1)R(t-\sigma_1)) \\ = {} & (r(t)w'(t))' + G_1(d)(r(t-\tau)w'(t-\tau))' \\ & + \lambda Q(t)G_1(R(t-\sigma_1))G_1(-r(t-\sigma_1)w'(t-\sigma_1)), \end{aligned} \tag{6}$$

for $t \geq t_3 > t_2 + \sigma_1$. Because $-r(t)w'(t)$ is nondecreasing, we can find a constant $c_1 > 0$ and $t_4 > t_3$ such that $-r(t)w'(t) \geq c_1$, for $t \geq t_4$. Accordingly, the last inequality becomes

$$\lambda Q(t)G_1(c_1)G_1(R(t-\sigma_1)) \leq -(r(t)w'(t))' - G_1(d)(r(t-\tau)w'(t-\tau))'$$

for $t \geq t_5 > t_4 + \sigma_1$ which on integration from t_5 to ∞, we get

$$\int_{t_5}^{\infty} Q(t)G_1(R(t-\sigma_1))dt < \infty,$$

a contradiction to our hypothesis (H_7).

Next, we suppose that $w'(t) > 0$ for $t \geq t_1$. If $w(t) < 0$, then $\lim_{t\to\infty} w(t)$ exists and $0 \neq \lim_{t\to\infty} w(t) = \lim_{t\to\infty} z(t)$ will imply that $z(t) < 0$, which is a contradiction to the fact that $z(t) > 0$. Let $\lim_{t\to\infty} w(t) = 0$. Consequently, $\lim_{t\to\infty} z(t) = 0$ provides $\lim_{t\to\infty} y(t) = 0$ due to $y(t) \leq z(t)$ for $t \geq t_2 > t_1$. Consider, $w(t) > 0$ for $t \geq t_2 > t_1$. By the Lemma 1 (i), it follows that $w(t) \geq CR(t)$ and $z(t) \geq w(t) \geq CR(t)$ for $t \geq t_2$. Accordingly, (6) yields

$$\lambda Q(t)G_1(C)G_1(R(t-\sigma_1)) \leq -(r(t)w'(t))' - G_1(d)(r(t-\tau)w'(t-\tau))'$$

for $t \geq t_2 + \sigma_1$. Integrating the above inequality from t_3 to ∞, we get

$$\int_{t_3}^{\infty} Q(t)G_1(R(t-\sigma_1))dt < \infty, \qquad t_3 > t_2 + 2\sigma_1$$

a contradiction.

If $y(t) < 0$, for $t \geq t_0 \geq 0$, then we set $x(t) = -y(t)$ to obtain $x(t) > 0$ for $t \geq t_0$ and

$$(r(t)(x(t) + p(t)x(t-\tau))')' + f_1(t)G_1(x(t-\sigma_1)) - f_2(t)G_2(x(t-\sigma_2)) = 0.$$

Proceeding as above we obtain a similar contradiction. This completes the proof of the theorem. ■

Theorem 2. *Let $-1 < d \leq p(t) \leq 0$. If (H_0), (H_1), (H_4) and*

$$(H_8) \qquad \int_0^{\infty} f_1(t)G_1(R(t-\sigma_1))dt = \infty,$$

hold, then every solution of (1) either oscillates or tends to zero as $t \to \infty$.

Proof. Let $y(t)$ be a nonoscillatory solution of (1) such that $y(t) > 0$ for $t \geq t_0 \geq 0$. Setting as in (3), we get (4) for $t \geq t_0 + \rho$. Accordingly, $w'(t)$ is a monotonic function on $[t_1, \infty)$ which concludes that either $w(t) > 0$ or $w(t) < 0$ for $t \geq t_2 > t_1$. Consider $w'(t) < 0$ and $w(t) < 0$ for $t \geq t_2$. Then $0 \neq \lim_{t\to\infty} w(t) = \lim_{t\to\infty} z(t)$ yields that $z(t) < 0$ for $t \geq t_2$. Hence $y(t) < y(t-\tau)$ for $t \geq t_3 > t_2$, that is, $y(t)$ is bounded on $[t_3, \infty)$. Consequently, $w(t)$ is bounded and $\lim_{t\to\infty} (r(t)w'(t))$ exists. Because $w(t)$ is monotonic, then $\lim_{t\to\infty} w(t) = L$, $L \in (-\infty, o)$ gives $\lim_{t\to\infty} z(t) = L$. We claim that $\liminf_{t\to\infty} y(t) = 0$. If not, there exists a constant $\gamma > 0$ and $t_4 > t_3$ such that $y(t) \geq \gamma$ for $t \geq t_4$. Integrating (4) from t_4 to ∞, we get

$$\int_{t_4}^{\infty} f_1(t)dt < \infty,$$

a contradiction to the fact that $R(t) \to 0$ as $t \to \infty$ and (H_8) implies that

$$\int_{0}^{\infty} f_1(t)dt = \infty. \tag{7}$$

So our claim holds. By Lemma 3, $L = 0$. Accordingly,

$$\begin{aligned} 0 &= \lim_{t\to\infty} z(t) = \limsup_{t\to\infty} \,[y(t) + p(t)y(t-\tau)] \\ &\geq \limsup_{t\to\infty} \,[y(t) + dy(t-\tau)] \\ &\geq \limsup_{t\to\infty} \, y(t) + \liminf_{t\to\infty} \,(dy(t-\tau)) \\ &= (1+d) \limsup_{t\to\infty} \, y(t) \end{aligned}$$

yields that $\lim_{t\to\infty} y(t) = 0$. Next, we consider the case $w(t) > 0$ for $t \geq t_2$. Let $\lim_{t\to\infty} w(t) = a$, $a \in [o, \infty)$. We claim that $y(t)$ is bounded. If not, there exists an increasing sequence $\{\eta_n\}_{n=1}^{\infty}$ such that $\eta_n \to \infty$ and $y(\eta_n) \to \infty$ as $n \to \infty$ and $y(\eta_n) = \max\{y(t) : t_2 \leq t \leq \eta_n\}$. Hence

$$\begin{aligned} w(\eta_n) &\geq y(\eta_n) + dy(\eta_n - \tau) - K(\eta_n) \\ &\geq (1+d)y(\eta_n) - K(\eta_n) \end{aligned}$$

implies that $w(\eta_n) \to \infty$ as $n \to \infty$, a contradiction to the fact that $\lim_{t\to\infty} w(t)$ exists. So our claim holds and accordingly, $\lim_{t\to\infty} (r(t)w'(t))$ exists. Using Lemma 1 (ii) with $u(t)$ replaced by $w(t)$ we get $w(t) \geq -r(t)w'(t)R(t)$ and hence

$$y(t) \geq w(t) \geq -r(t)w'(t)R(t), \quad t \geq t_3 > t_2.$$

Consquently, (4) becomes

$$f_1(t)G_1(R(t-\sigma_1))G_1(-r(t-\sigma_1)w'(t-\sigma_1)) \leq -(r(t)w'(t))',$$

for $t \geq t_4 > t_3 + \sigma_1$. Due to $r(t)w'(t)$ is nonincreasing, we can find a constant $b > 0$ and $t_5 > t_4 + \sigma_1$ such that $r(t-\sigma_1)w'(t-\sigma_1) \leq -b$ for $t \geq t_5$. Integrating the last inequality from t_5 to ∞, we get

$$\int_{t_5}^{\infty} f_1(t)G_1(R(t-\sigma_1))dt < \infty,$$

a contradiction to (H_8).

Assume that $w'(t) > 0$ for $t \geq t_1$. So we have two cases, $w(t) > 0$ and $w(t) < 0$. If the former holds then by Lemma 1 (i)

$$y(t) \geq w(t) \geq CR(t), \qquad t \geq t_2 > t_1,$$

and hence Eq.(4) can be written as

$$f_1(t)G_1(C\,R(t-\sigma_1)) \leq -(r(t)w'(t))'$$

for $t \geq t_3 > t_2 + \sigma_1$. Integrating the above inequality from t_3 to ∞, we get

$$\int_{t_3}^{\infty} f_1(t)G_1(R(t-\sigma_1))dt < \infty,$$

a contradiction to (H_8). Suppose the latter holds. Then $\lim_{t\to\infty} w(t)$ exists and $0 \neq \lim_{t\to\infty} w(t) = \lim_{t\to\infty} z(t)$ implies that $z(t) < 0$ for $t \geq t_2 > t_1$. Accordingly, $y(t)$ is bounded on $[t_3, \infty)$, $t_3 > t_2 + \rho$. Using the same type of reasoning as above, we obtain $\lim_{t\to\infty} y(t) = 0$. If $0 = \lim_{t\to\infty} w(t) = \lim_{t\to\infty} z(t)$, then we claim that $y(t)$ is bounded. Otherwise there is a contradiction that $w(t) > 0$ as $n \to \infty$. Proceeding as above we obtain $\lim_{t\to\infty} y(t) = 0$. ■

The case $y(t) < 0$ for $t \geq t_0 \geq 0$ is similar. Hence the theorem is proved.

Theorem 3. *Suppose that $-\infty < p_1 \leq p(t) \leq p_2 < -1$. If (H_0), (H_1), (H_4) and (H_8) hold, then every bounded solution of (1) either oscillates or tends to zero as $t \to \infty$.*

Proof. Let $y(t)$ be a bounded nonoscillatory solution of (1) such that $y(t) > 0$ for $t \geq t_0 \geq 0$. Then from (4), it follows that $w'(t) > 0$ or $w'(t) < 0$ for $t \geq t_1 > t_0 + \rho$, where $w(t)$ is given by (3). Consider $w'(t) < 0$ for $t \geq t_1$. Proceeding as in the proof of Theorem 2, we obtain $L = 0$. Consequently,

$$\begin{aligned}
0 &= \lim_{t\to\infty} z(t) = \liminf_{t\to\infty}[y(t) + p(t)y(t-\tau)] \\
&\leq \liminf_{t\to\infty}[y(t) + p_2 y(t-\tau)] \\
&\leq \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}(p_2 y(t-\tau)) \\
&\leq \limsup_{t\to\infty} y(t) + p_2 \limsup_{t\to\infty} y(t-\tau) \\
&= (1+p_2)\limsup_{t\to\infty} y(t)
\end{aligned}$$

implies that $\lim_{t\to\infty} y(t) = 0$, since $(1 + p_2) < 0$. Rest of the proof can be followed from the proof of the Theorem 2 and therefore the proof of the theorem is complete. ■

Example 1. Consider

$$(8)\qquad \left(e^{2t}\left(y(t)+e^{-2\pi}y(t-2\pi)\right)'\right)'+4e^{-4\pi}\left(e^{2t}+e^{-4t}\right)y(t-4\pi) - 4e^{-2(t+\pi)}y(t-2\pi)=0,$$

for $t \geq 0$, where $f_1(t) = 4e^{-4\pi}\left(e^{2t}+e^{-4t}\right)$ and $f_2(t) = 4e^{2(t+\pi)}$. Clearly, $R(t) = \frac{1}{2}e^{-2t}$, $Q(t) = 4\left[e^{2(t-4\pi)} + e^{-4(t-\pi)}\right]$ and (H_0), (H_7) hold. Eq.(8) satisfies all the conditions of Theorem 1. Hence every solution of (8) either oscillates or tends to zero as $t \to \infty$. In particular, $y(t) = e^{-t}\sin t$ is such a solution of (8).

Theorem 4. *Let $0 \leq d(t) \leq p < \infty$. If (H_0), (H_2), (H_3), (H_4), and*

$$(H_9)\qquad \int_0^\infty Q(t)dt = \infty$$

hold, then every solution of (1) either oscillates or tends to zero as $t \to \infty$.

Proof. Let $y(t)$ be a nonoscillatory solution of (1) such that $y(t) > 0$ for $t \geq t_0 \geq 0$. The case $y(t) < 0$ for $t \geq t_0 \geq 0$ can similarly be dealt with. Setting as in (3), we get (4) for $t \geq t_1 > t_0 + \rho$. Hence $r(t)w'(t)$ is a monotonic function on $[t_1, \infty)$. Assume that $w'(t) < 0$ for $t \geq t_1$. Integrating the inequality $(r(t)w'(t))' \leq 0$ from t to T, we get

$$w(t) \leq w(T) + r(T)w'(T)\int_T^t \frac{ds}{r(s)}$$

and hence $w(t) < 0$ due to (H_2). Following to the proof of the Theorem 1, we obtain a contradiction when $w(t) < 0$ for $t \geq t_2 > t_1$. Accordingly, $w'(t) > 0$ for $t \geq t_1$. If $w(t) < 0$, then $\lim_{t\to\infty} w(t)$ exists for which there is a contradiction when $0 \neq \lim_{t\to\infty} w(t) = \lim_{t\to\infty} z(t)$. Let $\lim_{t\to\infty} w(t) = 0$. Using the same type of reasoning as in the proof of Theorem 1, we obtain $\lim_{t\to\infty} y(t) = 0$. Suppose that $w(t) > 0$ for $t \geq t_1$. Consequently, there exists a constant $\alpha > 0$ such that $w(t) \geq \alpha$ for $t \geq t_2 > t_1$, that is, $z(t) \geq w(t) \geq \alpha$ for $t \geq t_2$. Accordingly, (5) yields that

$$\int_{t_3}^\infty Q(t)dt < \infty, \quad t_3 > t_2 + \sigma_1,$$

a contradiction to our assumption (H_9). This completes the proof of the theorem. ■

Remark 4. *In Theorem 4, G_1 could be linear, sublinear or superlinear. However, if we restrict τ and σ_1, G_1 could be sublinear only due to the following theorem.*

Theorem 5. *Let $0 \le p(t) \le d < \infty$, $r'(t) \ge 0$ and $\tau \le \sigma_1$. If (H_0), (H_2), (H_3), (H_4), (H_6) and*

$$(H_{10}) \qquad \int_{T+\sigma_1}^{\infty} Q(t)G_1(\beta(t-\sigma_1))dt = \infty$$

hold, there every solution of (1) either oscillates or tends to zero as $t \to \infty$.

Proof. Proceeding as in the proof of Theorem 4, we consider the case $w'(t) > 0$ and $w(t) > 0$ only for $t \ge t_1$. We note that $r'(t) \ge 0$ implies $w''(t) \le 0$. From (5) it follows that

$$\begin{aligned} 0 \ge\ & (r(t)w'(t))' + G_1(d)\,(r(t-\tau)\,w'(t-\tau))' \\ & + \lambda Q(t)G_1(\beta(t-\sigma_1))G_1(r(t-\sigma_1)w'(t-\sigma_1)) \end{aligned}$$

due to Lemma 2 for $t \ge t_2 > t_1$. Hence

$$\begin{aligned} \lambda Q(t)G_1(\beta(t-\sigma_1)) \le\ & -\ \left[G_1(r(t-\sigma_1)w'(t-\sigma_1))\right]^{-1}(r(t)w'(t))' \\ & -\ G_1(d)\left[G_1(r(t-\sigma_1)w'(t-\sigma_1))\right]^{-1} \\ & \times (r(t-\tau)w'(t-\tau))'. \end{aligned}$$

Because $\lim_{t\to\infty}(r(t)w'(t))$ exists, then use of (H_6) to the above inequality, we obtain

$$\int_{T+\sigma_1}^{\infty} Q(t)G_1(\beta(t-\sigma_1))dt < \infty$$

a contradiction to our hypothesis (H_{10}). Hence the theorem is proved. ■

Theorem 6. *Let $-1 < d \le p(t) \le 0$. If (H_0), (H_2), (H_4) and (7) hold, then a solution of (1) either oscillates or tends to zero as $t \to \infty$.*

Proof. Using the same type of reasoning as in the proof of the Theorem 4, we obtain $w(t) < 0$ for $t \ge t_2 > t_1$ when $w'(t) < 0$. Accordingly, $w(t)$ is monotonic function on $[t_2, \infty)$ and $0 \ne \lim_{t\to\infty} w(t) = \lim_{t\to\infty} z(t)$ exists. Following to Theorem 2 we get $\lim_{t\to\infty} y(t) = 0$.

Let $w'(t) > 0$ for $t \ge t_1$. If $w(t) < 0$ for $t \ge t_2 > t_1$, then we can use same arguments as in Theorem 2, to obtain $\lim_{t\to\infty} y(t) = 0$. Suppose that $w(t) > 0$ for $t \ge t_2 > t_1$. Then there exists a constant $\gamma > 0$ and $t_3 > t_2$ such that $w(t) \ge \gamma, t \ge t_3$. Consequently, $y(t) \ge w(t) \ge \gamma$ for $t \ge t_3$. Integrating (4) from $t_3 + \sigma_1$ to ∞, a contradiction is obtained to (7). Hence the theorem is proved. ■

Theorem 7. *Let* $-\infty < p_1 \le p(t) \le p_2 < -1$. *If* (H_0), (H_2), (H_4) *and (7) hold, then every bounded solution of (1) either oscillates or tends to zero as* $t \to \infty$.

The proof of the theorem can be followed from the Theorem 6 and Theorem 3.

4. Oscillation properties of Eq.(2)

In the following, we obtain sufficient conditions for oscillation of solutions of forced equation (2). Let

(H_{11}) there exists $F \in C([0,\infty), R)$ such that $F(t)$ changes sign, with $-\infty < \liminf_{t\to\infty}(F(t)) < 0 < \limsup_{t\to\infty} F(t) < \infty$, $rF' \in C([0,\infty), R)$ and $(rF')' = g$

$$(\mathrm{H}_{12}) \qquad F^+(t) = \max\{F(t), 0\} \text{ and } F^-(t) = \max\{-F(t), 0\}.$$

Theorem 8. *Let* $0 \le p(t) \le d < \infty$. *Assume that* (H_0), (H_3), (H_4), (H_5), (H_{11}) *and* (H_{12}) *hold. If*

$$(H_{13}) \qquad \int_{\sigma_1}^{\infty} Q(t)G_1(F^+(t-\sigma_1))dt = \infty = \int_{\sigma_1}^{\infty} Q(t)G_1(F^-(t-\sigma_1))dt$$

hold, then all solutions of (2) oscillate.

Proof. Let y(t) be a nonoscillatory solution of (2). Hence there exists $t_0 \ge 0$ such that $y(t) > 0$ or $y(t) < 0$ for $t \ge t_0$. Suppose that $y(t) > 0$ for $t \ge t_0$. Setting $z(t)$ and $w(t)$ as in (3), let

$$(9) \qquad U(t) = w(t) - F(t).$$

Thus Eq.(2) becomes

$$(10) \qquad \big(r(t)U'(t)\big)' = -f_1(t)G_1(y(t-\sigma_1)) \le 0, \not\equiv 0$$

for $t \ge t_1 > t_0 + \rho$. Accordingly, $U'(t)$ and $U(t)$ are monotonic functions. Assume that $U'(t) < 0$ for $t \ge t_1$. If $U(t) < 0$ for $t \ge t_2 > t_1$, then $z(t) < K(t) + F(t)$ and hence

$$\begin{aligned} 0 &= \liminf_{t\to\infty} z(t) \le \liminf_{t\to\infty}(K(t) + F(t)) \\ &\le \limsup_{t\to\infty} K(t) + \liminf_{t\to\infty} F(t) \\ &= \lim_{t\to\infty} K(t) + \liminf_{t\to\infty} F(t) < 0, \end{aligned}$$

a contradiction to the fact that $z(t) > 0$. Hence $U(t) > 0$ for $t \geq t_2$, that is, $z(t) > K(t) + F(t) \geq K(t) + F^+(t) > F^+(t)$ for $t \geq t_2$. Using Eq.(2) and (9) we obtain

$$\begin{aligned} 0 = &(r(t)\,u'(t))' + G_1(d)(r(t-\tau)u'(t-\tau))' + f_1(t)G_1(y(t-\sigma_1)) \\ &+ G_1(d)f_1(t-\tau)G_1(y(t-\sigma_1-\tau)) \end{aligned}$$

that is,

$$0 \geq (r(t)u'(t))' + G_1(d)(r(t-\tau)u'(t-\tau))' + \lambda Q(t)G_1(z(t-\sigma_1)) \tag{11}$$

due to (H_3) and (H_4). Thus

$$(r(t)U'(t))' + G_1(d)\,(r(t-\tau)\,U'(t-\tau))' + \lambda\,Q(t)\,G_1(F^+(t-\sigma_1)) \leq 0$$

for $t \geq t_3 > t_2$. We note that $\lim\limits_{t\to\infty} u(t)$ exists. If $y(t)$ is unbounded, then

$$U(t) = z(t) - K(t) - F(t) > y(t) - F(t) - K(t)$$

implies that $U(t)$ is unbounded. Consequently, $y(t)$ is bounded, on $[t_4, \infty)$, $t_4 > t_3$, that is, $\lim\limits_{t\to\infty}(r(t)U'(t))$ exists. Integrating the last inequality following to (11) from t_4 to ∞, we obtain

$$\int_{t_4}^{\infty} Q(t)G_1(F^+(t-\sigma_1))dt < \infty,$$

a contradiction to our hypothesis (H_{13}).

Next, we suppose that $U'(t) > 0$ for $t \geq t_1$. Then $\lim\limits_{t\to\infty}(r(t)U'(t))$ exists. Similar contradictions hold when we consider the cases $U(t) > 0$ and $U(t) < 0$ for $t \geq t_2 > t_1$.

If $y(t) < 0$ for $t \geq t_0$, then we set $x(t) = -y(t)$ to obtain $x(t) > 0$ for $t \geq t_0$ and

$$\begin{aligned} (r(t)(x(t) + p(t)x(t-\tau))')' &+ f_1(t)G_1(x(t-\sigma_1)) \\ &- f_2(t)G_2(x(t-\sigma_2)) = \tilde{g}(t), \end{aligned}$$

where $\tilde{g}(t) = -g(t)$. If $\tilde{F}(t) = -F(t)$, then $(r(t)\tilde{F}'(t))' = -g(t) = \tilde{g}(t)$ and $\tilde{F}(t)$ changes sign. Further $\tilde{F}^+(t) = F^-(t)$ and $\tilde{F}^-(t) = \tilde{F}^+(t)$. Proceeding as above we obtain a contradiction. Thus the proof of the theorem is complete. ■

Theorem 9. *Let* $-1 < d \le p(t) \le 0$. *Suppose that* (H_0), (H_4), (H_{11}), (H_{12}) *and*

$$(H_{14}) \qquad \int_{\sigma_1}^{\infty} f_1(t)G_1(F^-(t+\tau-\sigma_1))dt = \infty = \int_{\sigma_1}^{\infty} f_1(t)G_1(F^+(t-\sigma_1))dt$$

and

$$(H_{15}) \qquad \int_{\sigma_1}^{\infty} f_1(t)G_1(F^-(t-\sigma_1))dt = \infty = \int_{\sigma_1}^{\infty} f_1(t)G_1(F^+(t+\tau-\sigma_1))dt$$

hold, then (2) is oscillatory.

Proof. Suppose for contrary that $y(t)$ is a nonoscillatory solution of (2) such that $y(t) > 0$ for $t \ge t_0$. Setting as in (3) and (9), we get (10). Hence $U'(t)$ is a monotonic function on $[t_1, \infty)$, $t_1 > t_0+\rho$. Let $U'(t) < 0$ for $t \ge t_1$. Accordingly, $U(t)$ is a monotonic function and $\lim_{t\to\infty} U(t) = \lim_{t\to\infty} (z(t) - F(t))$ implies that $z(t) - F(t) < 0$ when $U(t) < 0$, that is, $z(t) < F(t)$ for $t \ge t_2 > t_1$. If $z(t) > 0$, then $F(t) > 0$ which is absurd. Hence $z(t) < 0$ for $t \ge t_2$. Ultimately, $z(t) < -F^-(t)$ for $t \ge t_2$ and

$$dy(t-\tau) \le p(t)y(t-\tau) < z(t) < -F^-(t)$$

yields that $y(t-\sigma_1) > F^-(t+\tau-\sigma_1)$ for $t \ge t_3 > t_2$. On the otherhand, $y(t)$ is bounded due to $z(t) < 0$, that is, $y(t) < y(t-\tau)$ and hence $\lim_{t\to\infty} (r(t)\,U'(t))$ exists. Integrating (10) from t_3 to ∞, we get

$$\int_{t_3}^{\infty} f_1(t)G_1(F^-(t+\tau-\sigma_1))dt < \infty,$$

a contradiction to our hypothesis. Next, we suppose that $U(t) > 0$ for $t \ge t_2 > t_1$. Hence $\lim_{t\to\infty} U(t) = \lim_{t\to\infty} (z(t) - F(t))$ implies that $z(t) - F(t) > 0$ if $\lim_{t\to\infty} U(t) \ne 0$, that is, $z(t) > F(t)$ for $t \ge t_2$. Ultimately, $y(t) > F^+(t)$ for $t \ge t_3 > t_2$. We claim that $y(t)$ is bounded. If not, there exists an increasing sequence $\{\eta_n\}_{n=1}^{\infty}$ such that $\eta_n \to \infty$ and $y(\eta_n) \to \infty$ as $n \to \infty$ and

$$y(\eta_n) = \max\{y(t) : t_3 \le t \le \eta_n\}.$$

Hence

$$\begin{aligned} U(\eta_n) &\ge y(\eta_n) + d\, y(\eta_n - \tau) - K(\eta_n) - F(\eta_n) \\ &\ge (1+d)y(\eta_n) - K(\eta_n) - F(\eta_n) \end{aligned}$$

implies that $U(\eta_n) \to \infty$ as $n \to \infty$, a contradiction to the fact that $\lim_{t\to\infty} U(t)$ exists. So our claim holds and $\lim_{t\to\infty}(r(t)U'(t))$ exists. Integrating (10) from t_3 to ∞, we obtain

$$\int_{t_3}^{\infty} f_1(t)G_1(F^+(t-\sigma_1))dt < \infty,$$

a contradiction to the hypothesis (H_{14}). If $\lim_{t\to\infty} U(t) = 0$, then $z(t)-F(t) > 0$ or $z(t) - F(t) < 0$ for $t \geq t_2$. In either case we have a contradiction.

Assume that $U'(t) > 0$ for $t \geq t_1$. Then $\lim_{t\to\infty}(r(t)U'(t))$ exists. Proceeding in the lines of the above argument, we obtain similar contradictions for the cases $U(t) < 0$ and $U(t) > 0$. $y(t) < 0$ for $t \geq t_0$ is similar. Hence the theorem is proved. ■

Theorem 10. *Let $-\infty < d \leq p(t) \leq -1$. If all the conditions of Theorem 9 hold, then every bounded solution of (2) is oscillatory.*

Proof. The proof follows from the Theorem 9 and hence the details are omitted. ■

Theorem 11. *Assume that $0 \leq p(t) \leq d < \infty$. If (H_0), (H_1), (H_3) - (H_5), (H_7), (H_{11}) and (H_{12}) hold, then (2) is oscillatory.*

Proof. Proceeding as in the proof of the Theorem 8, $U(t) < 0$ is not possible when $U'(t) < 0$ for $t \geq t_1$. Hence $U(t) > 0$ for $t \geq t_2 > t_1$. Using Lemma 1 (ii) with $u(t)$ is replaced by $U(t)$, we get $U(t) \geq -r(t)U'(t)R(t)$ and hence

$$\begin{aligned} z(t) &\geq -r(t)U'(t)R(t) + K(t) + F(t) \\ &\geq -r(t)U'(t)R(t) + K(t) + F^+(t) \\ &> -r(t)U'(t)R(t) \end{aligned}$$

for $t \geq t_2$. Further, $r(t)U'(t)$ is non-increasing. So we can find a constant $c_1 > 0$ and $t_3 > t_2$ such that $-r(t)U'(t) \geq -c_1$ for $t \geq t_3$. Hence inequality (6) becomes

$$\lambda Q(t)G_1(-c_1)G_1(R(t-\sigma_1)) \leq -(r(t)U'(t))' - G_1(d)(r(t-\tau)U'(t-\tau))',$$

where $w(t)$ is replaced by $U(t)$ for $t \geq t_4 > t_3 + \sigma_1$. Since $\lim_{t\to\infty} U(t)$ exists, we claim that $y(t)$ is bounded. Otherwise, following to Theorem 8., $U(t)$

is unbounded. Consequently, $\lim_{t\to\infty}(r(t)U'(t))'$ exists. Integrating the last inequality from t_4 to ∞, we obtain

$$\int_{t_4}^{\infty} Q(t)G_1(R(t-\sigma_1))dt < \infty,$$

a contradiction to (H_7).

Let $U'(t) > 0$ for $t \geq t_1$. The argument for the case $U(t) < 0$ is same. Consider the case $U(t) > 0$ for $t \geq t_2 > t_1$. By Lemma 1 (i), it follows that $U(t) \geq CR(t)$, that is,

$$z(t) \geq CR(t) + K(t) + F^{+}(t) > CR(t)$$

for $t \geq t_2$. Using the same type of reasoning as in the proof of the Theorem 1, we get a contradiction to our hypothesis (H_7). This completes the proof of the theorem. ■

Theorem 12. *If* $0 \leq p(t) \leq d < \infty$ *and* (H_0), (H_2), (H_3), (H_4), (H_9), (H_{11}) *and* (H_{12}) *hold, then every solution of (2) oscillates.*

Proof. Proceeding as in the proof of Theorem 8, we assume that $U'(t) < 0$ for $t \geq t_1$. Accordingly, $U(t) < 0$ for $t \geq t_2 > t_1$ due to (H_2). Using the same type of reasoning as in the proof of Theorem 8, $U(t) < 0$ is a contradiction. Hence $U'(t) > 0$ for $t \geq t_1$. Ultimately, $U(t) > 0$, $t \geq t_2 > t_1$. Since $U(t)$ is nondecreasing, there exists a constant $\alpha > 0$ and $t_3 > t_2$ such that $U(t) \geq \alpha$, $t \geq t_3$. Therefore

$$z(t) \geq \alpha + K(t) + F(t) \geq \alpha + K(t) + F^{+}(t) > \alpha$$

for $t \geq t_3$. Using the last inequality and then integrating (11) from t_4 to ∞, we get

$$\int_{t_4}^{\infty} Q(t)dt < \infty, \qquad t_4 > t_3 + \sigma_1$$

a contradiction to our hypothesis (H_9). This completes the proof of the theorem. ■

Theorem 13. *Assume that* $0 \leq p(t) \leq d < \infty$, $r'(t) \geq 0$ *and* $\tau \leq \sigma_1$. *If* (H_2), (H_3), (H_4), (H_6), (H_{10}), (H_{11}) *and* (H_{12}) *hold, then (2) is oscillatory.*

The proof of the the theorem can be followed from the Theorems 5 and 12 and hence the details are omitted.

Theorem 14. *Assume that $-1 < d \leq p(t) \leq 0$. If (H_0), (H_1), (H_4), (H_8), (H_{11}) and (H_{12}) hold, then (2) is oscillatory.*

Proof. Proceeding as in the proof of the Theorem 9 and using the same type of reasoning, we consider the case $U'(t) < 0$, $U(t) < 0$ and $z(t) < 0$ for $t \geq t_2$. Accordingly, $y(t) < y(t-\tau)$, that is, $y(t)$ is bounded on $[t_2, \infty)$. Hence $U(t)$ is bounded and $\lim_{t\to\infty}(r(t)U'(t))$ exists. Using the fact that $dy(t-\tau) < z(t) < -F^-(t)$ and $F(t)$ is bounded, we may conclude that $\liminf_{t\to\infty} y(t) \neq 0$. On the otherhand when (H_9) hold, $\liminf_{t\to\infty} y(t) = 0$, a contradiction. Consequently, $U(t) > 0$ for $t \geq t_2 > t_1$. Using Lemma 1 *(ii)*, we have $U(t) \geq -r(t)U'(t)R(t)$ and hence for $t \geq t_2$,

$$z(t) \geq -r(t)U'(t)R(t) + K(t) + F^+(t)$$

that is,

$$y(t) \geq -r(t)U'(t)R(t) + K(t) + F^+(t) > -r(t)U'(t)R(t).$$

Further, $r(t)U'(t)$ is nonincreasing. So we can find a constant $C_1 > 0$ and $t_3 > t_2$ such that $-r(t)U'(t) \geq -C_1$ for $t \geq t_3$. Hence for $t \geq t_3$, $y(t) > -C_1R(t)$. Integrating (10) from t_4 to ∞, we get

$$G_1(-C_1)\int_{t_4}^{\infty} f_1(t)G_1(R(t-\sigma_1))dt < -\int_{t_4}^{\infty}(r(t)U'(t))'dt,$$

$t_4 > t_3 + \sigma_1$. On the otherhand, $\lim_{t\to\infty} U(t)$ exists which implies that $y(t)$ is bounded. Otherwise, by Theorem 9, $U(\eta_n) \to \infty$ as $n \to \infty$. Thus $\lim_{t\to\infty}(r(t)U'(t))$ exists and the last integral becomes

$$\int_{t_4}^{\infty} f_1(t)G_1(R(t-\sigma_1))dt < \infty.$$

a contradiction to (H_8).

Let $U'(t) > 0$ for $t \geq t_1$. Then $\lim_{t\to\infty}(r(t)U'(t))$ exists. Similar contradictions can be obtained for $U(t) > 0$ and $U(t) < 0$ for $t \geq t_2 > t_1$. The case $y(t) < 0$ for $t \geq t_0$ is similar. Hence the proof of the theorem is complete. ■

Theorem 15. *Let $-\infty < d \leq p(t) \leq -1$. If all the conditions of Theorem 14 are satisfied, then every bounded solution of (2) is oscillatory.*

The proof follows from the Theorem 14.

Theorem 16. *If* $-1 < d \le p(t) \le 0$ *and* (H_0), (H_2), (H_4), (H_{11}), (H_{12}) *and (7) hold, then (2) is oscillatory.*

The proof of the theorem follows from the proof of the Theorems 14 and 12. Accordingly, the proof of the theorem is complete.

Theorem 17. *Suppose that* $-\infty < d \le p(t) \le -1$. *Let all the conditions of Theorem 16 be hold. Then every bounded solution of (2) is oscillatory.*

Remark 5. In Theorems 8 - 10, (H_1) and (H_2) are not required to show that Eq.(2) is oscillatory. This happened due to the analysis presented here. However, presence of $r(t)$ and to predict sufficient conditions with $r(t)$ is more interesting than the former ones.

Example 2. Consider

$$\left(e^t\left(y(t) - e^{-t}y(t-\pi)\right)'\right)' + f_1(t)y(t-2\pi) - f_2(t)y(t-\pi) = g(t), \tag{12}$$

for $t \ge 0$, where $R(t) = e^{-t}$, $f_1(t) = (e^t + e^{-t} + 1)$, $f_2(t) = e^{-t}$ and $g(t) = (1 - e^t)\sin t$. Clearly (H_8) hold. If we set

$$F(t) = \frac{1}{2}(1 + e^{-t})cost + \frac{1}{2}(1 - e^{-t})sint,$$

then it is easy to verify that $(r(t)F'(t))' = g(t)$ and (H_{11}), (H_{12}) hold. Eq.(12) satisfies all the conditions of Theorem 14. Hence every solution of (12) oscillates. Indeed, $y(t) = cost$ is such an oscillatory solution of (12).

Example 3. *Theorem 12 can be applied to*

$$\left(y(t) + e^{-t}y(t-\pi)\right)'' + f_1(t)y(t-2\pi) - f_2(t)y^3(t-4\pi) = g(t),$$

for $t \ge 0$, *where* $f_1(t) = (3e^{-t} + 1), f_2(t) = 4e^{-t}$ *and* $g(t) = 2e^{-t}cost + e^{-t}\sin 3t$. *Clearly,* (H_0), (H_2) *and* (H_9) *hold. If we set*

$$F(t) = e^{-t}\left(\frac{3}{50}\cos 3t - \frac{4}{50}\sin 3t - \sin t\right),$$

then it is easy to verify that $F''(t) = g(t)$ *and* (H_{11}), (H_{12}) *hold. Indeed,* $y(t) = \sin t$ *is such an oscillatory solution of the above equation.*

5. Existence of positive solutions

In this section, necessary conditions are obtained to show that Eq.(12) admits a positive bounded solution.

Theorem 18. *Let G_i, $= 1,2$ be Lipschitzian on the intervals of the form $[a,b]$, $0 < a < b < \infty$. Suppose that $g(t)$ satisfies (H_{11}) and (H_{12}). If*

$$\int_0^\infty A(t)f_i(t)dt < \infty, \qquad i = 1,2$$

where $A(t) = \int_{T_0}^{t} \frac{d\theta}{r(\theta)}$, then Eq.(2) admits a positive bounded solution.

Proof. The proof of the theorem is divided accordingly with respect to different ranges of $p(t)$.

Let $0 \le p(t) \le b_1 < 1$. It is possible to find T_0 large enough such that

$$M_1 \int_{T_0}^\infty A(t)f_1(t)dt < \frac{1-b_1}{10}, \qquad M_2 \int_{T_0}^\infty A(t)f_2(t)dt < \frac{1-b_1}{20},$$

where $M_1 = \max\{L_1, G_1(1)\}$, $M_2 = \max\{L_2, G_2(1)\}$ and L_1, L_2 are Lipschitz constants on $\left[\frac{1-b_1}{10}, 1\right]$. Let $F(t)$ be such that $-\frac{(1-b_1)}{20} \le F(t) \le \frac{1-b_1}{10}$ for $t \ge T_0$.

Let $BC([T_0,\infty), R)$ be the Banach space of all bounded real valued continuous functions $x(t)$, $t \ge T_0$ with supremum norm defined by

$$\|x\| = \sup\{|x(t)| : t \ge T_0\}.$$

Set

$$S = \left\{x \in X : \frac{1-b_1}{10} \le x(t) \le 1, \ \ t \ge T_0\right\}.$$

For $y \in S$, define

$$(Ty)(t) = \begin{cases} Ty(T_0+\rho), & T_0 \le t \le T_0+\rho \\ -p(t)y(t-\tau) + \frac{1+4b_1}{5} + F(t) \\ +A(t)\int_t^\infty [f_1(s)G_1(y(s-\sigma_1)) - F_2(s)G_2(y(s-\sigma_2))]ds \\ +\int_{T_0}^t A(s)[f_1(s)G_1(y(s-\sigma_1)) - F_2(s)G_2(y(s-\sigma_2))]ds, \\ & t \ge T_0+\rho. \end{cases}$$

Clearly, Ty is continuous. For every $t \ge T_0$,

$$Ty(t) \le \frac{1+4b_1}{5} + \frac{1-b_1}{10} + A(t)\int_t^\infty f_1(s)G_1(1)ds + \int_{T_0}^\infty A(s)f_1(s)G_1(1)ds$$

$$\leq \frac{1+4b_1}{5} + \frac{1-b_1}{10} + \int_t^\infty A(s)f_1(s)G_1(1)ds + \int_{T_0}^\infty A(s)f_1(s)G_1(1)ds$$

$$\leq \frac{1+4b_1}{5} + \frac{1-b_1}{10} + G_1(1)\int_{T_0}^\infty A(s)f_1(s)ds$$

$$\leq \frac{1+3b_1}{5} < 1.$$

We note that $A(t)$ is a nondecreasing function. Again for every $t \geq T_0$,

$$Ty(t) \geq -b_1 + \frac{1+4b_1}{5} - \frac{1-b_1}{20} - \frac{1-b_1}{20} = \frac{1-b_1}{10}.$$

Thus $T : S \to S$. Further, for $x, y \in S$,

$$\begin{aligned}
|Tx(t) - Ty(t)| \leq\ & b_1\|x-y\| \\
& + \left| A(t) \int_t^\infty f_1(s)[G_1(x(s-\sigma_1)) - G_1(y(s-\sigma_1))]ds \right| \\
& + \left| \int_{T_0}^\infty A(s)f_1(s)\left[G_1(x(s-\sigma_1)) - G_1(y(s-\sigma_1))\right]ds \right| \\
& + \left| A(t) \int_t^\infty f_2(s)\left[G_2(x(s-\sigma_1)) - G_2(y(s-\sigma_2))\right]ds \right| \\
& + \left| \int_{T_0}^\infty A(s)f_2(s)\left[G_2(x(s-\sigma_1)) - G_2(y(s-\sigma_1))\right]ds \right| \\
\leq\ & b_1\|x-y\| + L_1\|x-y\| \int_t^\infty A(s)f_1(s)ds \\
& + L_1\|x-y\| \int_{T_0}^\infty A(s)f_1(s)ds \\
& + L_2\|x-y\| \int_t^\infty A(s)f_2(s)ds + L_2\|x-y\| \int_{T_0}^\infty A(s)f_2(s)ds \\
\leq\ & b_1\|x-y\|
\end{aligned}$$

$$+\left[M_1\int_{T_0}^{\infty}A(s)f_1(s)ds+M_2\int_{T_0}^{\infty}A(s)f_2(s)ds\right]\|x-y\|$$
$$<\frac{3+17b_1}{20}\|x-y\|$$

implies that $\|Tx-Ty\|<\frac{3+17b_1}{20}\|x-y\|$. Thus T is a contraction. Consequently, T has a unique fixed point y in S. It is the required solution of (2).

In the other ranges of $p(t)$, the above procedure is same. Hence without details, the necessary informations are given below :

(ii) Let $-1<b_1\le p(t)\le 0$. Choose T_0 sufficiently large such that for $t\ge T_0$,

$$M_1\int_{T_0}^{\infty}A(t)f_1(t)dt<\frac{1+b_1}{10},\qquad M_2\int_{T_0}^{\infty}A(t)f_2(t)dt<\frac{1+b_1}{20},$$

and $-\frac{(1+b_1)}{20}\le F(t)\le\frac{1+b_1}{10}$. We set

$$S=\left\{x\in X:\frac{1+b_1}{10}\le x(t)\le 1,\quad t\ge T_0\right\}$$

and

$$(Ty)(t)=\begin{cases}Ty(T_0+\rho), & T_0\le t\le T_0+\rho\\ -p(t)y(t-\tau)+\frac{1+b_1}{5}+F(t) & \\ +A(t)\int_t^{\infty}[f_1(s)\,G_1(y(s-\sigma_1))-f_2(s)G_2(y(s-\sigma_2))]ds & \\ +\int_{T_0}^{t}A(s)\left[f_1(s)G_1(y(s-\sigma_1))-f_2(s)G_2(y(s-\sigma_2))\right]ds, & \\ & t\ge T_0+\rho.\end{cases}$$

(iii) Let $-1<b_1\le p(t)\le b_2<1$, $b_1<0$, $b_2>0$ be such that $b_2<1+5b_1$. Choose T_0 sufficiently large such that

$$M_1\int_{T_0}^{\infty}A(t)f_1(t)dt<\frac{b_1}{2}+\frac{1-b_2}{2},\qquad M_2\int_{T_0}^{\infty}A(t)f_2(t)dt<\frac{1-b_2}{20},$$

and $-\frac{(1-b_2)}{20}\le F(t)\le\frac{b_1}{2}+\frac{1-b_2}{10}$ for $t\ge T_0$. Here, we set

$$S=\left\{x\in X:\frac{1-b_2}{10}\le x(t)\le 1,\quad t\ge T_0\right\}$$

and

$$(Ty)(t) = \begin{cases} Ty(T_0+\rho), & T_0 \le t \le T_0+\rho \\ -p(t)y(t-\tau) + \frac{1+4b_2}{5} + F(t) \\ +A(t)\int\limits_t^\infty [f_1(s)G_1(y(s-\sigma_1)) - f_2(s)G_2(y(s-\sigma_2))]ds \\ +\int\limits_{T_0}^t A(s)[f_1(s)G_1(y(s-\sigma_1)) - f_2(s)G_2(y(s-\sigma_2))]ds, \\ & t \ge T_0+\rho. \end{cases}$$

(*iv*) $p(t) \equiv -1$. Let b_1 be such that $0 < b_1 < 1, b_1 \neq \frac{1}{2}$. We can find T_0 large enough such that

$$M_1 \int\limits_{T_0}^\infty A(t)f_1(t)dt < \frac{1-2b_1}{20}, \qquad M_2 \int\limits_{T_0}^\infty A(t)f_2(t)dt < \frac{1-2b_1}{40}$$

and $-\frac{(1-2b_1)}{40} \le F(t) \le \frac{1-2b_1}{20}$, for $t \ge T_0$. We set

$$S = \left\{x \in X : \frac{1-b_1}{20} \le x(t) \le b_1, \quad t \ge T_0\right\}$$

and

$$(Ty)(t) = \begin{cases} Ty(T_0+\rho), & T_0 \le t \le T_0+\rho \\ -y(t-\tau) + \frac{1-b_1}{10} + F(t) \\ +A(t)\int\limits_t^\infty [f_1(s)\, G_1(y(s-\sigma_1)) - F_2(S)G_2(y(s-\sigma_2))]ds \\ +\int\limits_{T_0}^t A(s)[f_1(s)G_1(y(s-\sigma_1)) - F_2(s)G_2(y(s-\sigma_2))]ds, \\ & t \ge T_0+\rho. \end{cases}$$

(*v*) $p(t) \equiv 1$. Let $-1 < b_1 < 0$ be such that $b_1 \neq -\frac{1}{2}$. Replacing $-b_1$ in the place of b_1 of the above settings of (*iv*), we obtain the needed.

(*vi*) Let $-\infty < b_1 \le p(t) \le b_2 < -1$. It is possible to find T_0, large enough such that

$$M_1 \int\limits_{T_0}^\infty A(t)f_1(t)dt < \frac{-b_2}{2(D-b_2)}, \qquad M_2 \int\limits_{T_0}^\infty A(t)f_2(t)dt < \frac{-b_2}{2(D-b_2)}$$

and $-\frac{1}{2(D-b_2)} \le F(t) \le \frac{1}{2(D-b_2)}$ for $t \ge T_0$, where $D > \max\left\{-b_1, b_2 + \frac{b_2}{1+b_2}\right\}$. We set

$$S = \left\{x \in X : \frac{-b_2}{D-b_2} \le x(t) \le K, \quad t \ge T_0\right\}$$

and

$$(Ty)(t) = \begin{cases} Ty(T_0+\rho), & T_0 \le t \le T_0+\rho \\ -\frac{y(t+\tau)}{p(t+\tau)} - \frac{D(2-b_2)}{p(t+\tau)(D-b_2)} + \frac{F(t+\tau)}{p(t+\tau)} \\ +\frac{A(t+\tau)}{p(t+\tau)} \int\limits_{t+\tau}^{\infty} [f_1(s)\, G_1(y(s-\sigma_1)) - f_2(s)G_2(y(s-\sigma_2))]ds \\ + \int\limits_{T_0}^{t+\tau} A(s)[f_1(s)G_1(y(s-\sigma_1)) - f_2(s)G_2(y(s-\sigma_2))]ds, \\ & t \ge T_0+\rho \end{cases}$$

where $K = \frac{2D-b_2(D+1)}{(b_2-D)(1+b_2)} > 0$.

(*vii*) Let $1 < b_1 \le p(t) \le b_2 < \frac{1}{2}b_1^2$. It is possible to find T_0, large enough such that

$$M_1 \int\limits_{T_0}^{\infty} A(t)f_1(t)dt < \frac{b_1-1}{8b_1} + \frac{b_1-1}{16b_2}, \qquad M_2 \int\limits_{T_0}^{\infty} f_2(t)A(t)dt < \frac{b_1-1}{16b_2}$$

and $-\frac{b_1-1}{16b_1b_2} \le F(t) \le \frac{b_1-1}{8b_1^2} + \frac{b_1-1}{16b_1b_2}$ for $t \ge T_0$. Set

$$S = \left\{ x \in X : \frac{b_1-1}{8b_1b_2} \le x(t) \le 1, \quad t \ge T_0 \right\}$$

and

$$(Ty)(t) = \begin{cases} Ty(T_0+\rho), & T_0 \le t \le T_0+\rho \\ -\frac{y(t+\tau)}{p(t+\tau)} + \frac{2b_1^2+b_1-1}{4b_1p(t+\tau)} + \frac{F(t+\tau)}{p(t+\tau)} \\ +\frac{A(t+\tau)}{p(t+\tau)} \int\limits_{t+\tau}^{\infty} [f_1(s)G_1(y(s-\sigma_1)) - f_2(s)G_2(y(s-\sigma_2))]ds \\ + \int\limits_{T_0}^{t+\tau} A(s)[f_1(s)G_1(y(s-\sigma_1)) - f_2(s)G_2(y(s-\sigma_2))]ds, \\ & t \ge T_0+\rho. \end{cases}$$

Hence the proof of the theorem is complete. ■

6. Summary

It is worth observation that both unforced and forced Eqs.(1) and (2) are studied under (H_1) and (H_2) keeping inview of the key assumptions (H_7), (H_8) and (H_10). The results concerning Eq.(1) are not completely oscillatory due to the analysis incorporated here. However, Eq.(2) provides complete oscillatory results. Of course influence of forcing term can be considered. It seems that some extra conditions are required to see that Eq.(1) is oscillatory.

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ARUN KUMAR TRIPATHY
DEPARTMENT OF MATHEMATICS
KAKATIYA INSTITUTE OF TECHNOLOGY AND SCIENCE
WARANGAL-506015, INDIA
e-mail: arun_tripathy70@rediffmail.com

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