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C. Carpintero, N. Rajesh and E. Rosas

ON A CLASS OF (γ, γ')-PREOPEN SETS IN A TOPOLOGICAL SPACE

Abstract. In this paper we have introduced the concept of (γ, γ')-preopen sets in topological space.

Key words: topological spaces, preopen set, (γ, γ')-open set, (γ, γ')-preopen set.

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1. Introduction

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real analysis concerns the various modified forms of continuity, seperation axioms etc. by utilizing generalized open sets. Kasahara [1] defined the concept of an operation on topological spaces and introduced the concept of γ-closed graphs of a function. In 1992, Umehara et. al. [4] introduced a new class of open sets called (γ, γ')-open sets into the field of General Topology. In this paper, we have introduced and studied the notion of (γ, γ')-preopen sets by using operations γ and γ' on a topological space (X, τ). We also introduced (γ, γ')-precontinuous functions and (γ, γ')-prehomeomorphisms and investigate some important properties.

2. Preliminaries

The closure and the interior of A of X are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A of X is said to be preopen [2] $A \subset \mathrm{Int}(\mathrm{Cl}(A))$.

Definition 1 ([1]). *Let (X, τ) be a topological space. An operation γ on the topology τ is function from τ on to power set $\mathcal{P}(X)$ of X such that $V \subset V^\gamma$ for each $V \in \tau$, where V^γ denotes the value of γ at V. It is denoted by $\gamma : \tau \to \mathcal{P}(X)$.*

Definition 2. *Let (X, τ) be a topological space. An operation γ is said to be regular [1] if, for every open neighborhood U and V of each $x \in X$, there exists an open neighborhood W of x such that $W^{\gamma} \subset U^{\gamma} \cap V^{\gamma}$.*

Definition 3. *A subset A of a topological space (X, τ) is said to be (γ, γ')-open set [4] if for each $x \in A$ there exist open neighbourhoods U and V of x such that $U^{\gamma} \cup V^{\gamma'} \subset A$. The complement of (γ, γ')-open set is called (γ, γ')-closed. $\tau_{(\gamma,\gamma')}$ denotes set of all (γ, γ')-open sets in (X, τ).*

Definition 4. *The point $x \in X$ is in the (γ, γ')-closure [4] of a set $A \subset X$ if $(U^{\gamma} \cup W^{\gamma'}) \cap A \neq \emptyset$ for every open neighbourhoods U and W of X. The (γ, γ')-closure of a set A is denoted by $\mathrm{Cl}_{(\gamma,\gamma')}(A)$.*

Definition 5 ([4])**.** *For a subset A of (X, τ), $\tau_{(\gamma,\gamma')} - \mathrm{Cl}(A)$ denotes the intersection of all (γ, γ')-closed sets containing A, that is, $\tau_{(\gamma,\gamma')} - \mathrm{Cl}(A) = \bigcap\{F : A \subset F, X \backslash F \in \tau_{(\gamma,\gamma')}\}$.*

Definition 6. *A topological space (X, τ) is said to be (γ, γ')-regular [4] if for each $x \in X$ and for every open neighborhood U of x there exist open neighborhoods W and S of x such that $W^{\gamma} \cup S^{\gamma'} \subset U$.*

Definition 7. *Let A be any subset of X. The $\tau_{(\gamma,\gamma')}$-$\mathrm{Int}(A)$ is defined as $\tau_{(\gamma,\gamma')}$-$\mathrm{Int}(A) = \bigcup\{U : U$ is a (γ, γ')-open set and $U \subset A\}$.*

3. (γ, γ')-preopen sets

Definition 8. *Let (X, τ) be a topological space and γ, γ' be operations on τ. A subset A of X is said to be (γ, γ')-preopen if $A \subset \tau_{(\gamma,\gamma')} - \mathrm{Int}(\tau_{(\gamma,\gamma')} - \mathrm{Cl}(A))$.*

Remark 1. The set of all (γ, γ')-preopen sets of a topological space (X, τ) is denoted as $\tau_{(\gamma,\gamma')} - PO(X)$.

Example 1 ([4])**.** Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}\}$. Let $\gamma : \tau \to \mathcal{P}(X)$ and $\gamma' : \tau \to \mathcal{P}(X)$ be operations defined as follows: for every $A \in \tau$, $A^{\gamma} = A \cup \{a\}$ and

$$A^{\gamma'} = \begin{cases} A, & \text{if} \quad A = \{a\}, \\ A \cup \{c\}, & \text{if} \quad A \neq \{a\}. \end{cases}$$

Then $\tau_{(\gamma,\gamma')} - PO(X) = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$.

Theorem 1. *If A is a (γ, γ')-open set in (X, τ), then it is (γ, γ')-preopen set.*

Proof. Clear. ∎

Remark 2. The converse of the above Theorem need not be true. From the Example 1, we have $\{a,b\}$ is (γ,γ')-preopen set but it is not (γ,γ')-open.

Remark 3. By Theorem 1 and Remark 2, we have $\tau_{(\gamma,\gamma')} \subset \tau_{(\gamma,\gamma')}\text{-}PO(X,\tau)$.

Remark 4. The following examples show that the concepts preopeness and (γ,γ')-preopeness set are independent.

Example 2. Let $X=\{a,b,c\}$ and $\tau=\{\emptyset,X,\{a\},\{b,c\}\}$. Let $\gamma:\tau\to\mathcal{P}(X)$ and $\gamma':\tau\to\mathcal{P}(X)$ be operations defined as follows: for every $A\in\tau$, $A^\gamma=A\cup\{a\}$ and

$$A^{\gamma'}=\begin{cases} A, & \text{if} \quad a\in A,\\ \mathrm{Cl}(A), & \text{if} \quad a\notin A.\end{cases}$$

Then $\tau_{(\gamma,\gamma')}-PO(X)=\{\emptyset,X,\{a\},\{a,b\},\{a,c\}\}$. Clearly, the set $\{b\}$ is preopen but not (γ,γ')-preopen in (X,τ).

Example 3. Let $X=\{a,b,c\}$ and $\tau=\{\emptyset,X,\{a\},\{b\},\{a,b\},\{a,c\}\}$. Let $\gamma:\tau\to\mathcal{P}(X)$ and $\gamma':\tau\to\mathcal{P}(X)$ be operations defined as follows:

$$A^{\gamma}=\begin{cases} A, & \text{if} \quad b\notin A,\\ \mathrm{Cl}(A), & \text{if} \quad b\in A.\end{cases}$$

and

$$A^{\gamma'}=\begin{cases} A\cup\{b\}, & \text{if} \quad b\notin A,\\ A\cup\{a\}, & \text{if} \quad b\in A.\end{cases}$$

Then $\tau_{(\gamma,\gamma')}-PO(X)=\mathcal{P}(X)\backslash\{\{c\}\}$. Clearly, the set $\{b,c\}$ is (γ,γ')-preopen but not preopen in (X,τ).

Theorem 2. *If (X,τ) is (γ,γ')-regular space, then the concept of (γ,γ')-preopen and preopen coincide.*

Proof. Follows from Proposition 2.6(i) of [4]. ■

Theorem 3. *Let γ and γ' be operations on τ and $\{A_\alpha\}_{\alpha\in\Delta}$ be the collection of (γ,γ')-preopen sets of (X,τ), then $\bigcup_{\alpha\in\Delta} A_\alpha$ is also a (γ,γ')-preopen set.*

Proof. Since each A_α is (γ,γ')-preopen and $A_\alpha\subset\bigcup_{\alpha\in\Delta}A_\alpha$, implies that $\bigcup_{\alpha\in\Delta}A_\alpha\subset\tau_{(\gamma,\gamma')}-\mathrm{Int}(\tau_{(\gamma,\gamma')}-\mathrm{Cl}(\bigcup_{\alpha\in\Delta}A_\alpha))$. Hence $\bigcup_{\alpha\in\Delta}A_\alpha$ is also a (γ,γ') -preopen set in (X,τ). ■

Remark 5. If A and B are any two (γ, γ')-preopen sets in (X, τ), then $A \cap B$ need not be (γ, γ')-preopen in (X, τ). From the Example 3, we have $\{a, c\}$ and $\{b, c\}$ are (γ, γ')-preopen set but their intersection is not a (γ, γ')-preopen set in (X, τ).

Lemma 1. *Let (X, τ) be a topological space and γ, γ' operations on τ and A be the subset of X. Then the following are holds good:*
(i) $\tau_{(\gamma,\gamma')} - \mathrm{Cl}(\tau_{(\gamma,\gamma')} - \mathrm{Cl}(A)) = \tau_{(\gamma,\gamma')} - \mathrm{Cl}(A)$.
(ii) $\tau_{(\gamma,\gamma')} - \mathrm{Int}(\tau_{(\gamma,\gamma')} - \mathrm{Int}(A)) = \tau_{(\gamma,\gamma')} - \mathrm{Int}(A)$.
(iii) $\tau_{(\gamma,\gamma')} - \mathrm{Cl}(A) = X \backslash \tau_{(\gamma,\gamma')} - \mathrm{Int}(X \backslash A))$.
(iv) $\tau_{(\gamma,\gamma')} - \mathrm{Int}(A) = X \backslash \tau_{(\gamma,\gamma')} - \mathrm{Cl}(X \backslash A))$.

Proof. Straightforward. ■

Lemma 2. *Let (X, τ) be a topological space and γ, γ' regular operations on τ and A be the subset of X. Then*
(i) for every (γ, γ')-open set G and every subset $A \subset X$ we have $\tau_{(\gamma,\gamma')} - \mathrm{Cl}(A) \cap G \subset \tau_{(\gamma,\gamma')} - \mathrm{Cl}(A \cap G)$.
(ii) for every (γ, γ')-closed set F and every subset $A \subset X$ we have $\tau_{(\gamma,\gamma')} - \mathrm{Int}(A \cup F) \subset \tau_{(\gamma,\gamma')} - \mathrm{Int}(A) \cup F$.

Proof. (i) Let $x \in \tau_{(\gamma,\gamma')} - \mathrm{Cl}(A) \cap G$, then $x \in \tau_{(\gamma,\gamma')} - \mathrm{Cl}(A)$ and $x \in G$. Let V be the (γ, γ')-open set containing x. Then by Proposition 2.7 of [4], $V \cap G$ is also (γ, γ')-open set containing x. Since $x \in \tau_{(\gamma,\gamma')} - \mathrm{Cl}(A)$, we have $(V \cap G) \cap A \neq \emptyset$. This implies that $V \cap (G \cap A) \neq \emptyset$. This is true for every V contnining x, hence by Proposition 3.3(i) of [4] $x \in \tau_{(\gamma,\gamma')} - \mathrm{Cl}(A \cap G)$. Therefore, $\tau_{(\gamma,\gamma')} - \mathrm{Cl}(A) \cap G \subset \tau_{(\gamma,\gamma')} - \mathrm{Cl}(A \cap G)$.
(ii) Follows from (i) and Lemma 1(iv). ■

Theorem 4. *Let (X, τ) be a topological space and γ, γ' regular operations on τ. Let A be a (γ, γ')-preopen and U be the (γ, γ')-open subset of X, then $A \cap U$ is also (γ, γ')-preopen set.*

Proof. Follows from the Proposition 2.7 of [4] and Lemma 1. ■

Definition 9. *Let (X, τ) be a topological space, a subset A of X is said to be:*
(i) (γ, γ')-dense if $\tau_{(\gamma,\gamma')} - \mathrm{Cl}(A) = X$.
(ii) (γ, γ')-nowhere dense if $\tau_{(\gamma,\gamma')} - \mathrm{Int}(\tau_{(\gamma,\gamma')} - \mathrm{Cl}(A)) = \emptyset$.

Theorem 5. *Let (X, τ) be a topological space and γ, γ' regular operations on τ. Then a subset N of X is (γ, γ')-nowhere dense if and only if any one of the following conditions holds:*
(i) $\tau_{(\gamma,\gamma')} - \mathrm{Cl}(X \backslash \tau_{(\gamma,\gamma')} - \mathrm{Cl}(N)) = X$.

(ii) $N \subset \tau_{(\gamma,\gamma')} - \mathrm{Cl}(X \backslash \tau_{(\gamma,\gamma')} - \mathrm{Cl}(N))$.

(iii) Every nonempty (γ,γ')-open set U contains a nonempty (γ,γ')-open set A disjoint with N.

Proof. *(i)* $\tau_{(\gamma,\gamma')} - \mathrm{Int}(\tau_{(\gamma,\gamma')} - \mathrm{Cl}(N)) = \emptyset$ if and only if $X \backslash (\tau_{(\gamma,\gamma')} - \mathrm{Cl}(X \backslash (\tau_{(\gamma,\gamma')} - \mathrm{Cl}(N))) = \emptyset$ if and only if $X \subset \tau_{(\gamma,\gamma')} - \mathrm{Cl}(X \backslash (\tau_{(\gamma,\gamma')} - \mathrm{Cl}(N)))$ if and only if $X = \tau_{(\gamma,\gamma')} - \mathrm{Cl}(X \backslash (\tau_{(\gamma,\gamma')} - \mathrm{Cl}(N)))$.

(ii) $N \subset X = \tau_{(\gamma,\gamma')} - \mathrm{Cl}(X \backslash (\tau_{(\gamma,\gamma')} - \mathrm{Cl}(N)))$ by *(i)*. Conversely, $N \subset \tau_{(\gamma,\gamma')} - \mathrm{Cl}(X \backslash (\tau_{(\gamma,\gamma')} - \mathrm{Cl}(N)))$, implies $\tau_{(\gamma,\gamma')} - \mathrm{Cl}(N) \subset \tau_{(\gamma,\gamma')} - \mathrm{Cl}(X \backslash (\tau_{(\gamma,\gamma')} - \mathrm{Cl}(N)))$. Since $X = \tau_{(\gamma,\gamma')} - \mathrm{Cl}(N) \cup (X \backslash (\tau_{(\gamma,\gamma')} - \mathrm{Cl}(N)))$, implies $X \subset \tau_{(\gamma,\gamma')} - \mathrm{Cl}(X \backslash (\tau_{(\gamma,\gamma')} - \mathrm{Cl}(N))) \cup (X \backslash (\tau_{(\gamma,\gamma')} - \mathrm{Cl}(N))) = \tau_{(\gamma,\gamma')} - \mathrm{Cl}(X \backslash (\tau_{(\gamma,\gamma')} - \mathrm{Cl}(N)))$. Hence $X = \tau_{(\gamma,\gamma')} - \mathrm{Cl}(X \backslash (\tau_{(\gamma,\gamma')} - \mathrm{Cl}(N)))$.

(iii) Let N be a (γ,γ')-nowhere dense subset of X, then $\tau_{(\gamma,\gamma')} - \mathrm{Int}(X \backslash \tau_{(\gamma,\gamma')} - \mathrm{Cl}(N)) = \emptyset$. This implies $\tau_{(\gamma,\gamma')} - \mathrm{Cl}(N)$ does not contain any (γ,γ')-open set. Hence for any nonempty (γ,γ')-open set U, $U \backslash (\tau_{(\gamma,\gamma')} - \mathrm{Cl}(N)) \neq \emptyset$. Thus by Proposition 2.7 of [4] $A = U \backslash \tau_{(\gamma,\gamma')} - \mathrm{Cl}(N)$ is a nonempty (γ,γ')-open set contained in U and disjoint with N. Conversely, if for any given nonempty (γ,γ')-open set U, there exists a nonempty (γ,γ')-open set A such that $A \subset U$ and $A \cap N = \emptyset$, then $N \subset X \backslash A$, $\tau_{(\gamma,\gamma')} - \mathrm{Cl}(N) \subset \tau_{(\gamma,\gamma')} - \mathrm{Cl}(X \backslash A) = X \backslash A$. Therefore, $U \backslash \tau_{(\gamma,\gamma')} - \mathrm{Cl}(N) \supset U \backslash (X \backslash A) = U \cap A = A \neq \emptyset$. Thus, $\tau_{(\gamma,\gamma')} - \mathrm{Cl}(N)$ does not contain any nonempty (γ,γ')-open set. This implies $\tau_{(\gamma,\gamma')} - \mathrm{Int}(X \backslash \tau_{(\gamma,\gamma')} - \mathrm{Cl}(N)) = \emptyset$. Hence N is (γ,γ')-nowhere dense set in X. ■

Theorem 6. *Let (X,τ) be a topological space and γ, γ' operations on τ. Then for every $x \in X$, $\{x\}$ is either (γ,γ')-preopen or (γ,γ')-nowhere dense set.*

Proof. Suppose $\{x\}$ is not (γ,γ')-preopen, then $\tau_{(\gamma,\gamma')} - \mathrm{Int}(X \backslash \tau_{(\gamma,\gamma')} - \mathrm{Cl}(\{x\})) = \emptyset$. This implies $\{x\}$ is (γ,γ')-nowhere dense set in X. ■

Definition 10. *A topological space X is said to be (γ,γ')-submaximal if every (γ,γ')-dense subset of X is (γ,γ')-open.*

Theorem 7. *Let (X,τ) be a topological space in which every (γ,γ')-preopen set is (γ,γ')-open, then (X,τ) is (γ,γ')-submaximal.*

Proof. Let A be a (γ,γ')-dense subset of (X,τ). Then $A \subset \tau_{(\gamma,\gamma')} - \mathrm{Int}(\tau_{(\gamma,\gamma')} - \mathrm{Cl}(A))$. This implies A is a (γ,γ')-preopen set. Therefore, from the assumption it is (γ,γ')-open. Hence (X,τ) is (γ,γ')-submaximal. ■

Definition 11. *Let A be a subset of a topological space (X,τ) and γ, γ' be operations on τ. Then a subset A of X is said to be (γ,γ')-preclosed if and only if $X \backslash A$ is (γ,γ')-preopen, equivalently a subset A of X is (γ,γ')-preclosed if and only if $\tau_{(\gamma,\gamma')} - \mathrm{Cl}(\tau_{(\gamma,\gamma')} - \mathrm{Int}(A)) \subset A$.*

Remark 6. The set of all (γ,γ')-preclosed sets of a topological space (X,τ) is denoted as $\tau_{(\gamma,\gamma')} - PC(X)$.

Definition 12. *Let A be subset of a topological space (X,τ) and γ, γ' be operations on τ. Then*

(i) the $\tau_{(\gamma,\gamma')}$-preclosure of A is defined as intersection of all (γ,γ')-preclosed sets containing A. That is, $\tau_{(\gamma,\gamma')}$-$p\,\mathrm{Cl}(A) = \bigcap\{F : F$ is (γ,γ')-preclosed and $A \subset F\}$.

(ii) the $\tau_{(\gamma,\gamma')}$-preinterior of A is defined as union of all (γ,γ')-preopen sets contained in A. That is, $\tau_{(\gamma,\gamma')}$-$p\,\mathrm{Int}(A) = \bigcup\{U : U$ is (γ,γ')-preopen and $U \subset A\}$.

The proof of the following theorem is obvious and therefore is omitted.

Theorem 8. *Let A be a subset of a topological space (X,τ) and γ, γ' be operations on τ. Then*

(i) $\tau_{(\gamma,\gamma')}$-$p\,\mathrm{Int}(A)$ is a (γ,γ')-preopen set contained in A.

(ii) $\tau_{(\gamma,\gamma')}$-$p\,\mathrm{Cl}(A)$ is a (γ,γ')-preclosed set containing A.

(iii) A is (γ,γ')-preclosed if and only if $\tau_{(\gamma,\gamma')}$-$p\,\mathrm{Cl}(A) = A$.

(iv) A is (γ,γ')-preopen if and only if $\tau_{(\gamma,\gamma')}$-$p\,\mathrm{Int}(A) = A$.

Remark 7. From the definitions, we have $A \subset \tau_{(\gamma,\gamma')}$-$p\,\mathrm{Cl}(A) \subset \tau_{(\gamma,\gamma')}$-$\mathrm{Cl}(A)$ for any subset A of (X,τ).

Theorem 9. *For a point $x \in X$, $x \in \tau_{(\gamma,\gamma')}$-$p\,\mathrm{Cl}(A)$ if and only if $V \cap A \neq \emptyset$ for all (γ,γ')-preopen set V of X containing x.*

Proof. Let E be the set of all $y \in X$ such that $V \cap A \neq \emptyset$ for every $V \in \tau_{(\gamma,\gamma')} - PO(X)$ and $y \in V$. Now to prove the theorem it is enough to prove that $E = \tau_{(\gamma,\gamma')} - p\,\mathrm{Cl}(A)$. Let $x \in \tau_{(\gamma,\gamma')}$-$p\,\mathrm{Cl}(A)$ and $x \notin E$. Then there exists a (γ,γ')-preopen set U of x such that $U \cap A = \emptyset$. This implies $A \subset U^c$. Hence $\tau_{(\gamma,\gamma')}$-$p\,\mathrm{Cl}(A) \subset U^c$. Therefore $x \notin \tau_{(\gamma,\gamma')}$-$p\,\mathrm{Cl}(A)$. This is a contradiction. Hence $\tau_{(\gamma,\gamma')} - p\,\mathrm{Cl}(A) \subset E$. Conversely, let F be a set such that $A \subset F$ and $X\backslash F \in \tau_{(\gamma,\gamma')}$-$PO(X)$. Let $x \notin F$, then we have $x \in X\backslash F$ and $(X\backslash F) \cap A = \emptyset$. This implies $x \notin E$. Hence $E \subset \tau_{(\gamma,\gamma')}$-$p\,\mathrm{Cl}(A)$. ■

Theorem 10. *Let (X,τ) be a topological space and γ, γ' be regular operations on τ and A be a subset of X. Then the following holds:*

(i) $\tau_{(\gamma,\gamma')} - p\,\mathrm{Cl}(A) = A \cup \tau_{(\gamma,\gamma')} - \mathrm{Cl}(\tau_{(\gamma,\gamma')} - \mathrm{Int}(A))$.

(ii) $\tau_{(\gamma,\gamma')} - p\,\mathrm{Int}(A) = A \cap \tau_{(\gamma,\gamma')} - \mathrm{Int}(\tau_{(\gamma,\gamma')} - \mathrm{Cl}(A))$.

Proof. *(i)* $\tau_{(\gamma,\gamma')} - \mathrm{Cl}(\tau_{(\gamma,\gamma')} - \mathrm{Int}(A \cup \tau_{(\gamma,\gamma')} - \mathrm{Cl}(\tau_{(\gamma,\gamma')} - \mathrm{Int}(A))) \subset \tau_{(\gamma,\gamma')} - \mathrm{Cl}(\tau_{(\gamma,\gamma')} - \mathrm{Int}(A) \cup \tau_{(\gamma,\gamma')} - \mathrm{Cl}(\tau_{(\gamma,\gamma')} - \mathrm{Cl} - \mathrm{Int}(A))) \subset \tau_{(\gamma,\gamma')} - \mathrm{Cl}(\tau_{(\gamma,\gamma')} - \mathrm{Int}(A)) \cup \tau_{(\gamma,\gamma')} - \mathrm{Cl}(\tau_{(\gamma,\gamma')} - \mathrm{Int}(A)) \subset A \cup (\tau_{(\gamma,\gamma')} - \mathrm{Cl}(\tau_{(\gamma,\gamma')} - \mathrm{Int}(A))$. Hence $A \cup \tau_{(\gamma,\gamma')} - \mathrm{Cl}(\tau_{(\gamma,\gamma')} - \mathrm{Int}(A))$ is a (γ,γ')- preclosed set containing A. Therefore, $\tau_{(\gamma,\gamma')}-p\operatorname{Cl}(A)\subset A\cup\tau_{(\gamma,\gamma')}-\operatorname{Cl}(\tau_{(\gamma,\gamma')}-\operatorname{Int}(A))$. Conversely, $\tau_{(\gamma,\gamma')}-\operatorname{Cl}(\tau_{(\gamma,\gamma')}-\operatorname{Int}(A))\subset\tau_{(\gamma,\gamma')}-\operatorname{Cl}(\tau_{(\gamma,\gamma')}-\operatorname{Int}(\tau_{(\gamma,\gamma')}-p\operatorname{Cl}(A))\subset\tau_{(\gamma,\gamma')}-p\operatorname{Cl}(A)$ since $\tau_{(\gamma,\gamma')}-p\operatorname{Cl}(A)$ is a (γ,γ')-preclosed set. Hence $\tau_{(\gamma,\gamma')}-p\operatorname{Cl}(A)=A\cup\tau_{(\gamma,\gamma')}-\operatorname{Cl}(\tau_{(\gamma,\gamma')}-\operatorname{Int}(A))$.

(*ii*) Follows from (*i*) and Lemma 1(i). ■

Remark 8. If the conditions of γ, γ' are omitted in the above Theorem, it may be false. See Example 3.

Corollary 1. *Let (X,τ) be a topological space and γ, γ' be regular operations on τ and A be a subset of X. Then the following holds good:*

(*i*) $\tau_{(\gamma,\gamma')}-p\operatorname{Int}(\tau_{(\gamma,\gamma')}-\operatorname{Cl}(A))=\tau_{(\gamma,\gamma')}-\operatorname{Int}(\tau_{(\gamma,\gamma')}-\operatorname{Cl}(A))$.

(*ii*) $\tau_{(\gamma,\gamma')}-p\operatorname{Cl}(\tau_{(\gamma,\gamma')}-\operatorname{Int}(A))=\tau_{(\gamma,\gamma')}-\operatorname{Cl}(\tau_{(\gamma,\gamma')}-\operatorname{Int}(A))$.

(*iii*) $\tau_{(\gamma,\gamma')}-\operatorname{Int}(\tau_{(\gamma,\gamma')}-p\operatorname{Cl}(A))=\tau_{(\gamma,\gamma')}-\operatorname{Int}(\tau_{(\gamma,\gamma')}-\operatorname{Cl}(\tau_{(\gamma,\gamma')}-\operatorname{Int}(A)))$.

(*iv*) $\tau_{(\gamma,\gamma')}-\operatorname{Cl}(\tau_{(\gamma,\gamma')}-p\operatorname{Int}(A))=\tau_{(\gamma,\gamma')}-\operatorname{Cl}(\tau_{(\gamma,\gamma')}-\operatorname{Int}(\tau_{(\gamma,\gamma')}-\operatorname{Cl}(A)))$.

Proof. (*i*) By Theorem 10(*ii*), $\tau_{(\gamma,\gamma')}-p\operatorname{Int}(\tau_{(\gamma,\gamma')}-\operatorname{Cl}(A))=\tau_{(\gamma,\gamma')}-\operatorname{Cl}(A)\cap\tau_{(\gamma,\gamma')}-\operatorname{Int}(\tau_{(\gamma,\gamma')}-\operatorname{Cl}(\tau_{(\gamma,\gamma')}-\operatorname{Cl}(A)))=\tau_{(\gamma,\gamma')}-\operatorname{Cl}(A)\cap\tau_{(\gamma,\gamma')}-\operatorname{Int}(\tau_{(\gamma,\gamma')}-\operatorname{Cl}(A))=\tau_{(\gamma,\gamma')}-\operatorname{Int}(\tau_{(\gamma,\gamma')}-\operatorname{Cl}(A))$.

(*ii*) By Theorem 10(i), $\tau_{(\gamma,\gamma')}-p\operatorname{Cl}(\tau_{(\gamma,\gamma')}-\operatorname{Int}(A))=\tau_{(\gamma,\gamma')}-\operatorname{Int}(A)\cup\tau_{(\gamma,\gamma')}-\operatorname{Cl}(\tau_{(\gamma,\gamma')}-\operatorname{Int}(\tau_{(\gamma,\gamma')}-\operatorname{Int}(A)))=\tau_{(\gamma,\gamma')}-\operatorname{Cl}(\tau_{(\gamma,\gamma')}-\operatorname{Int}(A))$.

(*iii*) and (*iv*) follows from (*i*) and (*ii*), respectively. ■

Theorem 11. *Let (X,τ) be a topological space and γ, γ' be regular operations on τ and A be a subset of X. Then $\tau_{(\gamma,\gamma')}-p\operatorname{Cl}(\tau_{(\gamma,\gamma')}-p\operatorname{Int}(A))=\tau_{(\gamma,\gamma')}-p\operatorname{Int}(A)\cup\tau_{(\gamma,\gamma')}-\operatorname{Cl}(\tau_{(\gamma,\gamma')}-\operatorname{Int}(A)))$.*

Proof. Since $\tau_{(\gamma,\gamma')}\subset\tau_{(\gamma,\gamma')}-PO(X)$, implies $\tau_{(\gamma,\gamma')}-\operatorname{Int}(A)\subset\tau_{(\gamma,\gamma')}-p\operatorname{Int}(A)\subset A$. Hence $\tau_{(\gamma,\gamma')}-\operatorname{Int}(\tau_{(\gamma,\gamma')}-p\operatorname{Int}(A))=\tau_{(\gamma,\gamma')}-\operatorname{Int}(A)$. By Theorem 10(i), $\tau_{(\gamma,\gamma')}-p\operatorname{Cl}(\tau_{(\gamma,\gamma')}-p\operatorname{Int}(A))=\tau_{(\gamma,\gamma')}-p\operatorname{Int}(A)\cap\tau_{(\gamma,\gamma')}-\operatorname{Cl}(\tau_{(\gamma,\gamma')}-\operatorname{Int}(\tau_{(\gamma,\gamma')}-p\int(A))=\tau_{(\gamma,\gamma')}-p\operatorname{Int}(A)\cup\tau_{(\gamma,\gamma')}-\operatorname{Cl}(\tau_{(\gamma,\gamma')}-\operatorname{Int}(A))$. ■

Theorem 12. *Let (X,τ) be a topological space and γ, γ' be operations on τ and A be a subset of X. Then V is (γ,γ')-preopen if and only if $V\subset\tau_{(\gamma,\gamma')}-p\operatorname{Int}(\tau_{(\gamma,\gamma')}-p\operatorname{Cl}(V))$.*

Proof. (*i*) Let V be (γ,γ')-preopen. Then $\tau_{(\gamma,\gamma')}-p\operatorname{Int}(V)=V$ and also $V\subset\tau_{(\gamma,\gamma')}-p\operatorname{Int}(\tau_{(\gamma,\gamma')}-p\operatorname{Cl}(V))$. Conversely, let $V\subset\tau_{(\gamma,\gamma')}-p\operatorname{Int}(\tau_{(\gamma,\gamma')}-p\operatorname{Cl}(V))$. Then $V\subset\tau_{(\gamma,\gamma')}-p\operatorname{Int}(\tau_{(\gamma,\gamma')}-\operatorname{Cl}(V))=\tau_{(\gamma,\gamma')}-\operatorname{Cl}(V)\cap\tau_{(\gamma,\gamma')}-\operatorname{Int}(\tau_{(\gamma,\gamma')}-\operatorname{Cl}(\tau_{(\gamma,\gamma')}-\operatorname{Cl}(V)))=\tau_{(\gamma,\gamma')}-\operatorname{Int}(\tau_{(\gamma,\gamma')}-\operatorname{Cl}(V))$. Hence, V is (γ,γ')-preopen. ■

Definition 13. *A subset A of (X,τ) is said to be (γ,γ')-pregeneralized closed if $\tau_{(\gamma,\gamma')}$-$p\operatorname{Cl}(A)\subset U$ whenever $A\subset U$ and U is (γ,γ')-preopen in (X,τ).*

Remark 9. In Example 2, the set $A=\{c\}$ is a (γ,γ')-pregeneralized closed and the set $B=\{a,b\}$ is not a (γ,γ')-pregeneralized closed.

Definition 14. *A topological space (X,τ) is said to be (γ,γ')-pre-$T_{1/2}$ if every (γ,γ')-pregeneralized closed set in (X,τ) is (γ,γ')-preclosed.*

Theorem 13. *A subset A of (X,τ) is (γ,γ')-pregeneralized closed if and only if $\tau_{(\gamma,\gamma')}$- $p\operatorname{Cl}(\{x\})\cap A\neq\emptyset$ holds for every $x\in\tau_{(\gamma,\gamma')}$-$p\operatorname{Cl}(A)$.*

Proof. Let U be any (γ,γ')-preopen set such that $A\subset U$. Let $x\in\tau_{(\gamma,\gamma')}$-$p\operatorname{Cl}(A)$. By assumption there exists $z\in\tau_{(\gamma,\gamma')}$-$p\operatorname{Cl}(\{x\})$ and $z\in A\subset U$. It follows from Theorem 9 that $U\cap\{x\}\neq\emptyset$. Hence $x\in U$. This implies $\tau_{(\gamma,\gamma')}$-$p\operatorname{Cl}(A)\subset U$. Therefore A is (γ,γ')-pregeneralized closed set in (X,τ). Conversely, suppose $x\in\tau_{(\gamma,\gamma')}$-$p\operatorname{Cl}(A)$ such that $\tau_{(\gamma,\gamma')}$-$p\operatorname{Cl}(\{x\})\cap A=\emptyset$. Since $\tau_{(\gamma,\gamma')}$-$p\operatorname{Cl}(\{x\})$ is (γ,γ')-preclosed set in (X,τ), $(\tau_{(\gamma,\gamma')}$-$p\operatorname{Cl}(\{x\}))^c$ is a (γ,γ')-preopen set of (X,τ). Since $A\subset(\tau_{(\gamma,\gamma')}$-$p\operatorname{Cl}(\{x\}))^c$ and A is (γ,γ')-pregeneralized closed, $\tau_{(\gamma,\gamma')}$-$p\operatorname{Cl}(A)\subset(\tau_{(\gamma,\gamma')}$-$p\operatorname{Cl}(\{x\}))^c$. This implies that $x\notin\tau_{(\gamma,\gamma')}$-$p\operatorname{Cl}(A)$. This is a contradiction. Hence $\tau_{(\gamma,\gamma')}$-$p\operatorname{Cl}(\{x\})$ $\cap A\neq\emptyset$. ■

Theorem 14. *A is a (γ,γ')-pregeneralized closed subset of a topological space (X,τ), if and only if $\tau_{(\gamma,\gamma')}$-$p\operatorname{Cl}(A)\backslash A$ does not contain a nonempty (γ,γ')-preclosed set.*

Proof. Suppose there exists a nonempty (γ,γ')-preclosed set F such that $F\subset\tau_{(\gamma,\gamma')}$-$p\operatorname{Cl}(A)\backslash A$. Let $x\in F$, $x\in\tau_{(\gamma,\gamma')}$-$p\operatorname{Cl}(A)$ holds. Then $F\cap A=\tau_{(\gamma,\gamma')}$-$p\operatorname{Cl}(F)\cap A\supset\tau_{(\gamma,\gamma')}$-$p\operatorname{Cl}(\{x\})\cap A\neq\emptyset$. Hence $F\cap A\neq\emptyset$. This is a contradiction. Conversely, suppose that $\tau_{(\gamma,\gamma')}$-$p\operatorname{Cl}(A)\backslash A$ does not contain a nonempty (γ,γ')-preclosed set. Let $A\subset U$ and U a (γ,γ')-preopen set in (X,τ), then $X\backslash U\subseteq X\backslash A$, follows $\tau_{(\gamma,\gamma')}$-$p\operatorname{Cl}(A)\cap(X\backslash U)\subseteq\tau_{(\gamma,\gamma')}$-$p\operatorname{Cl}(A)\cap(X\backslash A)=\tau_{(\gamma,\gamma')}$-$p\operatorname{Cl}(A)\backslash A)$. If we take $F=\tau_{(\gamma,\gamma')}$-$p\operatorname{Cl}(A)\cap(X\backslash U)$, F is a (γ,γ')-preclosed set and $F\subseteq\tau_{(\gamma,\gamma')}$-$p\operatorname{Cl}(A)\backslash A$. Therefore $F=\emptyset$, in consequence, $\tau_{(\gamma,\gamma')}$-$p\operatorname{Cl}(A)\subseteq U$ and follows that A is (γ,γ')-pregeneralized closed set. ■

Theorem 15. *For each $x\in X$, $\{x\}$ is (γ,γ')-preclosed or $\{x\}^c$ is (γ,γ')-pregeneralized closed set in (X,τ).*

Proof. Suppose that $\{x\}$ is not (γ,γ')-preclosed, then $X\backslash\{x\}$ is not (γ,γ')-preopen. Let U be any (γ,γ')-preopen set such that $\{x\}^c\subset U$. Since

$U = X$, $\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Cl}(\{x\}^c) \subset U$. Therefore, $\{x\}^c$ is (γ, γ')-pregeneralized closed. ■

Theorem 16. *A topological space (X, τ) is (γ, γ')-pre-$T_{1/2}$ space if and only if every singleton subset of X is (γ, γ')-preclosed or (γ, γ')-preopen in (X, τ).*

Proof. Let $x \in X$. Suppose $\{x\}$ is not (γ, γ')-preclosed. Then, it follows from assumption and Theorem 15 that $\{x\}$ is (γ, γ')-preopen. Conversely, Let F be a (γ, γ')-pregeneralized closed set in (X, τ). Let x be any point in $\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Cl}(F)$, then by assumption $\{x\}$ is (γ, γ')-preopen or (γ, γ')-preclosed.

Case (i): Suppose $\{x\}$ is (γ, γ')-preopen. Then by Theorem 13 we have $\{x\} \cap F \neq \emptyset$. This implies $\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Cl}(F) = F$; hence (X, τ) is (γ, γ')-pre-$T_{1/2}$.

Case (ii): Suppose $\{x\}$ is (γ, γ')-preclosed. Assume $x \notin F$, then $x \in \tau_{(\gamma,\gamma')}\text{-}p\operatorname{Cl}(F) \backslash F$. This is not possible by Theorem 14. Thus, we have $x \in F$. Therefore, $\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Cl}(F) = F$ and hence F is (γ, γ')-preclosed. It follows that (X, τ) is (γ, γ')-pre-$T_{1/2}$. ■

Remark 10. The space defined in Example 2, is (γ, γ')-pre-$T_{1/2}$ space.

3. (γ, γ')-precontinuous functions

Throughout this section let (X, τ) and (Y, σ) be two topological spaces and let $\gamma, \gamma' : \tau \to \mathcal{P}(X)$ and $\beta, \beta' : \sigma \to \mathcal{P}(Y)$ be operations on τ and σ, respectively.

Definition 15. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be (γ, γ')-precontinuous if for each $x \in X$ and each (β, β')-preopen set V containing $f(x)$ there exists a (γ, γ')-preopen set U such that $x \in U$ and $f(U) \subset V$.*

Theorem 17. *Let $f : (X, \tau) \to (Y, \sigma)$ be an (γ, γ')-precontinuous function. Then the following hold:*

(i) *$f(\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Cl}(A)) \subset \sigma_{(\beta,\beta')} - p\operatorname{Cl}(f(A))$ holds for every subset A of (X, τ).*

(ii) *for every (β, β')-preclosed set B of (Y, σ), $f^{-1}(B)$ is (γ, γ')-preclosed in (X, τ).*

Proof. (i) Let $y \in f(\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Cl}(A))$ and V be any (β, β')-preopen set containing y. Then there exists $x \in X$ and (γ, γ')-preopen set U such that $f(x) = y$ and $x \in U$ and $f(U) \subset V$. Since $x \in \tau_{(\gamma,\gamma')}\text{-}p\operatorname{Cl}(A)$, we have $U \cap A \neq \emptyset$ and hence $\emptyset \neq f(U \cap A) \subset f(U) \cap f(A) \subset V \cap f(A)$. This implies $x \in \sigma_{(\beta,\beta')}\text{-}p\operatorname{Cl}(f(A))$. Therefore we have $f(\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Cl}(A)) \subset \sigma_{(\beta,\beta')}\text{-}p\operatorname{Cl}(f(A))$.

(ii) Let B be a (β,β')-preclosed set in (Y,σ). Therefore, $\sigma_{(\beta,\beta')}\text{-}p\operatorname{Cl}(B) = B$. By using (i) we have $f(\tau_{(\gamma,\gamma')}\text{-}b\operatorname{Cl}(f^{-1}(B)) \subset \sigma_{(\beta,\beta')}\text{-}p\operatorname{Cl}(B) = B$. Therefore, we have $\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Cl}(f^{-1}(B)) = f^{-1}(B)$. Hence $f^{-1}(B)$ is (γ,γ')-preclosed. ■

Definition 16. *A function $f:(X,\tau)\to(Y,\sigma)$ is said to be (γ,γ')-preclosed if for any (γ,γ')-preclosed set A of (X,τ), $f(A)$ is a (β,β')-closed in Y.*

Theorem 18. *Suppose that f is (γ,γ')-precontinuous and (γ,γ')-preclosed function. Then,*

(i) for every (γ,γ')-pregeneralized closed set A of (X,τ), the image $f(A)$ is (β,β')-pregeneralized closed.

(ii) for every (β,β')-pregeneralized closed set B of (Y,σ), $f^{-1}(B)$ is (γ,γ')- pregeneralized closed.

Proof. (i) Let V be any (β,β')-preopen set in (Y,σ) such that $f(A)\subset V$. By using Theorem 17 (ii), $f^{-1}(V)$ is (γ,γ')-preopen in (X,τ). Since A is (γ,γ')-pregeneralized closed and $A\subset f^{-1}(V)$, we have $\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Cl}(A)\subset f^{-1}(V)$, and hence $f(\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Cl}(A))\subset V$. It follows that $f(\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Cl}(A))$ is a (β,β')-preclosed set in Y. Therefore, $\sigma_{(\beta,\beta')}\text{-}p\operatorname{Cl}(f(A))\subset \sigma_{(\beta,\beta')} - p\operatorname{Cl}(f(\tau_{(\gamma,\gamma')} - p\operatorname{Cl}(A))) = f(\tau_{(\gamma,\gamma')} - p\operatorname{Cl}(A))\subset V$. This implies $f(A)$ is (β,β')-pregeneralized closed.

(ii) Let U be a (γ,γ')-preopen set of (X,τ) such that $f^{-1}(B)\subset U$. Put $F=\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Cl}(f^{-1}(B))\cap U^c$. It follows that F is (γ,γ')-preclosed set in (X,τ). Since f is (γ,γ')-preclosed, $f(F)$ is (β,β')-preclosed in (Y,σ). Then $f(F)\subset f(\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Cl}(f^{-1}(B)\cap U^c))\subset \sigma_{(\beta,\beta')}\text{-}\operatorname{Cl}(f(f^{-1}(B))\cap f(U^c))\subset \tau_{(\gamma,\gamma')}\text{-}p\operatorname{Cl}(B)\backslash B$. This implies $f(F)=\emptyset$, and hence $F=\emptyset$. Therefore, $\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Cl}(f^{-1}(B))\subset U$. Hence $f^{-1}(B)$ is (γ,γ')-pregeneralized closed in (X,τ). ■

Theorem 19. *Let $f:(X,\tau)\to(Y,\sigma)$ be (γ,γ')-precontinuous and (γ,γ')-preclosed function. Then,*

(i) If f is injective and (Y,σ) is (β,β')-pre-$T_{1/2}$, then (X,τ) is (γ,γ')-pre-$T_{1/2}$ space.

(ii) If f is surjective and (X,τ) is (γ,γ')-pre-$T_{1/2}$, then (Y,σ) is (β,β')-pre-$T_{1/2}$.

Proof. Straightforward. ■

Definition 17. *A function $f:(X,\tau)\to(Y,\sigma)$ is said to be (γ,γ')-prehomeomorphism, if f is bijective, (γ,γ')-precontinuous and f^{-1} is (β,β')-precontinuous.*

Theorem 20. *Let $f:(X,\tau)\to(Y,\sigma)$ be (γ,γ')-prehomeomorphism. If (X,τ) is (γ,γ')-pre-$T_{1/2}$, then (Y,σ) is (β,β')-pre-$T_{1/2}$.*

Proof. Let $\{y\}$ be a singleton set of (Y, σ). Then, there exists a point x of X such that $y = f(x)$ and by Theorem 16 that $\{x\}$ is (γ, γ')-preopen or (γ, γ')-preclosed. By using Theorem 18(i), then $\{y\}$ is (β, β')-preclosed or (β, β')-preopen. By Theorem 16, (Y, σ) is (β, β')-pre-$T_{1/2}$ space. ■

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Carlos Carpintero
Universidad De Oriente
Nucleo De Sucre Cumana, Venezuela
e-mail: ccarpi@sucre.udo.edu.ve

Neelamegarajan Rajesh
Department of Mathematics
Kongu Engineering College
Perundurai, Erode-638 052, Tamilnadu, Indi
e-mail: nrajesh_topology@yahoo.co.in

Ennis Rosas
Universidad De Oriente
Nucleo De Sucre Cumana, Venezuela
e-mail: erosas@sucre.udo.edu.ve

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