F A S C I C U L I M A T H E M A T I C I

Nr 46 2011

Seshadev Padhi and Chuanxi Qian

ON ASYMPTOTIC PROPERTY OF SOLUTIONS OF HIGHER ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS*

Abstract. We establish sufficient conditions for the linear functional differential equation of n-th order of the forms

$$y^{(n)}(t) + p(t)y(g(t)) = 0$$

and

$$y^{(n)}(t) - p(t)y(g(t)) = 0$$

to have property A and B, where p and $g \in C([\sigma, \infty), (0, \infty))$, with $\sigma \in R$ and $g(t) \geq t$, and $n \geq 2$ is a positive integer.

Key words: oscillatory solution, nonoscillatory solution, property A, property B.

AMS Mathematics Subject Classification: 34C10,34K15.

1. Introduction

This paper deals with the oscillation and asymptotic property of nonoscillatory solutions of the functional differential equations of the forms

$$y^{(n)}(t) + p(t)y(g(t)) = 0 \tag{1}$$

and

$$y^{(n)}(t) - p(t)y(g(t)) = 0 \tag{2}$$

where p and $g \in C([\sigma, \infty), (0, \infty))$, with $\sigma \in R$ and $g(t) \geq t$, and $n \geq 2$ is a positive integer.

A continuous function $y \in [\sigma, \infty) \to R$ is said to be a solution of (1)(or (2)) if it is continuous, along with its derivatives up to $(n-1)$-th order and

* This work is supported by "Department of Science and Technology, New Delhi, Govt. of India, under BOYSCAST Programme vide Sanc. No. 100/IFD/5071/2004-2005. Dated 04.01.2005.

satisfies (1)(or (2)). A solution of (1)(or (2)) is said to be oscillatory if it has a sequence of zeros tending to infinity; Otherwise, the solution is said to be nonoscillatory. Equation (1) with $g(t) = t$ is said to be disconjugate on $[\sigma, \infty)$ if no nontrivial solution of the equation has more than $(n-1)$-zeros, counting multiplicities.

Let $y(t)$ be a nonoscillatory solution of (1)(or (2)). We may assume, without any loss of generality, that $y(t) > 0$ for $t \geq t_0 \geq \sigma$. Then $y(g(t)) > 0$ and $y^{(n)}(t) < 0$ ($y^{(n)}(t) > 0$) for $t \geq t_0$. Hence by Lemma 1.1 due to [9], it follows that there exists an integer $l, 0 \leq l \leq n-1$, such that $n + l$ is odd (even) and

$$
\begin{aligned}
y^{(i)}(t) &> 0, \quad i = 0, 1, 2, \ldots, l, \\
(-1)^{i+l} y^{(i)}(t) &> 0, \quad i = l+1, \ldots, n,
\end{aligned} \tag{3}
$$

for large t, say for $t \geq T \geq t_0$. Further, for $l \in \{1, 2, 3, \ldots, n-1\}, n + l$ odd, the following inequality holds for large t, say for $t \geq t_1 \geq T$,

$$
|y(t)| \geq \frac{(t - t_1)^{(n-1)}}{(n-1)(n-2)\ldots(n-l)} |y^{(n-1)}(2^{n-l-1}t)|, \quad t \geq t_1. \tag{4}
$$

Definition 1. *We say that Eq.(1) has property A if any of its solution is oscillatory when n is even and either is oscillatory or satisfies (3) for $l = 0$ when n is odd.*

If N denotes the set of all nonoscillatory solutions of (2) and N_l denotes the set of all nonoscillatory solutions of (2) satisfying (3). Then

$$
N = \begin{cases} N_1 \cup N_3 \cup \cdots \cup N_n, & \text{if } n \text{ is odd,} \\ N_0 \cup N_2 \cup \cdots \cup N_n, & \text{if } n \text{ is even.} \end{cases}
$$

Definition 2. *Equation (2) is said to have property B if $N = N_n$ for n odd, and $N = N_0 \cup N_n$ for n even.*

A vast literature exists on the oscillation, property A , property B and asymptotic behaviour of the nonoscillatory solutions of n-th order delay differential equations of the forms

$$
y^{(n)}(t) + q(t) y(\tau(t)) = 0 \tag{5}
$$

and

$$
y^{(n)}(t) - q(t) y(\tau(t)) = 0 \tag{6}
$$

where q and $\tau \in C([\sigma, \infty), [0, \infty)), \tau(t) \leq t$ and $\tau(t) \to \infty$ as $t \to \infty$. One may see the monograph due to Lakshmikantham et.al [13], Gyori and Ladas

[7] and the references cited therein for oscillation, property A and property B of Eq.(5) and (6) respectively. Higher order differential equations of the form (5) and (6) were studied by Koplatadze [11]. We note that the definitions given above for Eq.(1) also hold for Eq.(5) and (6). In a recent paper [16], the author studied the oscillation and property A of Eq.(5). The results obtained in [16] are different from earlier existing ones. Furthermore the sufficient conditions for the oscillation of (5) have been obtained while dealing with the property A of the equation. On the other hand, it seems that very few work has been done on the oscillation and asymptotic behaviour of nonoscillatory solutions of the advanced differential equations (1) and (2). The results we shall provide here, are different from the results due to Koplatadze et. al. [12]. This is dealt in Section 2. Further, some easily verifiable sufficient conditions for property B of (2) and (6) are given in Section 3.

The following lemma due to Kiguradze [9] is needed for our use in the sequel.

Lemma 1. *Let the inequality (3) hold for a certain* $l \in \{1, 2, 3, \dots, n-1\}$. *Then*

$$\int_{t_1}^{\infty} s^{n-l-1}|y^{(n)}(s)|\,ds < \infty, \tag{7}$$

$$y^{(i)}(t) \geq y^{(i)}(t_1) + \frac{1}{(l-i-1)!}\int_{t_1}^{t}(t-s)^{l-i-1}y^{(l)}(s)\,ds \tag{8}$$

for $t \geq t_1$, $i = 0, 1, 2, \dots, l-1$ *and*

$$y^{(l)}(t) \geq \frac{1}{(n-l-1)!}\int_{t}^{\infty}(s-t)^{n-l-1}|y^{(n)}(s)|\,ds \tag{9}$$

for $t \geq t_1$.

Lemma 2. *Let* t_0 *and* T *be such that* $T \geq t_0 \geq \sigma$ *and* $g(t) \geq t_0$ *for* $t \geq T$. *Assume that* $u : [t_0, \infty) \to (0, \infty)$, $w^* : [T, \infty) \times (0, \infty) \to (0, \infty)$, $H : [0, \infty) \times [0, \infty) \to [0, \infty)$ *and* $\phi, \psi : \Delta \to [0, \infty)$ *are continuous, where* $\Delta = \{(t, s); t \geq s \geq T\}$ *and each of* w^* *and* H *is nondecreasing in the second variable. Suppose that*

$$\int_{T}^{\infty} \psi^*(t)H(t, u(g(t)))\,dt < \infty$$

and

$$u(t) \geq w^*(t, u(t)) + \int_{T}^{t} \phi(t, s)\Big(\int_{s}^{\infty} \psi(\alpha, s)H(\alpha, u(g(\alpha)))\,d\alpha\Big)\,ds$$

for $t \geq T$, *where* $\psi^*(t) = \max\{\psi(t,s);\ s \in [T,t]\}$. *Then the integral equation*

$$u(t) = w^*(t, v(t)) + \int_T^t \phi(t,s)\Big(\int_s^\infty \psi(\alpha, s) H(\alpha, v(g(\alpha)))\, d\alpha\Big)\, ds,$$

$t \geq T$, *has a solution* $v \in C([t_0, \infty), (0,\infty))$ *satisfying*

$$w^*(t, v(t)) \leq v(t) \leq u(t), \quad t \geq T.$$

The proof of Lemma 2 is similar to that of Lemma 3 in Kusano and Naito [10] and hence is omitted.

Lemma 3 ([3]). *If* $z \in C^2([T,\infty), R)$ *and*

$$z(t) > 0, \quad z^{'}(t) > 0, \quad z^{''}(t) \leq 0$$

on $[T, \infty)$, *then for each* $k \in (0,1)$, *there is a* $T_k \geq T$ *such that*

$$z(\tau(t)) \geq k\frac{\tau(t)}{t} z(t)$$

for $t \geq T_k$.

2. Main results-I

In this section, sufficient conditions in terms of the coefficient functions have been obtained for property A and oscillation of (1).

Theorem 1. *Let* $t - g^{-1}(t) \to \infty$ *as* $t \to \infty$. *Suppose that for every* $l \in \{1, 2, \cdots, n-1\}$

$$\limsup_{t\to\infty} (g^{-1}(t)) \int_t^\infty (s-t)^{n-l-1} p(s)\, ds > (n-l-1)! \cdot l! \tag{10}$$

holds, then (1) has property A.

Proof. Suppose, for the sake of contradiction, that (1) does not have property A. Then there exists a nonoscillatory solution $y(t)$ of (1) and a real $t_1 \geq \sigma$ such that for $t \geq t_1$, (3) holds for some $l \in \{1, 2, \cdots, n-1\}$. Without any loss of generality, we may assume that $y(t) > 0$ for $t \geq t_1$. Then $y^{(n)}(t) < 0$ and $y(g(t)) > 0$ for $t \geq t_1$. Putting $i = 0$ in (8), we obtain

$$y(t) \geq \frac{1}{(l-1)!} \int_{t_1}^t (t-s)^{l-1} y^{(l)}(s)\, ds, \quad t \geq t_1. \tag{11}$$

We can find a $t_2 \geq t_1$ such that $t - g^{-1}(t) > t_1$ for $t \geq t_2$. Thus, for $t \geq t_2$

$$\begin{aligned} y(t) &\geq \frac{1}{(l-1)!}\int_{t-g^{-1}(t)}^{t}(t-s)^{l-1}y^{(l)}(s)\,ds \\ &\geq \frac{y^{(l)}(t)}{(l-1)!}\int_{t-g^{-1}(t)}^{t}(t-s)^{l-1}\,ds \\ &\geq \frac{(g^{-1}(t))^l}{l!}y^{(l)}(t), \end{aligned}$$

which using (35) gives

$$\begin{aligned} y(t) &\geq \frac{(g^{-1}(t))^l}{l!\cdot(n-l-1)!}\int_t^\infty (s-t)^{n-l-1}p(s)y(g(s))\,ds \\ &\geq \frac{(g^{-1}(t))^l}{l!\cdot(n-l-1)!}y(g(t))\int_t^\infty (s-t)^{n-l-1}p(s)\,ds \\ &\geq \frac{(g^{-1}(t))^l}{l!\cdot(n-l-1)!}y(t)\int_t^\infty (s-t)^{n-l-1}p(s)\,ds \end{aligned}$$

which contradicts (10). Hence Eq.(1) has property A. Thus the theorem is proved. ■

Remark 1. Theorem 1 holds strictly for the advanced differential equations, that is, the theorem does not hold when $g(t) = t$. In [11], several sufficient conditions for the property A were obtained for functional differential equations. Our result Theorem 1 cannot be compared with the results in [11]. Corollary 6.2 in [11] gives a sufficient condition where no advanced term is required, while our theorem 1 requires the advanced term $g(t)$. Our result here is also different from the results obtained in [4] and [5].

Theorem 2. *Suppose that $g(t) > t$ and for every $l \in \{1, 2, \cdots, n-1\}$,*

$$\limsup_{t\to\infty}\int_{g^{-1}(t)}^{t}(t-s)^{l-1}\int_s^\infty (u-s)^{n-l-1}p(u)\,du\,ds > (l-1)!\cdot(n-l-1)! \tag{12}$$

holds. Then (1) has property A.

Proof. Let $y(t)$ be a nonoscillatory solution of (1). Without any loss of generality, we may assume that $y(t) > 0$ for $t \geq t_0 > \sigma$. Then $y^{(n)}(t) < 0$ and $y(g(t)) > 0$ for $t \geq t_0$. Thus there exists an integer $l, 0 \leq l \leq n-1$ such that $n+l$ odd and (3) holds for some $t \geq t_1 \geq t_0$. Let $l \in \{1, 2, \cdots, n-1\}$. Then from (9) and (11), we have

$$y(t) \geq \frac{1}{(l-1)!\cdot(n-l-1)!}\int_{t_1}^{t}(t-s)^{l-1}\int_s^\infty (u-s)^{n-l-1}p(u)y(g(u))\,du\,ds.$$

We can find a $t_2 \geq t_1$ so that $g^{-1}(t) > t_1$ for $t \geq t_2$. Thus, for $t \geq t_2$ the above integral inequality yields

$$1 \geq \frac{1}{(l-1)! \cdot (n-l-1)!} \int_{t_1}^{t} (t-s)^{l-1} \int_{s}^{\infty} (u-s)^{n-l-1} p(u)\, du\, ds,$$

which contradicts (12). Hence $l = 0$. Consequently, (1) has property A. This completes the proof of the theorem. ■

Remark 2. In [1], Agarwal and Grace have proved the following different result (see Theorem 10 with $\alpha = 1$ and $f(y) = y$ in [1]): Let $\int_t^\infty p(s)ds < \infty$. If

$$\limsup_{t\to\infty} t^{n-1} \int_t^\infty p(s)\, ds > (n-1)!, \tag{13}$$

then for n even, every solution of (1) oscillates. In addition, if

$$\int^\infty s^{n-1} \int_s^\infty p(u)\, du\, ds = \infty \tag{14}$$

holds, then for n odd, every solution $y(t)$ of (1) either oscillates or $y^{(i)}(t) \to 0$ monotonically as $t \to \infty$, $i = 0, 1, \cdots, n-1$.

Example 1. By Theorem 2

$$y'''(t) + \frac{3}{t^3} y(2t) = 0, \quad t \geq 2$$

has property A. However, since (13) fails to hold, Theorem 3.5 in [1] cannot be applied to this example.

Example 2. Consider the equation

$$y'''(t) + \frac{7}{t^3} y(2t) = 0, \quad t \geq 1. \tag{15}$$

By Theorem 2, (15) has property A. On the other hand, Theorem 1 cannot be applied to (15).

Remark 3. Let the advanced argument $g(t) = 2t$ in (15) be replaced by $g(t) = \frac{11t}{10}$. Then we get

$$y'''(t) + \frac{7}{t^3} y\left(\frac{11t}{10}\right) = 0, \quad t \geq 1. \tag{16}$$

It is easy to check that the condition (12) fails to hold to the Eq.(16). It seems that if $g(t)$ is closer to t, then (12) fails to hold and hence Theorem 2 cannot be applied to (16). However, by Theorem 1, (16) has property A.

Theorem 3. *Let $g^{'}(t) > 0$. If for every $l \in \{1, 2, \cdots, n-1\}$,*

$$\int^{\infty} H_l(t)\, dt = \infty, \tag{17}$$

holds, where

$$H_{n-1}(t) = t^{n-1} g^{'}(t) p(g(t)) - \frac{(n-1) \cdot (n-1)! \cdot 2^{n-4} . t^{n-3}}{g^{'}(t) g^{n-2}(t)} \tag{18}$$

and

$$H_l(t) = \frac{t^l g^{'}(t)}{(n-l-2)!} \int_{g(t)}^{\infty} (s - g(t))^{n-l-2} p(s)\, ds - \frac{l \cdot l! \cdot t^{l-2} \cdot 2^{l-3}}{g^{'}(t) g^{l-1}(t)} \tag{19}$$

for $l \in \{1, 2, \cdots, n-2\}$, then for n even, every solution of (1) oscillates and for n odd every solution of (1) either oscillates or tends to zero as $t \to \infty$.

Proof. Suppose that (1) does not have property A. Then (1) admits a nonoscillatory solution $y(t)$ such that (3) holds for $l \in \{1, 2, \cdots, n-1\}$ for some $t \geq t_1 \geq \sigma$. We may assume, without any loss of generality, that $y(t) > 0$ for $t \geq t_1$.

Let $l = n-1$. Setting $z(t) = \frac{t^{n-1} y^{(n-1)}(g(t))}{y(g(t))}$ we see that $z(t) > 0$ for $t \geq t_1$ and

$$z^{'}(t) \leq -t^{n-1} p(g(t)) g^{'}(t) + \frac{n-1}{t} z(t) - \frac{y^{'}(g(t))}{y(g(t))} g^{'}(t) z(t). \tag{20}$$

Putting $i = 1$ and $l = n-1$ in (8), we obtain

$$y^{'}(t) \geq \frac{1}{(n-2)!} (t - t_1)^{n-2} y^{(n-1)}(t).$$

Hence for $t \geq 2t_1$, we get

$$y^{'}(t) \geq \frac{1}{(n-2)! \cdot 2^{n-2}} t^{n-2} y^{(n-1)}(t).$$

Thus for $t \geq t_2 > 2t_1$, we have

$$\frac{y^{'}(g(t))}{y(g(t))} \geq \frac{g^{n-2}(t)}{(n-2)! \cdot 2^{n-2}} \frac{z(t)}{t^{n-1}}. \tag{21}$$

Using (21) in (20), we have

$$z^{'}(t) \leq -F_{n-1}(t), \tag{22}$$

where

$$F_{n-1}(t) = t^{n-1}p(g(t))g^{'}(t) - \frac{n-1}{t}z(t) + \frac{g^{'}(t)g^{n-2}(t)}{(n-2)! \cdot 2^{n-2} \cdot t^{n-1}}z^2(t),$$

which attains the minimum $H_{n-1}(t)$ as given in (18). Integration of (22) from t_2 to t yields $z(t) < 0$ for large t, a contradiction.

Next, let $l \in \{1, 2, \cdots, n-2\}$. Setting $z_1(t) = \frac{t^l y^{(l)}(g(t))}{y(g(t))}$, we observe that $z_1(t) > 0$ for $t \geq t_1$ and

$$z_1^{'}(t) = \frac{t^l g^{'}(t) y^{(l+1)}(g(t))}{y(g(t))} + \frac{l}{t}z_1(t) - \frac{g^{'}(t)y^{'}(g(t))z_1(t)}{y(g(t))}. \tag{23}$$

Putting $i = 1$ in (8), we obtain for $t \geq t_1$

$$y^{'}(t) \geq \frac{t^{l-1}}{(l-1)! \cdot 2^{l-1}} y^{(l)}(t)$$

and hence for $t \geq t_2 \geq 2t_1$, the above inequality yields

$$\frac{y^{'}(g(t))}{y(g(t))} \geq \frac{g^{l-1}(t)}{t^l \cdot (l-1)! \cdot 2^{l-1}} z_1(t). \tag{24}$$

Next putting $i = l+1$ and $k = n$ and $(t_2 \leq) t < s$ and letting $s \to \infty$ in

$$y^{(i)}(t) = \sum_{j=i}^{k-1} \frac{(t-s)^{j-i}}{(j-i)!} y^{(j)}(s) + \frac{1}{(k-i-1)!} \int_s^t (t-u)^{k-i-1} y^{(k)}(u)\, du, \tag{25}$$

we obtain

$$\frac{y^{(l+1)}(g(t))}{y(g(t))} \leq -\frac{1}{(n-l-2)!} \int_{g(t)}^{\infty} (s - g(t))^{n-l-2} p(s)\, ds. \tag{26}$$

Using (24) and (26) in (23), we obtain

$$z_1^{'}(t) \leq -F_l(t) \tag{27}$$

where

$$\begin{aligned} F_l(t) &= \frac{g^{'}(t)g^{l-1}(t)}{t^l \cdot (l-1)! \cdot 2^{l-1}} z_1^2(t) - \frac{l}{t}z_1(t) \\ &\quad + \frac{t^l g^{'}(t)}{(n-l-2)!} \int_{g(t)}^{\infty} (s - g(t))^{n-l-2} p(s)\, ds \end{aligned}$$

which attains the minimum $H_l(t)$ as given in (19). Hence integration of (27) from t_2 to t yields a contradiction. ■

The theorem is proved for n even. Clearly, $l = 0$ implies that n is odd. Our theorem will be completed if we can show that $y(t) \to 0$ as $t \to \infty$. Since $l = 0$, then $\lim_{t\to\infty} y(t) = \lambda$, $0 \le \lambda < \infty$. If $0 < \lambda < \infty$, then for $0 < \epsilon < \lambda$, there exists a $t_3 \ge t_2$ such that $y(g(t)) > \lambda - \epsilon$ for $t \ge t_3$. Now putting $i = 0$, $k = n$ and $s > t = t_3$ and letting $s \to \infty$ in (25), we obtain

$$y(t_3) > (\lambda - \epsilon) \int_{t_3}^{\infty} (u - t_3)^{n-1} p(u)\, du$$

which in turn gives

$$\int_{t_3}^{\infty} (u - t_3)^{n-1} p(u)\, du < \infty.$$

On the other hand, the integral condition (17) with $l = n - 1$ yields that $\int_{t_3}^{\infty} t^{n-1} p(t)\, dt = \infty$, a contradiction. Hence $\lambda = 0$. Thus the theorem is proved.

Example 3. By Theorem 3,

$$y^{'''}(t) + \frac{24(t+1)^2}{t^5} y(t+1) = 0, \quad t \ge 1 \tag{28}$$

has property A. In particular, $y(t) = \frac{1}{t^2}$ is a nonoscillatory solution of the equation with $y(t) \to 0$ as $t \to \infty$.

Remark 4. If we set $z(t) = \frac{g^{n-1}(t) y^{(n-1)}(g(t))}{y(g(t))}$ for $l = n - 1$ and $z_1(t) = \frac{g^l(t) y^{(l)}(g(t))}{y(g(t))}$ for $l = 1, 2, \cdots, n-2$ and proceed as in the proof of Theorem 3, we obtain:

Theorem 4. *Let $g^{'}(t) > 0$. If for every $l \in \{1, 2, \cdots, n-1\}$,*

$$\int^{\infty} K_l(t)\, dt = \infty$$

where

$$K_{n-1}(t) = g^{n-1}(t) g^{'}(t) p(g(t)) - \frac{(n-1) \cdot (n-1)! \cdot 2^{n-4}}{g(t) g^{'}(t)}$$

and

$$K_l(t) = \frac{g^l(t) g^{'}(t)}{(n-l-2)!} \int_{g(t)}^{\infty} (s - g(t))^{n-l-2} p(s)\, ds - \frac{l \cdot l! \cdot 2^{l-3}}{g(t) g^{'}(t)}$$

then the conclusion of Theorem 3 holds.

Remark 5. When $g(t) = t$, then $H_l(t) = K_l(t), 1 \leq l \leq n-1$. On the other hand, if $g(t) > t$, from the second terms of the right hand sides of $H_l(t)$ and $K_l(t), 1 \leq l \leq n-1$, it follows that the conditions of Theorems 3 and 4 are different. If $g(t) = t$ and $n = 2$, then Theorem 3 or Theorem 4 solves the following open problem proposed by Kiguradze [9].

Problem: Let $M_{n^*} = \max(\lambda(\lambda-1)(\lambda-2)\dots(\lambda-n+1))$. If

$$\int^{\infty} t^{n-1}\big[p(t) - \frac{M_{n^*}}{t^n}\big]\, dt = \infty,$$

then (1) with $g(t) = t$ has property A.

Remark 6. In [12], Koplatadze et al. has obtained a sufficient condition for the property A of (1). This is given below: for example, see. It states that

Theorem 5 (Theorem 2.2, [12]). *Let $g(t)$ be nondecreasing, n be even, $\int^{\infty} t^{n-1}p(t)\, dt = \infty$ and*

$$\limsup_{t\to\infty}\{g(t)\int_{g(t)}^{\infty} s^{n-2}p(s)\, ds + \int_t^{g(t)} s^{n-1}p(s)\, ds + (g(t))^{-1}\int_0^t s^{n-1}g(s)p(s)\, ds\} > (n-1)! \tag{29}$$

hold, then (1) has property A.

The following example shows that our Theorem 3 can be applied where Theorem 5 fails to hold.

Example 4. Consider the equation

$$y''(t) + \frac{1}{2.1\, t^2}y(\frac{11t}{10}) = 0, \quad t \geq 1.$$

Clearly $p(t) = \frac{2.1}{t^2}$, $g(t) = \frac{11t}{10} > t$, $g^{-1}(t) = \frac{10t}{11}$ and $(g(t))^{-1} = \frac{10}{11t}$. Here $n = 2$. Hence $l = 1$. Now a simple calculation shows that

$$H_1(t) = \frac{10}{11t}\left\{\frac{1}{2.1} - \frac{1}{4}\right\}$$

holds, which in turn implies that (17) holds. Hence Theorem 3 can be applied to this example. On the other hand,

$$g(t)\int_{g(t)}^{\infty} s^{n-2}p(s)\, ds + \int_t^{g(t)} s^{n-1}p(s)\, ds + (g(t))^{-1}\int_0^t s^{n-1}g(s)p(s)\, ds = \frac{2.0953}{2.1} < 1$$

implies that (29) fails to hold. Consequently, Theorem 5 cannot be applied to this example.

Theorem 6. *Let $n \geq 3$ and $g(t) > t$. Further suppose that the third order ordinary differential equation*

$$u^{'''} + G_l(t)u = 0 \tag{30}$$

is oscillatory for $l = 1, 2, \cdots, n-1$, where

$$G_{n-1}(t) = \frac{1}{(n-3)!}(g(t) - t)^{n-3}p(t) \tag{31}$$

and

$$G_l(t) = \frac{1}{(l-1)! \cdot (n-l-2)!}(g(t) - t)^{l-2}(\int_t^\infty (s-t)^{n-l-2}p(s)\,ds) \tag{32}$$

for $l = 1, 2, \cdots, n-2$. Then (1) has property A.

Remark 7. Clearly, there are third order differential equations which admit both oscillatory and nonoscillatory solutions. A third order ordinary differential equation is said to be oscillatory if it has an oscillatory solution; otherwise, it is called nonoscillatory.

Proof of Theorem 6. Let $y(t)$ be a solution of (1). If $y(t)$ is oscillatory, then there is nothing to prove. Let $y(t)$ be nonoscillatory. We may assume, without any loss of generality, that $y(t) > 0$ for $t \geq t_0 > \sigma$. Then $y(g(t)) > 0$ and $y^{(n)}(t) \leq 0$ for $t \geq t_0$. There exists a positive integer $l \in \{0, 1, 2, \cdots, n-1\}$ such that $n + l$ is odd and (3) holds for some $t \geq t_1 \geq t_0$. We claim that $l = 0$. If not, then $l \in \{1, 2, \cdots, n-1\}$. First, suppose that $l = n-1$. Let $x(t) = y^{(n-3)}(t)$. Set $z(t) = \frac{x^{'}(t)}{x(t)}$. Then $z(t) > 0$ for $t \geq t_1$ and $z^{'}(t) = \frac{x^{''}(t)}{x(t)} - z^2(t)$. Further, assuming $u(t) = \exp(\int_{t_1}^t z(s)\,ds)$ and using

$$y(g(t)) \geq \frac{1}{(n-3)!}(g(t) - t)^{n-3}y^{(n-3)}(t)$$

we obtain

$$u^{'''} + G_{n-1}(t)u \leq 0$$

for $t \geq t_1$. This in turn implies, by Lemma 4 in [6], that (30) with $l = n-1$ is disconjugate, a contradiction. Next, suppose that $l \in \{1, 2, \cdots, n-2\}$. Putting $i = l+1, k = n$ and $(t_1 \leq) t < S$ and letting $s \to \infty$ in (25), we obtain

$$y^{(l+1)}(t) + \frac{1}{(n-l-2)!}(\int_t^\infty (s-t)^{n-l-2}p(s)\,ds)y(g(t)) \leq 0.$$

This inequality and

$$y(g(t)) \geq \frac{(g(t)-t)^{l-2}}{(l-1)!} y^{(l-2)}(t)$$

imply that $v(t) = y^{(l-2)}(t)$ is a solution of

$$v''' + G_l(t)v \leq 0$$

with $v(t) > 0, v'(t) > 0, v''(t) > 0$ and $v'''(t) < 0$ for $t \geq t_1$. This in turn implies by Lemma 4 in [6], that (30) with $l = 1, 2, \cdots, n-2$ is disconjugate, a contradiction. Hence $l = 0$. This completes the proof of the theorem. ■

Example 5. Since the third order ordinary differential equation

$$x''' + e^{\frac{1}{2}}x = 0$$

is oscillatory, then by Theorem 6, the equation

$$y''' + e^{\frac{1}{2}}y(t + \frac{1}{2}) = 0 \tag{33}$$

has property A. In particular, $y(t) = e^{-t}$ is a nonoscillatory solution of (33).

The next comparison theorem gives property A of Eq.(1).

Theorem 7. *If the ordinary differential equation*

$$x^{(n)}(t) + p(t)x(t) = 0 \tag{34}$$

has property A, then (1) has property A.

The proof of the theorem is in the lines of Theorem 2.7 due to Parhi and Padhi [15] for the delay differential equations. However, for completeness, we retain the proof of the theorem.

Proof of Theorem 7. If (1) does not have property A, then there exists a nonoscillatory solution $y(t)$ of (1) and an integer $l, 1 \leq l \leq n-1$ such that $n + l$ is odd and $y(t)$ satisfies (3) for some $t \geq t_1 \geq \sigma$.

From (9) we have, for $t \geq t_1$

$$y^{(l)}(t) \geq \frac{1}{(n-l-1)!} \int_t^\infty (s-t)^{n-l-1} p(s) y(g(s)))\, ds. \tag{35}$$

Further, putting $i = 0$ in (8), we obtain

$$y(t) \geq \lambda + \frac{1}{(l-1)!} \int_{t_1}^t (t-s)^{l-1} y^{(l)}(s)\, ds, \tag{36}$$

where $\lambda = y(t_1) > 0$. Hence from (35) and (36), we obtain

$$y(t) \geq \lambda + \frac{1}{(l-1)! \cdot (n-l-1)!} \int_{t_1}^{t} (t-s)^{l-1} \int_{s}^{\infty} (u-s)^{n-l-1} p(u) y(g(u))\, du\, ds.$$

Since $g(t) > t$, then the above integral inequality gives

$$y(t) \geq \lambda + \frac{1}{(l-1)! \cdot (n-l-1)!} \int_{t_1}^{t} (t-s)^{l-1} \int_{s}^{\infty} (u-s)^{n-l-1} p(u) y(u)\, du\, ds.$$

Further, since

$$\begin{aligned}\int_{t_1}^{\infty} (u-t_1)^{n-l-1} p(u) y(u)\, du &\leq \int_{t_1}^{\infty} (u-t_1)^{n-l-1} p(u) y(g(u))\, du \\ &\leq (n-l-1)! y^{(l)}(t_1) < \infty,\end{aligned}$$

then by Lemma 2, it follows that the integral equation

$$\begin{aligned}x(t) = \lambda + &\frac{1}{(l-1)! \cdot (n-l-1)!} \\ &\times \int_{t_1}^{t} (t-s)^{l-1} \int_{s}^{\infty} (u-s)^{n-l-1} p(u) x(u)\, du\, ds, \quad t \geq t_1\end{aligned}$$

has a solution $x \in C([t_1, \infty), (0, \infty))$ satisfying $\lambda \leq x(t) \leq y(t)$. Repeated integration of the above integral equation yields that $x(t)$ is a positive solution of (34) such that (3) holds for some $l \in \{1, 2, \cdots, n-1\}$, which contradicts the assumption of the theorem. Hence (1) has property A. This completes the proof of the theorem. ∎

Remark 8. Let $p(t) = p > 0$ be a constant and $g(t) = t + g$, $g > 0$ be a constant. Then (1) becomes

$$y^{(n)}(t) + p y(t+g) = 0. \tag{37}$$

The characteristic equation associated with (37) is given by

$$\lambda^n + p e^{g\lambda} = 0. \tag{38}$$

Let $F(\lambda) = \lambda^n + p e^{g\lambda}$. Then $F(0) = p > 0$ and $F^{'}(\lambda) = n\lambda^{n-1} + pg e^{g\lambda} > 0$ for $\lambda \geq 0$. Furthermore, for n even, $F(\lambda) > 0$ for $\lambda < 0$. Hence (38) has no real root when n is even. Consequently, all solutions of Eq.(37) oscillates when n is even. Next, suppose that n is odd. Then $\lim_{\lambda \to -\infty} F(\lambda) = -\infty$ implies that (38) has a negative real root. This in turn implies that (37) has a nonoscillatory solution which together with all its derivatives tend to zero as $t \to \infty$. It would be interesting to improve the above observation to the advanced differential equation (1). However we have not succeeded yet.

3. Main Results-II

This section deals with the asymptotic behaviour of nonoscillatory solutions of the advanced differential equations of the forms (2) and (6).

Theorem 8. *If for every $l \in \{1, 2, \cdots, n-2\}$ with $n+l$ even, the following conditions*

$$\int_{\sigma}^{\infty} t^{n-l-2} p(t)\, dt < \infty \quad \text{and} \quad \int_{\sigma}^{\infty} t^{n-l-1} p(t)\, dt = \infty \tag{39}$$

hold, then (2) has property B.

Proof. Suppose that (2) does not have property B. Then there exists a nonoscillatory solution $y(t)$ of (2) such that (3) is satisfied for $l \in \{1, 2, \cdots, n-2\}, n+l$ even. Without any loss of generality, we may assume that $y(t) > 0$ and (3) holds for $t \geq t_0 \geq \sigma$.

Repeated integration of (2) from $t(\geq t_0)$ to ∞ yields

$$y^{(l)}(t) \leq y^{(l)}(t_1) - \frac{y(g(t_1))}{(n-l-2)!} \int_{t_1}^{t} \int_{s}^{\infty} (u-s)^{n-l-2} p(u)\, du\, ds. \tag{40}$$

A simple integration shows that (39) implies

$$\lim_{t\to\infty} \int_{t_1}^{t} \int_{s}^{\infty} (u-s)^{n-l-2} p(u)\, du\, ds = \infty.$$

Hence from (40) it follows that $y^{(l)}(t) < 0$ for large t, a contradiction. Hence (2) has property B. This completes the proof of the theorem. ■

Example 6. The equation

$$y'''(t) - \frac{1}{t^2} y(t+1) = 0, \quad t \geq 1$$

has property B, by Theorem 8.

Remark 9. Consider the equation

$$y'''(t) + \frac{1}{3t^2} y(2t) = 0, \quad t \geq 1.$$

Theorem 8 can be applied to this example, while Theorem 6.12 due to [11] cannot be applied to this example. Thus Theorem 8 cannot be treated as a particular case of Theorem 6.12 in [11].

If we proceed as in the lines of the proof of Theorem 8, we obtain the following result for the delay differential equation (6):

Theorem 9. *If for every $l \in \{1, 2, \cdots, n-2\}$ with $n+l$ even,*

$$\int_\sigma^\infty t^{n-l-2} q(t)\, dt < \infty$$

and

$$\int_\sigma^\infty t^{n-l-1} q(t)\, dt = \infty$$

hold, then (6) has property B.

Since Theorem 8 and Theorem 9 hold for their corresponding ordinary differential equations, it is interesting to find out sufficient conditions for the property B of (2) and (6) by retaining the functions $g(t)$ and $\tau(t)$.

Theorem 10. *Let $g(t) > t$. If for every $l \in \{1, 2, \cdots, n-2\}$ with $n+l$ even*

$$\limsup_{t\to\infty} (t - g^{-1}(t))^{l-1} (g^{-1}(t) - T)^2 \int_{2g^{-1}(t)}^\infty (s - 2g^{-1}(t))^{n-l-2} p(s)\, ds > 2 \cdot (l-1)! \cdot (n-l-2)! \tag{41}$$

holds for every $T \geq \sigma$, then (2) has property B.

Proof. Suppose, for the sake of contradiction, that (2) does not have property B. Then there exists a nonoscillatory solution $y(t)$ of (2) such that (3) is satisfied for $l \in \{1, 2, \cdots, n-2\}, n+l$ even. Without any loss of generality, we may assume that $y(t) > 0$ and (3) holds for $t \geq t_0 \geq \sigma$.

Integrating $\frac{dy^{(l)}(t)}{dt} = y^{(l+1)}(t)$ from $t (\geq t_0)$ to $2t$, we obtain

$$y^{(l)}(t) \geq -t y^{(l+1)}(2t).$$

Further integration from t_0 to t gives

$$y^{(l-1)}(t) \geq -\frac{(t-t_1)^2}{2} y^{(l+1)}(2t). \tag{42}$$

Next, putting $i = l+1$, $k = n$ and $s > t (\geq t_0)$ and letting $s \to \infty$ in (25), we get

$$y^{(l+1)}(t) \leq -\frac{y(g(t))}{(n-l-2)!} \int_t^\infty (s-t)^{n-l-2} p(s)\, ds. \tag{43}$$

Hence from (42) and (43), we get

$$y^{(l-1)}(t) \geq \frac{(t-t_1)^2}{2 \cdot (n-l-2)!} y(g(t)) \int_{2t}^\infty (s-2t)^{n-l-2} p(s)\, ds. \tag{44}$$

Next, using

$$y(t) \geq \frac{(t - g^{-1}(t))^{l-1}}{(l-1)!} y^{(l-1)}(g^{-1}(t)),$$

and (44), we obtain

$$1 \geq \frac{(t - g^{-1}(t))^{l-1}(g^{-1}(t) - t_1)^2}{2 \cdot (n-l-2)! \cdot (l-1)!} \int_{2g^{-1}(t)}^{\infty} (s - 2g^{-1}(t))^{n-l-2} p(s)\, ds.$$

Taking lim sup both sides in the above inequality, we get a contradiction to the above assumption. The proof is complete. ■

We have the following result for Eq.(6) similar to Theorem 10:

Theorem 11. *Let $\tau(t) < t$ and if for every $l \in \{1, 2, \cdots, n-2\}$*

$$\limsup_{t\to\infty} (\tau(t) - T)^2 (t - \tau(t))^{l-1} \int_{2\tau^{-1}(t)}^{\infty} (s - 2\tau(t))^{n-l-2} p(s)\, ds > 2 \cdot (l-1)! \cdot (n-l-2)!$$

for every $T \geq \sigma$, then (6) has property B.

Using Lemma3 we can obtain the following result:

Theorem 12. *Let $\tau(t) < t$ and $l \in \{1, 2, \cdots, n-2\}$. If for every $\mu \in (0,1)$ and $T \geq \sigma$*

$$\limsup_{t\to\infty} \frac{\tau(t)(t-T)^2(t-\tau(t))^{l-1}}{t} \int_{2t}^{\infty} (s - 2t)^{n-l-2} p(s)\, ds > \frac{2 \cdot (n-l-2)! \cdot (l-1)!}{\mu}$$

holds, then (6) has property B.

Remark 10. Theorem 11 holds also when $n = 3$ and $\tau(t) = t$. In fact, for $n = 3$ and $\tau(t) \leq t$, it is easy to prove the following result (see [14]).

Theorem 13. *Let $n = 3$, $\tau(t) \leq t$ and*

$$\limsup_{t\to\infty} (t - T)^2 \int_{2\tau^{-1}(t)}^{\infty} p(s)\, ds > 2$$

for every $T \geq \sigma$, then (6) has property B.

Clearly, for the specific case $n = 3$, Theorem 11 is not as good as Theorem 13. Thus Theorem 11 is yet to be improved.

Acknowledgement. The authors are thankful to the referee for his/her suggestions in revising the manuscript.

References

[1] Agarwal R.P., Grace S.R., Oscillation theorems for certain functional differential equations of higher order, *Math. Comp. Modelling,* 39(2004), 1185-1194.

[2] Bainov D.D., Mishev D.P., *Oscillation Theory for Neutral Differential Equations with Delay,* Adam Hilger, New York 1990.

[3] Erbe L., Kong Q., Zhang B.G., Oscillation Theory for Functional Differential Equations, *Marcel Dekker Inc.,* New York 1995.

[4] Graef J.R., Koplatadze R., Kvinikadze G., Nonlinear functional differential equations with properties A and B, *J. Math. Anal. Appl.,* 306(2005), 136-160.

[5] Gramatikopolous M.K., Koplatadze R., Kvinikadze G.V., Linear functional differential equations with property A, *J. Math. Anal. Appl.,* 284 (2003), 294-314.

[6] Gregus M., Gera M., Some results in the theory of a third order linear differential equations, *Ann. Polonici Math.,* XLII(1983), 93-102.

[7] Gyori I., Ladas G., *Oscillation Theory of Delay Differential Equations,* Clarendon Press, Oxford 1991.

[8] Hanan M., Oscillation criteria for third order linear differential equations, *Pacific J. Math.,* 11(1961), 919-944.

[9] Kiguradze I.T., Chanturia T.A., *Asymptotic Properties of Solutions of Nonautonomous Ordinary Differential Equations,* Kluwer Academic Pub., London 1993.

[10] Kusano T., Naito M., Comparison theorems for functional differential equations with deviating arguments, *J. Math. Soc. Japan,* 33(1981), 509-532.

[11] Koplatadze R.G., On oscillatory properties of solutions of functional differential equations, *Mem. Differential Equations Math. Phys.,* 3(1994), 1-179.

[12] Koplatadze R., Kvinikadze G., Stavroulakis I.P., Properties A and B of n-th order linear differential equations with deviating arguments, *Georgian Math. J.,* 6(6)(1999), 553-566.

[13] Ladde G.S., Lakshmikantham V., Zhang B.G., *Oscillation Theory of Differential Equations with Deviating Arguments,* Marcel Dekker, INC. New York and Bassel 1987.

[14] Parhi N., Padhi S., Asymptotic behaviour of solutions of third order delay differential equations, *Indian J.Pure and Appl. Math.,* 33(10)(2002), 1609-1620.

[15] Parhi N., Padhi S., Asymptotic behaviour of solutions of delay differential equations of n-th order, *Arch. Math.* (*BRNO*)*,* 37(2001), 81-101.

[16] Padhi S., Asymptotic behaviour of solutions to n-order functional differential equations, *Elec. J. Diff. Equs.,* 65(2005), 1-14.

Seshadev Padhi
Department of Mathematics and Statistics
Mississippi State University
Mississippi state, MS 39762, U.S.A.

Permanent address:
Department of Applied Mathematics
Birla Institute of Technology
Mesra, Ranchi-835 215, India
e-mail: ses_2312@yahoo.co.in

Chuanxi Qian
Department of Mathematics and Statistics
Mississippi State University
Mississippi state, MS 39762, U.S.A.

Received on 09.03.2010 and, in revised form, on 28.07.2010.