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N. Rajesh, M. Caldas, S. Jafari and S.P. Moshokoa

QUASI δ-β-CONTINUOUS FUNCTIONS

Abstract. The aim of this paper is to introduce and characterize a new class of functions called quasi-δ-β-continuous functions in ideal topological spaces by using δ-β-open sets.

Key words: topological spaces, δ-β-open sets, quasi δ-β-continuous functions.

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1. Introduction

Functions and of course continuous functions stand among the most important notions in the whole of mathematical science. Many different forms of continuous functions have been introduced over the years. Various interesting problems arise when one considers continuity. Its importance is significant in various areas of mathematics and related sciences. Recently, Hatir and Noiri [3] have introduced a weak form of open sets called δ-β-open sets. The aim of this paper is to introduce and characterize a new class of functions called quasi-δ-β-continuous functions in topological spaces by using δ-β-open sets.

2. Preliminaries

Let A be a subset of a topological space (X,τ). We denote the closure of A and the interior of A by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A of a topological space (X,τ) is said to be regular open [9] if $A = \mathrm{Int}(\mathrm{Cl}(A))$. A set $A \subset X$ is said to be δ-open [10] if it is the union of regular open sets of X. The complement of a regular open (resp. δ-open) set is called regular closed (resp. δ-closed). The intersection of all δ-closed sets of (X,τ) containing A is called the δ-closure [10] of A and is denoted by $\mathrm{Cl}_\delta(A)$. A point $x \in X$ is called a θ-cluster point of A if $\mathrm{Cl}(V) \cap A \neq \varnothing$ for every open set V of X containing x. The set of all θ-cluster points of A is called the θ-closure of A [10] and is denoted by $\mathrm{Cl}_\theta(A)$. If $A = \mathrm{Cl}_\theta(A)$, then A is said to be θ-closed [10]. The complement of θ-closed set is said to be θ-open [10]. The union of all θ-open sets contained in a subset A is called the θ-interior of A and is denoted by $\mathrm{Int}_\theta(A)$ [10]. A subset A of a topological space (X,τ) is said to be δ-β-open [3]) if $S \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}_\delta(S)))$. The complement of a δ-β-open set is called δ-β-closed [3]. The intersection of all δ-β-closed sets containing S is called the δ-β-closure of S and is denoted by ${}_\beta\mathrm{Cl}_\delta(S)$. The δ-β-interior of S is defined by the union of all δ-β-open sets contained in S and is denoted by ${}_\beta\mathrm{Int}_\delta(S)$. The set of all δ-β-open sets of (X,τ) is denoted by $\delta\beta O(X)$. The set of all δ-β-open sets of (X,τ) containing a point $x \in X$ is denoted by $\delta\beta O(X,x)$. A subset B_x of a topological space (X,τ) is said to be a δ-β-neighbourhood [4] of a point $x \in X$ if there exists a δ-β-open set U such that $x \in U \subset B_x$.

Definition 1. *A function $f:(X,\tau)\to(Y,\sigma)$ is said to be:*

(i) *δ-β-continuous [4] at a point $x \in X$ if for each open subset V in Y containing $f(x)$, there exists $U \in \delta\beta O(X,x)$ such that $f(U) \subset V$;*

(ii) *δ-β-continuous [4] if it has this property at each point of X.*

Definition 2. *A function $f:(X,\tau)\to(Y,\sigma)$ is said to be:*

(i) *almost δ-β-continuous [2] at a point $x \in X$ if for each open subset V in Y containing $f(x)$, there exists $U \in \delta\beta O(X,x)$ such that $f(U) \subset \mathrm{Int}(\mathrm{Cl}(V))$;*

(ii) *almost δ-β-continuous [2] if it has this property at each point of X.*

Definition 3. *A function $f:(X,\tau)\to(Y,\sigma)$ is said to be faintly δ-β-continuous [1] if for each $x \in X$ and for each θ-open set V of Y containing $f(x)$, then there exists $U \in \delta\beta O(X,x)$ such that $f(U) \subset V$.*

Definition 4. *A topological space (X,τ) is said to be:*

(i) *δ-β-T_1 [4] if for each pair of distinct pointsx and y of X, there exists δ-β-open sets and U and V such that $x \in U$, $y \notin U$ and $x \notin V$, $y \in V$.*

(ii) *δ-β-T_2 [4] if for each pair of distinct points x and y of X, there exists δ-β-open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \varnothing$.*

(iii) *almost regular [7] if for any regular closed set F of X and any point $x \in X\backslash F$ there exist disjoint open sets U and V such that $x \in U$ and $F \subset V$.*

3. Quasi δ-β-continuous functions

Definition 5. *A function $f:(X,\tau)\to(Y,\sigma)$ is said to be quasi δ-β-continuous if for each $x \in X$ and each open set V of Y containing $f(x)$ there exists $U \in \delta\beta O(X,x)$ such that $f(U) \subset \mathrm{Cl}(V)$.*

Theorem 1. *If a function $f:(X,\tau)\to(Y,\sigma)$ is almost δ-β-continuous, then it is quasi δ-β-continuous.*

Proof. Let $x \in X$ and $V \subset Y$ be an open set with $f(x) \in V$. Then since $f(x) \in V \subset \mathrm{Cl}(V)$, $f(x) \in \mathrm{Int}(\mathrm{Cl}(V))$, which is regular open. Since

f is almost δ-β-continuous, there exists $U \in \delta\beta O(X,x)$ such that $f(U) \subset \mathrm{Int}(\mathrm{Cl}(V)) \subset \mathrm{Cl}(V)$. Therefore, f is quasi δ-β-continuous. ■

Remark 1. The converse of Theorem 1 is not true in general as can be seen from the following example.

Example 1. Let $X = \{a,b,c\}$, $\tau = \{\varnothing, \{b\}, \{c\}, \{b,c\}, X\}$ and $\sigma = \{\varnothing, \{a\}, \{a,b\}, X\}$. Then the identity function $f : (X,\tau) \to (X,\sigma)$ is quasi δ-β-continuous but not almost δ-β-continuous.

Corollary 1. *Every δ-β-continuous function is quasi δ-β-continuous.*

Theorem 2. *If $f : (X,\tau) \to (Y,\sigma)$ is a quasi δ-β-continuous function and Y is almost regular, then f is almost δ-β-continuous.*

Proof. Let $x \in X$ and let V be any open set of Y containing $f(x)$. By the almost regularity of Y, there exists a regular open set G of Y such that $f(x) \in G \subset \mathrm{Cl}(G) \subset \mathrm{Int}(\mathrm{Cl}(V))$ [[7], Theorem 2.2]. Since f is quasi δ-β-continuous, there exists $U \in \delta\beta O(X,x)$ such that $f(U) \subset \mathrm{Cl}(G) \subset \mathrm{Int}(\mathrm{Cl}(V))$. Therefore, f is almost δ-β-continuous. ■

Theorem 3. *For a function $f : (X,\tau) \to (Y,\sigma)$, the following statements are equivalent:*

(i) *f is quasi δ-β-continuous;*
(ii) ${}_\beta\mathrm{Cl}_\delta(f^{-1}(\mathrm{Int}(\mathrm{Cl}_\theta(A)))) \subset f^{-1}(\mathrm{Cl}_\theta(A))$ *for every subset A of Y;*
(iii) ${}_\beta\mathrm{Cl}_\delta(f^{-1}(\mathrm{Int}(\mathrm{Cl}(B)))) \subset f^{-1}(\mathrm{Cl}(B))$ *for every open set B of Y;*
(iv) ${}_\beta\mathrm{Cl}_\delta(f^{-1}(\mathrm{Int}(C))) \subset f^{-1}(C)$ *for every regular closed set C of Y;*
(v) ${}_\beta\mathrm{Cl}_\delta(f^{-1}(D)) \subset f^{-1}(\mathrm{Cl}(D))$ *for every open set D of Y;*
(vi) $f^{-1}(E) \subset {}_\beta\mathrm{Int}_\delta(f^{-1}(\mathrm{Cl}(E)))$ *for every open set E of Y.*

Proof. (i) $\Rightarrow$ (ii): Let A be a subset of Y and $x \in X \backslash f^{-1}(\mathrm{Cl}_\theta(A))$. Then $x \notin f^{-1}(\mathrm{Cl}_\theta(A))$, that is, $f(x) \notin \mathrm{Cl}_\theta(A)$. This means that the existence of an open set W of Y containing $f(x)$ such that $A \cap \mathrm{Cl}(W) = \varnothing$. Hence $\mathrm{Cl}_\theta(A) \cap W = \varnothing$. So, $W \subset Y \backslash \mathrm{Cl}_\theta(A)$, that is, $\mathrm{Cl}(W) \subset \mathrm{Cl}(Y \backslash \mathrm{Cl}_\theta(A))$. Since f is quasi δ-β-continuous, there exists $U \in \delta\beta O(X,x)$ such that $f(U) \subset \mathrm{Cl}(W) \subset \mathrm{Cl}(Y \backslash \mathrm{Cl}_\theta(A))$. So $f(U) \cap (Y \backslash \mathrm{Cl}(Y \backslash \mathrm{Cl}_\theta(A))) = \varnothing$. Then $f(U) \cap \mathrm{Int}(\mathrm{Cl}_\theta(A)) = \varnothing$ and hence $U \cap f^{-1}(\mathrm{Int}(\mathrm{Cl}_\theta(A))) = \varnothing$. This shows that $x \notin {}_\beta\mathrm{Cl}_\delta(f^{-1}(\mathrm{Int}(\mathrm{Cl}_\theta(A))))$. Therefore, ${}_\beta\mathrm{Cl}_\delta(f^{-1}(\mathrm{Int}(\mathrm{Cl}_\theta(A)))) \subset f^{-1}(\mathrm{Cl}_\theta(A))$.

(ii) $\Rightarrow$ (iii): This implication is follows from the fact that, $\mathrm{Cl}_\theta(A)$=$\mathrm{Cl}(A)$ for every open set B of Y.

(iii) $\Rightarrow$ (iv): Let C be a regular closed subset of Y. Then ${}_\beta\mathrm{Cl}_\delta(f^{-1}(\mathrm{Int}(C)))$ $= {}_\beta\mathrm{Cl}_\delta(f^{-1}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(C))))) \subset f^{-1}(\mathrm{Cl}(\mathrm{Int}(C))) = f^{-1}(C)$.

(iv) $\Rightarrow$ (v): Let D be an open subset of Y. Then $\mathrm{Cl}(D)$ is regular closed in Y. So, ${}_\beta\mathrm{Cl}_\delta(f^{-1}(D)) = {}_\beta\mathrm{Cl}_\delta(f^{-1}(\mathrm{Int}(D))) \subset {}_\beta\mathrm{Cl}_\delta(f^{-1}(\mathrm{Int}(\mathrm{Cl}(D)))) \subset f^{-1}(\mathrm{Cl}(D))$, by ($iv$).

$(v) \Rightarrow (vi)$: Let $x \in f^{-1}(E)$. Then $f(x) \in E$ and since $E \cap (Y \setminus \mathrm{Cl}(E)) = \varnothing$, $f(x) \notin \mathrm{Cl}(Y \setminus \mathrm{Cl}(E))$ where $x \notin f^{-1}(\mathrm{Cl}(Y \setminus \mathrm{Cl}(E)))$. Openness of $(Y \setminus \mathrm{Cl}(E))$ gives from (v) that $x \notin_\beta \mathrm{Cl}_\delta(f^{-1}(Y \setminus \mathrm{Cl}(E)))$. This implies the existence of $U \in \delta\beta O(X, x)$ such that $U \cap f^{-1}(Y \setminus \mathrm{Cl}(E)) = \varnothing$; that is, $f(U) \cap (Y \setminus \mathrm{Cl}(E)) = \varnothing$. Which assures that $f(U) \subset \mathrm{Cl}(E)$ and hence $U \subset f^{-1}(\mathrm{Cl}(E))$. Thus $x \in U \subset f^{-1}(\mathrm{Cl}(E))$ and this indicates that x is a δ-β-interior point of $f^{-1}(\mathrm{Cl}(E))$. Consequently, $f^{-1}(E) \subset {}_\beta\,\mathrm{Int}_\delta(f^{-1}(\mathrm{Cl}(E)))$.

$(vi) \Rightarrow (i)$: Let $x \in X$ and V be an open subset of Y containing $f(x)$ by (vi), $x \in f^{-1}(V) \subseteq {}_\beta\,\mathrm{Int}_\delta(f^{-1}(\mathrm{Cl}(V)))$. Let $U = {}_\beta\,\mathrm{Int}_\delta(f^{-1}(\mathrm{Cl}(V)))$. Then $U \in \delta\beta O(X, x)$. Now, $f(U) = f({}_\beta\mathrm{Int}_\delta(f^{-1}(\mathrm{Cl}(V)))) \subseteq f(f^{-1}(\mathrm{Cl}(V))) \subset \mathrm{Cl}(V)$. This shows that f is quasi δ-β-continuous. ■

Theorem 4. *The following statements are equivalent for a function $f : (X, \tau) \to (Y, \sigma)$:*

(i) f is quasi δ-β-continuous;

(ii) $f({}_\beta\mathrm{Cl}_\delta(A)) \subset \mathrm{Cl}_\theta(f(A))$ for each subset A of X;

(iii) ${}_\beta\,\mathrm{Cl}_\delta(f^{-1}(B)) \subset f^{-1}(\mathrm{Cl}_\theta(B))$ for each subset B of Y;

(iv) ${}_\beta\,\mathrm{Cl}_\delta(f^{-1}(\mathrm{Int}(\mathrm{Cl}_\theta(B)))) \subset f^{-1}(\mathrm{Cl}_\theta(B))$ for every subset B of Y.

Proof. $(i) \Rightarrow (ii)$: Let A be any subset of X and $x \in_\beta \mathrm{Cl}_\delta(A)$. Then $f(x) \in f({}_\beta\mathrm{Cl}_\delta(A))$ Suppose that V be an open set of Y containing $f(x)$. Then there exists $U \in \delta\beta O(X, x)$ such that $f(U) \subset \mathrm{Cl}(V)$. Since $x \in_\beta \mathrm{Cl}_\delta(A)$, $U \cap A \neq \emptyset$ and hence $\emptyset \neq f(U) \cap f(A) \subset \mathrm{Cl}(V) \cap f(A)$. Therefore, we have $f(x) \in \mathrm{Cl}_\theta(f(A))$ and hence $f({}_\beta\mathrm{Cl}_\delta(A)) \subset \mathrm{Cl}_\theta(f(A))$.

$(ii) \Rightarrow (iii)$: Let B be any subset of Y. We have $f({}_\beta\mathrm{Cl}_\delta(f^{-1}(B))) \subset \mathrm{Cl}_\theta(B)$ and hence ${}_\beta\,\mathrm{Cl}_\delta(f^{-1}(B)) \subset f^{-1}(\mathrm{Cl}_\theta(B))$.

$(iii) \Rightarrow (iv)$: Let B be any subset of Y. Since $\mathrm{Cl}_\theta(B)$ is closed in Y we have ${}_\beta\,\mathrm{Cl}_\delta(f^{-1}(\mathrm{Int}(\mathrm{Cl}_\theta(B)))) \subset f^{-1}(\mathrm{Cl}_\theta(B)) = f^{-1}(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}_\theta(B))))$ $f^{-1}(\mathrm{Cl}_\theta(B))$.

$(iv) \Rightarrow (i)$: Let V be any open subset of Y. Then $V \subset \mathrm{Int}(\mathrm{Cl}(V)) = \mathrm{Int}(\mathrm{Cl}_\theta(V))$. Then ${}_\beta\,\mathrm{Cl}_\delta(f^{-1}(V)) \subset f^{-1}(\mathrm{Cl}(V))$. It follows from Theorem 3 that is quasi δ-β-continuous. ■

Theorem 5. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function and Y be regular. Then the following statements are equivalent:*

(i) f is δ-β-continuous;

(ii) $f^{-1}(\mathrm{Cl}_\theta(B))$ is δ-β-closed in X for every subset B of Y;

(iii) f is quasi δ-β-continuous;

(iv) f is faintly δ-β-continuous.

Proof. $(i) \Rightarrow (ii)$: Since $\mathrm{Cl}_\theta(B)$ is closed in Y for every subset B of Y then by (i), $f^{-1}(\mathrm{Cl}_\theta(B))$ is δ-β-closed in X.

$(ii) \Rightarrow (iii)$: Clear.

$(iii) \Rightarrow (iv)$: Let F be any θ-closed set of Y. By Theorem 4 we have ${}_\beta\,\mathrm{Cl}_\delta(f^{-1}(F)) \subset f^{-1}(\mathrm{Cl}_\theta(F)) = f^{-1}(F)$. Therefore, $f^{-1}(F)$ is δ-β-closed in X and it follows that f is faintly δ-β-continuous.

$(iv) \Rightarrow (i)$: Let V be any open set of Y. Since Y is regular, V is θ-open in Y. By (iv), $f^{-1}(V)$ is δ-β-open in X. This shows that, f is δ-β-continuous. ■

Definition 6. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be weakly continuous [6] if for each $x \in X$ and an open set V in Y containing $f(x)$, there exists an open set U of X containing x such that $f(U) \subset \mathrm{Cl}(V)$.*

Theorem 6. *If $f : (X, \tau) \to (Y, \sigma)$ is δ-β-continuous and $g : (Y, \sigma) \to (Z, \eta)$ is weakly continuous, then the composition $g \circ f : (X, \tau) \to (Z, \eta)$ is quasi δ-β-continuous.*

Proof. Let $x \in X$ and W be an open subset of Z containing $g(f(x))$. Since g is weakly continuous, then there exists an open set V of Y containing $f(x)$ such that $g(V) \subset \mathrm{Cl}(W)$. Again since f is δ-β-continuous, there exists $U \in \delta\beta O(X, x)$ such that $f(U) \subset V$. Then $(g \circ f)(U) \subset g(V) \subset \mathrm{Cl}(W)$. This shows that $g \circ f : (X, \tau) \to (Z, \eta)$ is quasi δ-β-continuous. ■

Theorem 7. *If $f : (X, \tau) \to (Y, \sigma)$ is quasi δ-β-continuous and $g : (Y, \sigma) \to (Z, \eta)$ is continuous, then the composition $g \circ f : (X, \tau) \to (Z, \eta)$ is quasi δ-β-continuous.*

Proof. Let $x \in X$ and W be an open subset of Z containing $g(f(x))$ then $g^{-1}(W)$ is an open set of Y containing $f(x)$. Since f is quasi δ-β-continuous, there exists $U \in \delta\beta O(X, x)$ such that $f(U) \subset \mathrm{Cl}(g^{-1}(W))$. Since g is continuous, we obtain $(g \circ f)(U) \subset g(\mathrm{Cl}(g^{-1}(W))) \subset \mathrm{Cl}(W)$. Thus, $g \circ f$ is quasi δ-β-continuous. ■

Lemma 1. *[4] Let A and B be subsets of a topological space (X, τ). If $A \in \delta\beta O(X)$ and B is δ-open in (X, τ), then $A \cap B \in \delta\beta O(B)$.*

Lemma 2. *If $A \subset B \subset X$ and B is δ-open in (X, τ), then ${}_\beta\,\mathrm{Cl}_\delta(A) \cap B = {}_\beta\,\mathrm{Cl}_{\delta B}(A)$, where ${}_\beta\,\mathrm{Cl}_{\delta B}(A)$ denotes the δ-β-closure of A in the subspace B.*

Theorem 8. *If $f : (X, \tau) \to (Y, \sigma)$ is quasi δ-β-continuous and A is a δ-open subset of X, then the restriction $f_{|_A} : (A, \tau_{|_A}) \to (Y, \sigma)$ is quasi δ-β-continuous.*

Proof. Let $x \in A$ and V be an open subset of Y containing $f(x)$ by hypothesis there exists $U \in \delta\beta O(X, x)$ such that $f(U) \subset \mathrm{Cl}(V)$. Let $U = G \cap A$, where $G \in \delta\beta O(X, x)$. Then by Lemma 1, $U \in \delta\beta O(A)$. Also $(f_{|_A})(U) = (f_{|_A})(G \cap A) = f(G \cap A) \subset f(G) \subset \mathrm{Cl}(V)$. This shows that $f_{|_A} : (A, \tau_{|_A}) \to (Y, \sigma)$ is quasi δ-β-continuous. ■

Definition 7. *The graph $G(f)$ of a function $f : (X, \tau) \to (Y, \sigma)$ is said to be quasi δ-β-closed if for each $(x, y) \in (X \times Y)\backslash G(f)$, there exist $U \in \delta\beta O(X, x)$ and an open set V of Y containing y such that $(U \times \mathrm{Cl}(V)) \cap G(f) = \varnothing$.*

Lemma 3. *The graph $G(f)$ of $f : (X, \tau) \to (Y, \sigma)$ is quasi δ-β-closed in $X \times Y$ if and only if for each $(x, y) \in (X \times Y)\backslash G(f)$, there exist $U \in \delta\beta O(X, x)$ and an open set V of Y containing y such that $f(U) \cap \mathrm{Cl}(V) = \varnothing$.*

Proof. It follows immediately from the Definition 7. ■

Theorem 9. *If $f : (X, \tau) \to (Y, \sigma)$ is quasi δ-β-continuous and Y is a Urysohn space, then the graph $G(f)$ of f is quasi δ-β-closed in $X \times Y$.*

Proof. Let $(x, y) \notin G(f)$, then $y \neq f(x)$. Since Y is Urysohn, there exist open sets V_1 and V_2 of Y containing $f(x)$ and y, respectively, such that $\mathrm{Cl}(V_1) \cap \mathrm{Cl}(V_2) = \varnothing$. Since f is quasi δ-β-continuous, there exists $U \in \delta\beta O(X, x)$ such that $f(U) \subset \mathrm{Cl}(V_1)$ and consequently $f(U) \cap \mathrm{Cl}(V_2) = \varnothing$. This shows that graph $G(f)$ is quasi δ-β-closed in $X \times Y$. ■

Definition 8. *A topological space (X, τ) is said to be δ-β-connected [4] if it is not the union of two nonempty disjoint δ-β-open sets.*

Theorem 10. *If (X, τ) is a δ-β-connected space and $f : (X, \tau) \to (Y, \sigma)$ is a quasi δ-β-continuous function with the quasi δ-β-closed graph $G(f)$, then f is constant.*

Proof. Suppose that f is not constant. There exist disjoint points $x, y \in X$ such that $f(x) = f(y)$. Since $(x, f(x)) \notin G(f)$, by Lemma 3 there exist open sets U and V containing x and $f(x)$, respectively, such that $f(U) \cap \mathrm{Cl}(V) = \emptyset$. Since f is quasi δ-β-continuous, there exists $G \in \delta\beta O(X, y)$ such that $f(G) \subset Cl(V)$. Since U and V are disjoint δ-β-open sets of (X, τ), it follows that (X, τ) is not δ-β-connected. Therefore, f is constant. ■

Theorem 11. *Let $f : (X, \tau) \to (Y, \sigma)$ be a quasi δ-β-continuous injective function. If (Y, σ) is Urysohn, then (X, τ) is δ-β-T_2.*

Proof. Since f is injective, for any pair of distinct points $x_1, x_2 \in X$, $f(x_1) \neq f(x_2)$. Since (Y, σ) is Urysohn, there exist $V_1, V_2 \in \sigma$ such that $f(x_1) \in V_1$, $f(x_2) \in V_2$ and $\mathrm{Cl}(V_1) \cap \mathrm{Cl}(V_2) = \varnothing$. This gives $f^{-1}(\mathrm{Cl}(V_1)) \cap f^{-1}(\mathrm{Cl}(V_2))) = \varnothing$. Since f is quasi δ-β-continuous $x_i \in f^{-1}(V_i) \subset {}_{\beta}\mathrm{Int}_{\delta}(f^{-1}(\mathrm{Cl}(V_i))$, $i = 1,2.$, by Theorem 3 and this indicates that (X, τ) is δ-β-T_2. ■

Theorem 12. *If for each pair of distinct x_1 and x_2 in a topological space (X,τ) there exists a function f of X into a Hausdorff space Y such that*

(i) $f(x_1) \neq f(x_2)$,

(ii) f *is quasi δ-β-continuous at* x_1 *and*

(iii) f *is almost δ-β-continuous at* x_2,

then X is δ-β-T_2.

Proof. Since Y is Hausdorff, there exist open sets V_1 and V_2 of Y such that $f(x_1) \in V_1$, $f(x_2) \in V_2$, and $V_1 \cap V_2 = \emptyset$; hence $\mathrm{Cl}(V_1) \cap \mathrm{Int}(\mathrm{Cl}(V_2)) = \emptyset$. Since f is quasi δ-β-continuous at x_1, there exists $U_1 \in \delta\beta O(X, x_1)$ such that $f(U_1) \subset \mathrm{Cl}(V_1)$. Since f is almost δ-β-continuous at x_2, there exists $U_2 \in \delta\beta O(X, x_2)$ such that $f(U_2) \subset \mathrm{Int}(\mathrm{Cl}(V_2))$. Therefore, we obtain $U_1 \cap U_2 = \emptyset$. This shows that X is δ-β-T_2. ■

Lemma 4. *[4] Let A be a subset of a space (X,τ). Then*

(i) $A \subset B \Rightarrow_\beta \mathrm{Int}_\delta(A) \subset_\beta \mathrm{Int}_\delta(B)$;

(ii) $A \subset B \Rightarrow_\beta \mathrm{Cl}_\delta(A) \subset_\beta \mathrm{Cl}_\delta(B)$;

(iii) ${}_\beta\,\mathrm{Int}_\delta(X \backslash A) = X \backslash_\beta \mathrm{Cl}_\delta(A)$;

(iv) ${}_\beta\,\mathrm{Cl}_\delta(X \backslash A) = X \backslash_\beta \mathrm{Int}_\delta(A)$;

(v) $x \in_\beta \mathrm{Cl}_\delta(A)$ *if and only if* $A \cap U \neq \varnothing$ *for each* $U \in \delta\beta O(X, x)$;

(vi) A *is δ-β-closed in (X,τ) if and only if* $A = {}_\beta\,\mathrm{Cl}_\delta(A)$;

(vii) ${}_\beta\,\mathrm{Cl}_\delta(A)$ *is δ-β-closed in (X,τ).*

Theorem 13. *If $f : (X,\tau) \to (Y,\sigma)$ is quasi δ-β-continuous and A is θ-closed in $X \times Y$, then $p_X(A \cap G(f))$ is δ-β-closed in X, where p_X represents the projection of $X \times Y$ onto X.*

Proof. Let A be a θ-closed subset of $X \times Y$ and $x \in {}_\beta\,\mathrm{Cl}_\delta(p_X(A \cap G(f)))$. Let $U \in \tau$ containing x and $V \in \sigma$ containing $f(x)$. Since f is quasi δ-β-continuous, by Theorem 3, $x \in f^{-1}(V) \subseteq {}_\beta\,\mathrm{Int}_\delta(f^{-1}(\mathrm{Cl}(V)))$. Then $U \cap {}_\beta\,\mathrm{Int}_\delta(f^{-1}(\mathrm{Cl}(V))) \cap p_X(A \cap G(f))$ contains some point z of X. This implies that $(z, f(z)) \in A$ and $f(z) \in \mathrm{Cl}(V)$. Thus we have $\varnothing \neq (U \times \mathrm{Cl}(V)) \cap A = \mathrm{Cl}(U \times V) \cap A$ and hence $(x, f(x)) \in \mathrm{Cl}_\theta(A)$. Since A is θ-closed, $(x, f(x)) \in A \cap G(f)$ and $x \in p_X(A \cap G(f))$ by Lemma 4, $p_X(A \cap G(f))$ is δ-β-closed in (X,τ). ■

Lemma 5 ([4])**.** *The product of two δ-β-open sets is δ-β-open.*

Lemma 6. *Let $\{A_\alpha : \alpha \in \lambda\}$ be a family of subsets in a topological space (X,τ). Then* ${}_\beta\,\mathrm{Int}_\delta(\bigcup_{\alpha\in\lambda} A_\alpha) \supset \bigcup_{\alpha\in\lambda} {}_\beta\,\mathrm{Int}_\delta(A_\alpha)$.

Proof. Since for each α, $\bigcup_{\alpha\in\lambda} A_\alpha \supset A_\alpha \supset_\beta \mathrm{Int}_\delta(A_\alpha)$, the Lemma follows from (i) of Lemma 4. ■

Theorem 14. *Let (X_1, τ_1), (X_2, τ_2) and (X, τ) be topological spaces. Define a function $f : (X, \tau) \to (X_1 \times X_2, \tau_1 \times \tau_2)$ by $f(x) = (f(x_1), f(x_2))$. Then $f_i : X \to (X_i, \tau_i)$ $(i = 1, 2)$ is quasi δ-β-continuous if f is quasi δ-β-continuous.*

Proof. It suffices to prove that $f_1 : X \to (X_i, \tau_i)$ $(i = 1, 2)$ is quasi δ-β-continuous. Let U_1 be open in X_1. Then $U_1 \times U_2$ is open in $X_1 \times X_2$. Then clearly $f_1^{-1}(U_1) = f^{-1}(U_1 \times U_2)$ and $f_1^{-1}(\mathrm{Cl}(U_1)) = f^{-1}(\mathrm{Cl}(U_1) \times X_2)$. Hence the quasi δ-β-continuous of f gives, by Theorem 3, $f_1^{-1}(U_1) = f_1^{-1}(U_1 \times X_2) \subset {}_\beta\mathrm{Int}_\delta(f^{-1}(\mathrm{Cl}(U_1 \times X_2))) = {}_\beta\mathrm{Int}_\delta(f^{-1}(\mathrm{Cl}(U_1) \times X_2)) = {}_\beta\mathrm{Int}_\delta(f_1^{-1}(\mathrm{Cl}(U_1))$. This implies that $f_1 : X \to (X_i, \tau_i)$ is quasi δ-β-continuous. ■

Theorem 15. *If $f_1 : (X_1, \tau) \to (Y, \sigma)$ is quasi δ-β-continuous, $f_2 : (X_2, \tau) \to (Y, \sigma)$ is almost δ-β-continuous and (Y, σ) is Hausdorff, then the set $\{(x_1, x_2) \in X_1 \times X_2 | f_1(x_1) = f_2(x_2)\}$ is δ-β-closed in $X_1 \times X_2$.*

Proof. Let $A = \{(x_1, x_2) \in X_1 \times X_2 | f(x_1) = f(x_2)\}$. If $(x_1, x_2) \in (X_1 \times X_2) \backslash A$, then we have $f(x_1) \neq f(x_2)$. Since Y is Hausdorff, there exist disjoint open sets V_1 and V_2 in Y such that $f(x_1) \in V_1$ and $f(x_2) \in V_2$ and $\mathrm{Cl}(V_1) \cap \mathrm{Int}(\mathrm{Cl}(V_2)) = \varnothing$. Since f_1 (resp. f_2) is quasi δ-β-continuous (resp. almost δ-β-continuous), there exist $U_1 \in \delta\beta O(X_1, x_1)$ such that $f(U_1) \subset \mathrm{Cl}(V_1)$ (resp. $U_2 \in \delta\beta O(X_2, x_2)$ such that $f({}_\beta\mathrm{Cl}_\delta(U_2)) \subset (\mathrm{Int}(\mathrm{Cl}(V_2))))$. Therefore, we obtain $(x_1, x_2) \in U_1 \times U_2 \subset (X_1 \times X_2) \backslash A$. ■

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References

[1] Jafari S., Rajesh N., On faintly δ-β-continuous functions, (*submitted*).
[2] Jafari S., Rajesh N., Almost δ-β-continuous functions, (*submitted*).
[3] Hatir E., Noiri T., Decomposition of continuity and complete continuity, *Acta Math. Hungar.,* 113(4)(2006), 281-287.
[4] Hatir E., Noiri T., On δ-β-continuous functions, *Chaos, Solutions and Fractals,* 42(1)(2009), 205-211.
[5] Kuratowski K., *Topology,* Academic Press, New York 1966.
[6] Levine N., A decomposition of continuity in topological topological spaces, *Amer. Math. Monthly,* 68(1961), 44-46.
[7] Singal M.K., Arya S.P., On almost regular spaces, *Glasnik Mat.,* 4(24) (1969), 89-99.
[8] Staum R., The algebra of bounded continuous fuctions into a nonarchimedean field, *Pacific J. Math.,* 50(1974), 169-185.
[9] Stone M., Applications of the theory of boolean rings to general topology, *Trans. Amer. Math. Soc.,* 41(1937), 374-381.

[10] Velicko N.V., *H*-closed topological spaces, *Trans. Amer. Math. Soc.,* 78(2) (1968), 103-118.

Neelamegarajan Rajesh
Department of Mathematics
Kongu Engineering College
Perundurai, Erode-638 052, Tamilnadu, India
e-mail: nrajesh_topology@yahoo.co.in

Miguel Caldas
Departamento de Matematica Aplicada
Universidade Federal Fluminense
Rua Mario Santos Braga, s/n
24020-140, Niteroi, RJ Brasil
e-mail: gmamccs@vm.uff.br

Saeid Jafari
College of Vestsjaelland South
Herrestraede 11, 4200 Slagelse, DENMARK
e-mail: jafari@stofanet.dk

Seithuti Philemon Moshokoa
Department of Mathematical Sciences
University of South Africa
P.O. Box 392, Pretoria, 0003, South Africa
e-mail: moshosp@unisa.ac.za

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