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A NOTE ON ω-LIMIT SET OF A TREE MAP*

Abstract. Let T be a tree and $f : T \to T$ be continuous. Denote by $P(f)$ and $\omega(x, f)$ the set of periodic points of f and ω-limit set of x under f respectively. Write $\Lambda(f) = \bigcup_{x \in T} \omega(x, f)$. In this paper, we show that if $x \in \Lambda(f) - \overline{P(f)}$, then $\omega(x, f)$ is an infinite minimal set.

Key words: tree map, periodic point, ω-limit set, minimal set.

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1. Introduction

In this paper, let $\mathbf{N}$ denote the set of all positive integers. Write $\mathbf{Z}^+ = \mathbf{N} \cup \{0\}$ and $\mathbf{N}_n = \{1, 2, \cdots, n\}$. Let (X, d) be a compact metric space, denote by $C^0(X)$ the set of all continuous maps from X to X. For any $A \subset X$, we use $\#(A)$, $\overline{A}$, $\text{int}(A)$, ∂A to denote the number of elements, the closure, the interior and the boundary of A, respectively. Let $f \in C^0(X)$, $x \in X$, $r > 0$, write $B(x, r) = B(x, r, d) = \{y \in X : d(y, x) < r\}$, $O(x, f) = \{f^n(x) : n \in \mathbf{Z}^+\}$, $\omega(x, f) = \bigcap_{n=0}^{\infty} \overline{O(f^n(x), f)}$. $O(x, f)$, $\omega(x, f)$ are called the orbit and ω-limit set of x under f, respectively.

$$\begin{aligned}
F(f) &= \{x \in X : f(x) = x\}, \\
P_n(f) &= \{x \in X : f^n(x) = x, f^i(x) \neq x (1 \leq i \leq n-1)\}, \\
R(f) &= \{x \in X : x \in \omega(x, f)\}, \\
AP(f) &= \{x \in X : \text{ for any } \varepsilon > 0, \text{ there exists } N \in \mathbf{N} \text{such that for any} \\
&\qquad m \in \mathbf{Z}^+, d(x, f^{m+k}(x)) < \varepsilon \text{ for some } 0 < k \leq N\}, \\
SAP(f) &= \{x \in X : \text{ for any } \varepsilon > 0, \text{ there exists } N \in \mathbf{N} \text{ such that} \\
&\qquad d(x, f^{kN}(x)) < \varepsilon \text{ for any } k \in \mathbf{N}\}, \\
\Omega(f) &= \{x \in X : \text{ there exist } \{x_i\} \subset X \text{ and } \{n_i\} \subset \mathbf{N} \text{ with } x_i \to x, \\
&\qquad n_i \to \infty \text{ such that } f^{n_i}(x_i) \to x\}.
\end{aligned}$$

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$F(f)$, $P_n(f)$, $R(f)$, $AP(f)$, $SAP(f)$, $\Omega(f)$ are called the set of fixed points, the set of periodic points with period n, the set of recurrent points, the set of almost periodic points, the set of strongly almost periodic points and the set of non-wandering points of f, respectively. Write $P(f) = \bigcup_{n=1}^{\infty} P_n(f)$ and $\Lambda(f) = \bigcup_{x\in X} \omega(x, f)$, which are called the set of periodic points of and ω-limit set of f, respectively. It is known that for any $\in C^0(X)$,

$$F(f) \subset P(f) \subset SAP(f) \subset AP(f) \subset R(f) \subset \Lambda(f) \subset \Omega(f).$$

By a tree we mean a connected compact one-dimensional branched manifold containing no copies homeomorphic to the circle. A subtree of T is a subset of T, which is a tree itself. Let $x \in T$, denote by $V(x)$ the number of connected components of $T - \{x\}$. If $V(x) = 1$, then x is called an end point of T. If $V(x) \geq 3$, then x is called a branched point of T. Write $E(T) = \{x \in T : V(x) = 1\}$ and $O = \{x \in T : V(x) \geq 3\}$. The closure every connected components of $T - O$ is called edge of T and A connected subset of any edge of T is called an interval of T. For any $A \subset T$, we use $[A]$ to denote the smallest connected closed subset of T containing A. For any $x, y \in T$, write $[x, y] = [\{x, y\}]$, $(x, y] = [x, y] - \{x\}$, $[x, y) = [x, y] - \{y\}$, $(x, y) = [x, y] - \{x, y\}$, denote by $T_x(y)$ the connected component of $T - \{x\}$ containing y.

Definition 1. *Let $(a, b) \subset T$ be an interval. Given an orient on (a, b) such that $a < b$. (a, b) is called of increasing type if $f^n(c) \in (c, b)$ for any $c \in (a, b)$ and any $n \in \mathbf{N}$ with $f^n(c) \in (a, b)$ and decreasing type if $f^n(c) \in (a, c)$ for any $c \in (a, b)$ and any $n \in \mathbf{N}$ with $f^n(c) \in (a, b)$.*

Now we are in the position to state the main result of the paper.

Theorem 1. *Let $f \in C^0(T)$. If $x \in \Lambda(f) - \overline{P(f)}$, then $\omega(x, f)$ is an infinite minimal set.*

2. Proof of the main theorem

Lemma 1 ([2])**.** *Let $f \in C^0(T)$, $[c, d] \subset T$ and Y be a subtree of T.*
(a) If $f([c, d]) \supset [c, d]$ and $[c, d] \cap (d, f(d)] = \emptyset$, then $F(f) \cap [c, d] \neq \emptyset$.
(b) If $(a, f(a)] \cap Y \neq \emptyset$ for all $a \in Y - F(f)$, then $F(f) \neq \emptyset$.

Lemma 2 ([3])**.** *Let $J \subset T - \overline{P(f)} - O$ be connected. If there exist $c \in J$ and $n \in \mathbf{N}$ such that $f^n(c) \in J$, then $f^{nk}(c) \notin T_{f^n(c)}(c)$ for any $k \geq 2$.*

Lemma 3 ([3])**.** *Let $f \in C^0(T)$, then $\Lambda(f)$ is a closed subset of T.*

Lemma 4 ([3], [4])**.** *Let J be a connected components of $T - \overline{P(f)} - O$. If $x \in J \bigcap \Omega(f)$, then $f^n(x) \notin J$ for any $n \in \mathbf{N}$.*

Lemma 5 ([3]). *Let (a,b) be a connected components of $T - \overline{P(f)} - O$. Given a director on (a,b) such that $a < b$, then (a,b) is of increasing type or decreasing type.*

Proof. Assume on the contrary that (a,b) neither be of increasing type nor be of decreasing type, then there exist $c, d \in (a,b)$, $c \neq d$ and $n_1, n_2 \in \mathbf{N}$ such that $c < f^{n_1}(c) < b$ and $a < f^{n_2}(d) < d$. By Lemma 2 we see $f^{n_1 n_2}(c) \in \overline{T_{f^{n_1}(c)}(b)}$ and $f^{n_1 n_2}(d) \in \overline{T_{f^{n_2}(d)}(a)}$. It follows from Lemma 1 that $[c,d] \bigcap F(f^{n_1 n_2}) \neq \emptyset$, which implies $[c,d] \bigcap P(f) \neq \emptyset$, a contradiction. ■

Lemma 6 ([3]). *Let $f \in C^0(T)$. If $x \in \Lambda(f) - \overline{P(f)} - O$, then $O(x,f)$ is an infinite set.*

Lemma 7 ([3]). *Let $f \in C^0(T)$, then $x \in \Omega(f)$ if and only if there exist $x_n \longrightarrow x$ and $k_n \longrightarrow +\infty$ such that $f^{k_n}(x_n) = x$.*

Lemma 8 ([1]). *Let $f \in C^0(T)$, then $x \in AP(f)$ if and only if $x \in \omega(x,f)$ and $\omega(x,f)$ is a minimal set.*

Proof of Theorem 1. Let $T - O$ has m' connected components and $\#(O) = k'$. It follows from Lemma 3 that $\overline{P(f)} \subset \Lambda(f)$. Since $f(\Lambda(f)) = \Lambda(f)$, we obtain $f(\Lambda(f) - \overline{P(f)}\,) = \Lambda(f) - \overline{P(f)}$. Let $x \in \Lambda(f) - \overline{P(f)}$, then there exist $x_{2m'+k'}, x_{2m'+k'-1}, \cdots, x_1 \in \Lambda(f) - \overline{P(f)}$ such that $f(x_i) = x_{i-1}$ for $1 \leq i \leq 2m' + k'$ and $x_i \neq x_j$ for any $0 \leq i < j \leq 2m' + k'$, where $x_0 = x$. Thus, there exist the connected component I of $T - O$ and $i,\ j,\ k \in \{0,\ 1, \cdots, 2m' + k'\}$ such that $x_i, x_j, x_k \in I$ with $x_j \in (x_i, x_k)$. Let (a,b) be the connected component of $I - \overline{P(f)}$ containing x_j, then it follows from Lemma 4 that $x_i \notin (a,b)$ and $x_j \notin (a,b)$. Given an orient on (a,b) such that $a < b$. Set $\alpha = x_j$. According to Lemma 5, We may assume without loss of generality that (a,b) is of increasing type. ■

Claim 1. $[\alpha, b], f([\alpha,b]), \cdots, f^s([\alpha,b]), \cdots$ *are pairwise disjoint.*

Proof of Claim 1. Assume on the contrary that there exist $0 \leq s < t$ such that $f^s([\alpha,b]) \bigcap f^t([\alpha,b]) \neq \emptyset$, then $W = \bigcup_{l=0}^{\infty} f^l([\alpha,b])$ has only finitely many connected components. By Lemma 6 we see that $O(\alpha,f)$ is an infinite set, it follows that $f^r(\alpha) \in \text{int}(W)$ for some $r \in \mathbf{Z}^+$, which implies that there exists a neighborhood U of α such that $f^r(U) \subset W$. Since $\alpha \in \Lambda(f) \subset \Omega(f)$, it follows from Lemma 7 that $f^l(y) = \alpha$ for some $y \in U$ and some $l > r$. By $f^r(y) \in f^r(U) \subset W$ we have that $f^{l_1}(u) = f^r(y)$ for some $l_1 \in \mathbf{Z}^+$ and some $u \in [\alpha,b]$. Thus $\alpha = f^l(y) = f^{l-r}(f^{l_1}(u)) = f^{l-r+l_1}(u)$, which contradicts the fact that (a,b) is of increasing type. Claim 1 is proven. ■

By Claim 1 we see $d(f^n(\alpha), f^n(b)) \longrightarrow 0(\ n \longrightarrow +\infty)$, which implies $\omega(\alpha,f) = \omega(b,f)$.

Claim 2. $b \in \omega(\alpha, f)$.

Proof of Claim 2. It follows from Claim 1 that $b \notin P(f)$, therefore $b \in \overline{P(f)} \backslash P(f)$. For any $p \in (b, x_k) \bigcap P(f)$. Let $p \in P_m(f)$ for some $m \in \mathbf{N}$. Write $g = f^m$, then $\alpha \in \Lambda(g) - \overline{P(g)}$ and (a, b) is of increasing type for g. Thus there exist $z \in (a, b)$ and $k_n \in \mathbf{N}$ with $1 \leq k_1 < k_2 < \cdots$ such that $g^{k_n}(z) \longrightarrow \alpha$ and $z < g^{k_1}(z) < g^{k_2}(z) < \cdots < \alpha$. Now we show $[b, p] \bigcap O(\alpha, g) \neq \emptyset$.

Assume on the contrary that $[b, p] \bigcap O(\alpha, g) = \emptyset$. We claim $\#([b, p] \bigcap O(b, g)) \leq 2$. Indeed, if there exist $n_1 < n_2 < n_3$ such that $g^{n_1}(b)$, $g^{n_2}(b)$, $g^{n_2}(b) \in [b, p]$, then $g^{n_i}([\alpha, b]) \bigcap [b, p] \neq \emptyset$, $i = 1, 2, 3$. It follows from Claim 1 that $g^{n_{i_0}}([\alpha, b]) \subset [b, p]$ for some $i_0 \in \{1, 2, 3\}$, which implies $[b, p] \bigcap O(\alpha, g) \neq \emptyset$, a contradiction. Hence, there exists $N_1 \in \mathbf{N}$ such that $g^n(b) \notin [b, p]$ for any $n \geq N_1$. Since $z < g^{k_1}(z) < g^{k_2}(z) < \cdots < \alpha$, $(1 \leq k_1 < k_2 < \cdots)$ and $(a, b) \bigcap \overline{P(g)} = \emptyset$, we have $g^{k_{n+1}-k_n}(b) \in T_b(x_k)$ for $n \in \mathbf{N}$. In a similar fashion, we can show $g^{k_{n+1}-k_n}(\alpha) \in T_\alpha(x_k)$ for any $n \in \mathbf{N}$. Note that $\alpha \in \Lambda(f) \subset \Omega(f)$, it follows from Lemma 4 that $g^n(\alpha) \notin (a, b)$ for any $n \in \mathbf{N}$. Thus we have $g^{k_{n+1}-k_n}(\alpha)$, $g^{k_{n+1}-k_n}(b) \in T_b(x_k)$ for any $n \in \mathbf{N}$ and there exists $N \in \mathbf{N}$ such that $g^{k_{n+1}-k_n}(b) \in T_p(x_k)$ for all $n \geq N$. By Claim 1 we can choose a connected component J of $T_p(x_k) - O$ and $m_1, m_2 \geq k_{N+1} - k_N$ such that $g^{m_1}([\alpha, b]), g^{m_2}([\alpha, b]) \subset J$ and $g^{m_1}([\alpha, b]) \subset [p, g^{m_2}(b)]$. Hence $g^t(z) \in [p, g^{m_2}(b)]$ for aome $t \in \mathbf{N}$. For any $k_n > t$, we have

$$\begin{aligned} g^{k_{n+1}-k_n+m_2}([g^{k_n}(z), \alpha]) &\supset g^{m_2}([g^{k_{n+1}}(z), g^{k_{n+1}-k_n}(\alpha)]) \\ &\supset g^{m_2}([b, p]) \supset [p\,, g^{m_2}(b)] \end{aligned}$$

since $g^{k_{n+1}}(z) < b < p < g^{k_{n+1}-k_n}(\alpha)$. Thus, there exists $u \in [g^{k_n}(z), \alpha]$ such that $g^{k_{n+1}-k_n+m_2}(u) = g^t(z)$ and $g^{k_n}(z) = g^{k_{n+1}+m_2-t}(u)$, which contradicts the fact that (a, b) is of increasing type for g.

Since $b \in \overline{P(g)} \backslash P(f)$, $(a, b) \bigcap \overline{P(g)} = \emptyset$ and p is arbitrary, $b \in \omega(\alpha, g) \subset \omega(\alpha, f)$. Claim 2 is proven. ■

Claim 3. $b \in SAP(f)$.

Proof of Claim 3. For any $0 < \varepsilon < \frac{1}{2} \min\{d(b, \alpha), d(b, x_k)\}$, it follows from the proof of Claim 2 that $b \in \omega(\alpha, g)$, which implies $g^l(\alpha) \in B(b, \varepsilon)$ for some $l \in \mathbf{N}$. Since $g^l(\alpha) \notin (\alpha, b]$, we get $g^l(\alpha) \in B(b, \varepsilon) \backslash (\alpha, b]$, then there exists a neighborhood U of α such that $g^l(U) \subset B(b, \varepsilon) \backslash (\alpha, b]$. Note $g^{k_n}(z) \longrightarrow \alpha$, we have $g^r(z) \in B(b, \varepsilon) \backslash (\alpha, b]$ for some $r \in \mathbf{N}$. Take $k_n > r$, then there exists $p_0 \in P(f)$ such that $p_0 \in (b, g^r(z)) \bigcap g^{k_{n+1}-k_n}([g^{k_n}(z), \alpha])$. We may assume $p_0 \in P_q(g)$, then $g^q(p_0) = p_0$. Now we show $d(g^{qt}(b), b) < \varepsilon$ for any $t \in \mathbf{N}$, which implies $b \in SAP(f)$.

Assume on the contrary that $g^{qt}(b) \notin B(b, \varepsilon)$ for some $t \in \mathbf{N}$.

If $g^{qt}(b) \in T_b(x_k)$, then $g^{qt}(b) \in T_b(x_k) \backslash B(b, \varepsilon)$ and

$$g^{qt+k_{n+1}-k_n}([g^{k_n}(z), \alpha]) \supset g^{qt}([g^{k_{n+1}}(z), p_0]) \supset g^{qt}([b, p_0]) \ni g^r(z).$$

Thus, there exists $v \in [g^{k_n}(z), \alpha]$ such that $g^{qt+k_{n+1}-k_n}(v) = g^r(z)$ and $g^{k_n}(z) = g^{qt+k_{n+1}-r}(v)$, which contradicts the fact that (a, b) is of increasing type for g.

If $g^{qt}(b) \notin T_b(x_k)$, then it follows from Claim 1 that $g^{qt}(\alpha) \notin \overline{T_\alpha(b)}$. Since $g^{qt}(\alpha) \notin (a, b)$, we obtain $g^{qt}(\alpha) \notin T_a(\alpha)$ and

$$g^{qt+k_{n+1}-k_n}([g^{k_n}(z), \alpha]) \supset g^{qt}([g^{k_{n+1}}(z), p_0]) \supset g^{qt}([\alpha, p_0]) \supset [a, p_0] \ni z,$$

Thus, there exists $w \in [g^{k_n}(z), \alpha]$ such that $g^{qt+k_{n+1}-k_n}(w) = z$, which contradicts the fact that (a, b) is of increasing type for g. Claim 3 is proven. ■

By Claim 1, Claim 2 and Claim 3, we obtain $\omega(x, f) = \omega(\alpha, f) = \omega(b, f)$ and $b \in \omega(x, f) \bigcap SAP(f) \subset \omega(x, f) \bigcap AP(f)$. It follows from Lemma 8 that $\omega(b, f)$ is a minimal set, which implies that $O(b, f) \subset \omega(b, f)$ and b is not an eventually periodic point. Thus, $\omega(x, f)$ is an infinite minimal set.

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