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CHARACTERIZATIONS OF THE EXPONENTIAL DISTRIBUTION BY GEOMETRIC COMPOUND

Abstract. Under the reliability conditions IFRA/DFRA (NBU/NWU), the exponential distribution is characterized by stochastic ordering properties which link the geometric compound with record values (spacing of record values). The index of record values is random.

Key words: characterization, exponential distribution, geometric compounding model, stochastic order, record values.

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1. Introduction

Let X be a nonnegative and nondegenerate random variable with distribution function $F(x) = P(X \leq x)$, $x \geq 0$. Let $\overline{F}(x) = P(X > x)$, $x \geq 0$ denote the survival function of X.

We say that

a) $F \in EXP$ if

$$\overline{F}(x) = e^{-\lambda x} \quad \text{for} \quad x > 0, \ \ \lambda > 0; \tag{1}$$

b) $F \in IFRA$ if $-\frac{1}{x} \ln \overline{F}(x)$ is nondecreasing in $x > 0$;

c) $F \in DFRA$ if $-\frac{1}{x} \ln \overline{F}(x)$ is nonincreasing in $x > 0$;

d) $F \in NBU$ if

$$\overline{F}(x+y) \leq \overline{F}(x) \cdot \overline{F}(y) \quad \text{for all} \quad x, y > 0; \tag{2}$$

e) $F \in NWU$ if

$$\overline{F}(x+y) \geq \overline{F}(x) \cdot \overline{F}(y) \quad \text{for all} \quad x, y > 0. \tag{3}$$

Note that

$$EXP \subset IFRA \subset NBU,$$

$$EXP \subset DFRA \subset NWU.$$

It is known (see[2]) that

$$F \in \mathit{IFRA} \ \text{ if and only if}$$

(4) $$\overline{F}(\alpha x) \geq \left[\overline{F}(x)\right]^{\alpha} \quad \text{for all} \quad 0 < \alpha < 1 \ \text{ and } \ x > 0,$$

and $F \in \mathit{DFRA}$ if and only if

(5) $$\overline{F}(\alpha x) \leq \left[\overline{F}(x)\right]^{\alpha} \quad \text{for all} \quad 0 < \alpha < 1 \ \text{ and } \ x > 0.$$

Let X and Y be two nonnegative random variables. We say

a) that X has the same distribution as Y (denoted $X \overset{d}{=} Y$) if

$$P(X > x) = P(Y > x) \quad \text{for all} \quad x \geq 0;$$

b) that X is smaller than Y in the stochastic order (denoted $X \leq_{st} Y$) if

$$P(X > x) \leq P(Y > x) \quad \text{for all} \quad x \geq 0.$$

Let $E(X)$ be the expected value of X.

Lemma 1 ([5]). *Suppose that* $X \leq_{st} Y$ *and* $F(f(X)) = E(f(Y)) \in (-\infty, \infty)$ *for some strictly increasing function* f *on* $[0, \infty)$. *Then* $X \overset{d}{=} Y$.

We say that υ is geometrically distributed if

(6) $$P(\upsilon = k) = p(1-p)^{k-1}, \quad k = 1, 2, \ldots, \quad \text{for some} \quad 0 < p < 1.$$

Let X, X_1, X_2, ... be a sequence of independent and identically distributed (i.i.d.) nonnegative and nondegenerate random variables with common distribution function F. Assume that υ, independent of X_1, X_2, ..., is geometrically distributed with parameter $p \in (0, 1)$. Then the random sum $S_\upsilon = \sum_{n=1}^{\upsilon} X_n$ is called a geometric compound of the sequence X_1, X_2, ...

The geometric compounding model is useful in risk theory, queueing theory, reliability and distribution theory.

Under the compounding model, it is known ([1]) that

(7) $$pS_\upsilon \overset{d}{=} X \quad \text{if and only if} \quad F \in \mathit{EXP}.$$

If $E(X) < \infty$, then $E(S_\upsilon) = E(\upsilon)E(X) = \frac{1}{p}E(X)$. Hence

(8) $$E(pS_\upsilon) = E(X).$$

2. Characterizations of the exponential distribution

Let X_1, X_2, ... be a sequence of i.i.d. nonnegative random variables with common continuous distribution function F. Define the sequence of record times $L(1)$, $L(2)$, ... as follows: $L(1) = 1$, $L(n{+}1) = \min\{j : j > L(n),\ X_j > X_{L(n)}\}$, for $n = 1, 2, \ldots$. Then $X_{L(1)}$, $X_{L(2)}$, ... is called the record values of the sequence X_1, X_2, ... Write $R_n = X_{L(n)}$, $n = 1, 2, \ldots$.

Lemma 2. *Let $X, X_1, X_2, \ldots$ be a sequence of i.i.d. nonnegative random variables with common continuous distribution function F. Let υ be a geometric random variable (6) independent of X_1, X_2, ...*

(*a*) *If $F \in$ IFRA, then $pR_\upsilon \leq_{st} X$.*

(*b*) *If $F \in$ IFRA, then $X \leq_{st} pR_\upsilon$.*

Proof. (*a*) It is known ([4]) that

$$P(R_\upsilon > x) = \left[\overline{F}(x)\right]^p \quad \text{for all} \quad x > 0.$$

Hence

$$P(pR_\upsilon > x) = \left[\overline{F}\left(\frac{x}{p}\right)\right]^p \quad \text{for} \quad x > 0. \tag{9}$$

Since $F \in$ *IFRA*, it follows from (4) that

$$\left[\overline{F}\left(\frac{x}{p}\right)\right]^p \leq \overline{F}\left(p \cdot \frac{x}{p}\right) = \overline{F}(x) \quad \text{for} \quad x > 0.$$

Hence

$$P(pR_\upsilon > x) \leq \overline{F}(x) = P(X > x) \quad \text{for all } x > 0, \text{ i.e. } pR_\upsilon \leq_{st} X.$$

(*b*) Suppose $F \in$ *DFRA*. From (5) we obtain

$$\left[\overline{F}\left(\frac{x}{p}\right)\right]^p \geq \overline{F}\left(p \cdot \frac{x}{p}\right) = \overline{F}(x) \quad \text{for } x > 0.$$

By formula (9) we conclude that $P(pR_\upsilon > x) \geq \overline{F}(x) = P(X > x)$ for $x > 0$. Namely, $X \leq_{st} pR_\upsilon$. ∎

Using stochastic inequalities, we have the following characterization results.

Theorem 1. *Assume that there are satisfied the assumptions of Lemma 2.*

(*a*) *If $F \in$ IFRA and $S_\upsilon \leq_{st} R_\upsilon$, then $F \in$ EXP.*

(*b*) *If $F \in$ IFRA, $E(X) < \infty$ and $R_\upsilon \leq_{st} S_\upsilon$, then $F \in$ EXP.*

Proof. (a) Suppose that $F \in \mathit{IFRA}$. Then $E(X) < \infty$ ([2]). Suppose that $S_\upsilon \leq_{st} R_\upsilon$. Then $pS_\upsilon \leq_{st} pR_\upsilon \leq_{st} X$ by Lemma 2 (a). From (8) and Lemma 1 we obtain $X \stackrel{d}{=} pS_\upsilon$. This implies that $F \in \mathit{EXP}$.

(b) Suppose that $F \in \mathit{DFRA}$, $E(X) < \infty$ and $R_\upsilon \leq_{st} S_\upsilon$. Then by Lemma 2 ($b$), we have $X \leq_{st} pR_\upsilon \leq_{st} pS_\upsilon$. From (8) and Lemma 1 we conclude that $X \stackrel{d}{=} pS_\upsilon$ and hence $F \in \mathit{EXP}$. The proof of the theorem is complete. ∎

Lemma 3. *Let X, X_1, X_2, ... be a sequence of i.i.d. nonnegative random variables with common continuous distribution function F. Let $N \geq 1$ be an integer-valued random variable independent of X_1, X_2, ...*

(a) *If $F \in \mathit{NBU}$, then $R_{N+1} - R_N \leq_{st} X$.*

(b) *If $F \in \mathit{NWU}$, then $X \leq_{st} R_{N+1} - R_N$.*

Proof. Recall the Markov property of record values ([1])

$$P\left(R_{n+1} > x \mid R_n = y\right) = \frac{\overline{F}(x)}{\overline{F}(y)} \quad \text{for} \quad x \geq y.$$

Then we have

$$\begin{aligned} P\left(R_{N+1} - R_N > x\right) &= \sum_{n=1}^{\infty} P\left(N = n\right) P\left(R_{n+1} - R_n > x\right) \\ &= \sum_{n=1}^{\infty} P\left(N = n\right) \int_Q P\left(R_{n+1} > x + z \mid R_n = z\right) dH_n(z) \\ &= \sum_{n=1}^{\infty} P\left(N = n\right) \int_Q \frac{\overline{F}(x+z)}{\overline{F}(z)} dH_n(z) \quad \text{for all} \quad x > 0, \end{aligned}$$

in which H_n denotes the distribution function of R_n and Q is its support.

(a) Suppose that $F \in \mathit{NBU}$. From (2) we obtain

$$P\left(R_{N+1} - R_N > x\right) \leq \sum_{n=1}^{\infty} P\left(N = n\right) \int_Q \overline{F}(x) dH_n(z) = \overline{F}(x) \quad \text{for} \ \ x > 0,$$

i.e. $R_{N+1} - R_N \leq_{st} X$.

(b) Suppose that $F \in \mathit{NWU}$. From (3) it follows that

$$P\left(R_{N+1} - R_N > x\right) \geq \sum_{n=1}^{\infty} P\left(N = n\right) \int_Q \overline{F}(x) dH_n(z) = \overline{F}(x) \quad \text{for} \ \ x > 0,$$

i.e. $X \leq_{st} R_{N+1} - R_N$. ∎

Theorem 2. *Suppose that the assumptions of Lemmas 2 and 3 are satisfied.*

(a) *If* $F \in NBU$ *and* $pS_v \leq_{st} R_{N+1} - R_N$*, then* $F \in EXP$*.*

(b) *If* $F \in NWU$ *and* $E(X) < \infty$ *and* $R_{N+1} - R_N \leq_{st} pS_v$*, then* $F \in EXP$*.*

Proof. (a) Suppose that $F \in NBU$. Then $E(X) < \infty$ ([2]). Suppose that $pS_v \leq_{st} R_{N+1} - R_N$. Then by Lemma 3 ($a$), we have $pS_v \leq_{st} R_{N+1} - R_N \leq_{st} X$. From (8), Lemma 1 and (7) we obtain $F \in EXP$.

(b) Suppose that $F \in NWU$, $E(X) < \infty$ and $R_{N+1} - R_N \leq_{st} pS_v$. Then by Lemma 3 ($b$), it may be concluded that $X \leq_{st} R_{N+1} - R_N \leq_{st} pS_v$. Next, analogously as in the proof of part (a), we get that $F \in EXP$. This completes the proof. ■

Corollary 1 ([3], Theorem 2.4)**.** *Suppose that the assumptions of Lemma 2 are satisfied.*

(a) *If* $F \in NBU$ *and* $pS_v \leq_{st} R_{n+1} - R_n$ *for some* $n \geq 1$*, then* $F \in EXP$*.*

(b) *If* $F \in NWU$*,* $E(X) < \infty$ *and* $R_{n+1} - R_n \leq_{st} pS_v$ *for some* $n \geq 1$*, then* $F \in EXP$*.*

References

[1] Azlarov T.A., Volodin N.A., *Characterization Problems Associated with the Exponential Distribution,* Tashkent, 1982, (in Russian).

[2] Barlow R.E., Proschan F., *Statistical Theory of Reliability and Life Testing,* Holt, Rinehart and Winston, Inc., 1975.

[3] Hu C.-Y., Lin G.D., Characterizations of the exponential distribution by stochastic ordering properties of the geometric compound, *Ann. Inst. Statist. Math.,* 55(3)(2003), 499-506.

[4] Kakosyan A.V., Klebanov L.B., Melamed J.A., *Characterization of Distributions by the Method of Intensively Monotone Operators,* Springer- Verlang, Berlin, Heidelberg, New York, Tokyo, 1984.

[5] Lin G.D., Hu C.-Y., Characterizations of distributions via the stochastic ordering property of random linear forms, *Statist. Probab. Lett.,* 51(2001), 93-99.

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