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MEAN-VALUE TYPE EQUALITIES WITH INTERCHANGED FUNCTION AND DERIVATIVE

Abstract. According to a new mean value-theorem, if a function f satisfies the classical conditions ensuring the existence and uniqueness of Lagrange's mean, then there also exists a unique mean M such that

$$\frac{f(x)-f(y)}{x-y}=M\left(f'(x),f'(y)\right).$$

The main result gives necessary and sufficient conditions for the equality

$$\frac{f'(x)-f'(y)}{x-y}=M\left(f(x),f(y)\right).$$

The relevant equality for the Lagrange mean-value theorem is also considered.

Key words: mean-value theorem, mean.

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1. Introduction

Recall that a function $M:I^2\to\mathbb{R}$ is a *mean* in an interval $I\subseteq\mathbb{R}$, if

$$\min(x,y)\le M(x,y)\le\max(x,y),\quad x,y\in I.$$

If for all $x,y\in I, x\neq y$, these inequalities are strict, M is called *strict*; and *symmetric,* if $M(x,y)=M(y,x)$. Clearly, any mean M is *reflexive*, i.e.

$$M(x,x)=x,\quad x\in I.$$

It is obvious that every reflexive function $M:I^2\to\mathbb{R}$ which is (strictly) increasing with respect to each variable is a (strict) mean in I.

In [1] the following counterpart of the Lagrange mean-value theorem has been proved. *If a real function f defined on an interval I is differentiable,* *and f' is one-to-one, then there exists a unique mean function $M : f'(I) \times f'(I) \to f'(I)$ such that*

$$\frac{f(x)-f(y)}{x-y} = M\left(f'(x), f'(y)\right), \quad x, y \in I, \;\; x \neq y, \tag{1}$$

The following problem was considered in [2]. Given a mean M, determine all differentiable real functions $f : I \to \mathbb{R}$ such that equality (1) is satisfied.

At this background one can ask if in equation (1), for some functions f and means M, the roles of f and f' can be interchanged, that is if there exist a function f and a mean M such that

$$\frac{f'(x)-f'(y)}{x-y} = M\left(f(x), f(y)\right), \quad x, y \in I, \;\; x \neq y.$$

In section 1 we give an answer to this question. The main result, Theorem 2, says that, under natural regularity assumptions, this equation is satisfied if, and only if,

$$f(x) = ae^{-x} + be^{x}$$

for some $a, b \in \mathbb{R}$, $a^2 + b^2 > 0$, and

$$M(u,v) := \frac{\sqrt{u^2+c} - \sqrt{v^2+c}}{\log\left(u + \sqrt{u^2+c}\right) - \log\left(u + \sqrt{v^2+c}\right)}, \quad u \neq v,$$

where $c = 4ab$. Moreover the maximal domain of the mean M, depending on a and b, is established.

In section 2 we consider the equation

$$\frac{f'(x)-f'(y)}{x-y} = f(M(x,y)), \quad x, y \in I, \;\; x \neq y, \tag{2}$$

which appears when we interchange the roles of the function and its derivative in the Lagrange mean-value theorem. It turnes out that in this case the function f must be of the same form as in case of equation (1). The respective family of means has more complicated form (2).

Let us note that, in particular, the sinh and cosh satisfy both equations.

2. Equation (1)

We shall need the following easy to verify

Remark 1. Let I be an interval. If a function $M : I^2 \to \mathbb{R}$ is such that

1. for any fixed $y \in I$,
$(-\infty, y) \cap I \ni x \to M(x,y)$ is (strictly) increasing, and $M(x,y) \leq y$ for

$x < y$,
$(y, \infty) \cap I \ni x \to M(x, y)$ is (strictly) increasing, and $y \le M(x, y)$ for $x > y$;

2. for any fixed $x \in I$,
$(-\infty, x) \cap I \ni y \to M(x, y)$ is (strictly) increasing, and $M(x, y) \le x$ for $y < x$,
$(x, \infty) \cap I \ni y \to M(x, y)$ is (strictly) increasing, and $x \le M(x, y)$ for $y > x$,

3. If M is reflexive then M is an (strictly) increasing (and strict) mean in I.

Applying this remark we prove

Theorem 1. (*i*) *For any $c > 0$, the function $M_c : \mathbb{R}^2 \to \mathbb{R}$ defined by*

$$M_c(u, v) := \begin{cases} \frac{\sqrt{u^2+c}-\sqrt{v^2+c}}{\log \frac{u+\sqrt{u^2+c}}{v+\sqrt{v^2+c}}} & \text{for } u \neq v \\ u & \text{for } u = v \end{cases} \tag{3}$$

is a strict, symmetric, continuous, and increasing mean in $\mathbb{R}$; moreover

$$\lim_{c\to\infty} M_c(u, v) = A(u, v) \quad \text{for all } u, v \in \mathbb{R},$$

$$\lim_{c\to 0} M_c(u, v) = L(u, v) \quad \text{for all } u, v \in \mathbb{R} \text{ such that } uv > 0,$$

where $A(u, v) := \frac{u+v}{2}$, and $L : \left\{(0, \infty)^2 \cup (-\infty, 0)^2\right\} \to \mathbb{R}\backslash\{0\}$ given by

$$L(u, v) := \begin{cases} \frac{u-v}{\log \frac{u}{v}} & \text{for } u \neq v \\ u & \text{for } u = v \end{cases}, \qquad uv > 0,$$

is the logarithmic mean in each of the intervals $(0, \infty)$ and $(-\infty, 0)$.

(*ii*) *for any $c < 0$, the function $M_c : \left\{(\sqrt{-c}, \infty)^2 \cup (-\infty, \sqrt{-c})^2\right\} \to \mathbb{R}\backslash\{0\}$ defined by formula* (3) *is a symmetric, continuous, strictly increasing mean in; moreover*

$$\begin{matrix} \lim_{c\to\infty} M_c(u, v) = A(u, v) \\ \lim_{c\to 0} M_c(u, v) = L(u, v) \end{matrix}, \qquad \text{for all } u, v \text{ such that } uv > 0.$$

Proof. Assume that $c > 0$ and take $u, v \in \mathbb{R}$, $u \neq v$. A simple calculation gives

$$\frac{\partial M_c}{\partial u}(u, v) = \frac{u\left[\log\left(u+\sqrt{u^2+c}\right)-\log\left(v+\sqrt{v^2+c}\right)\right]-\sqrt{u^2+c}+\sqrt{v^2+c}}{\sqrt{u^2+c}\left[\log\left(u+\sqrt{u^2+c}\right)-\log\left(v+\sqrt{v^2+c}\right)\right]}.$$

Thus $\frac{\partial M_c}{\partial u}(u,v) > 0$ iff

$$\begin{aligned}&\frac{u\left[\log\left(u+\sqrt{u^2+c}\right)-\log\left(v+\sqrt{v^2+c}\right)\right]-\sqrt{u^2+c}+\sqrt{v^2+c}}{\log\left(u+\sqrt{u^2+c}\right)-\log\left(v+\sqrt{v^2+c}\right)}\\ &\qquad = u-\frac{\sqrt{u^2+c}-\sqrt{v^2+c}}{\log\left(u+\sqrt{u^2+c}\right)-\log\left(v+\sqrt{v^2+c}\right)}\\ &\qquad = u-M_c(u,v)>0.\end{aligned}$$

Put $g(t) := \sqrt{t^2+c}$, $h(t) := \log\left(t+\sqrt{t^2+c}\right)$. Since

$$\frac{g'(t)}{h'(t)} = \frac{\frac{t}{\sqrt{t^2+c}}}{\frac{t+\frac{t}{\sqrt{t^2+c}}}{t+\sqrt{t^2+c}}} = t,$$

by the Cauchy mean-value theorem,

$$M_c(u,v) = \frac{\sqrt{u^2+c}-\sqrt{v^2+c}}{\log\left(u+\sqrt{u^2+c}\right)-\log\left(v+\sqrt{v^2+c}\right)}$$

is located between v and u, that is

$$\min(u,v) = v < M_c(u,v) < u = \max(u,v).$$

It follows that $u - M_c(u,v) > 0$ and, consequently, $\frac{\partial M_c}{\partial u}(u,v) > 0$. This proves that for any fixed v, the function

$$M_c(\cdot, v) \text{ is increasing in the interval } (-\infty, v).$$

In the same way we can verify that conditions 1-2 of Remark 1 are satisfied.

Since $M(u,u) = u$ for all $u \in \mathbb{R}$, by Remark 1, the function M is an increasing mean. The continuity of M_c at every point (u,v) such that $u \neq v$ is obvious. Since, for any $w \in \mathbb{R}$, by the Cauchy mean-value theorem and the definition of M_c on the diagonal,

$$\begin{aligned}\lim_{v\to w, v\to w} M_c(u,v) &= \lim_{v\to w, v\to w}\frac{\sqrt{u^2+c}-\sqrt{v^2+c}}{\log\left(u+\sqrt{u^2+c}\right)-\log\left(u+\sqrt{v^2+c}\right)}\\ &= \lim_{v\to w, v\to w} M_c(u,v) = w = M_c(w,w),\end{aligned}$$

the mean M_c is continuous everywhere. The symmetry of M_c is obvious.

Since the proof in the case when $c < 0$ is similar, we omit it. ■

The main result of his section reads as follows.

Theorem 2. *Let $I \subseteq \mathbb{R}$ be an interval. Suppose that $f : I \to \mathbb{R}$ is a differentiable function, $f'(x) \neq 0$ for $x \in I$, $J := f(I)$ and $M : J^2 \to J$ is a mean that is continuous on the diagonal $\Delta := \{(u,u) : u \in J\}$. Then*

$$\frac{f'(x) - f'(y)}{x - y} = M\left(f(x), f(y)\right), \quad x, y \in I, \;\; x \neq y, \tag{4}$$

if, and only if, there are $a, b \in \mathbb{R}$, $a^2 + b^2 > 0$, such that

$$f(x) = ae^{-x} + be^{x}, \quad x \in I.$$

Moreover, assuming that I is a maximal interval such that $f'(x) \neq 0$ for all $x \in I$,

(i) if $a = 0$ and $b > 0$, or $a > 0$ and $b = 0$, then $I = \mathbb{R}$, $J = (0, \infty)$ and

$$M\left(u, v\right) = L(u, v), \quad u, v > 0, \;\; u \neq v;$$

(ii) if $a = 0$ and $b < 0$, or $a < 0$ and $b = 0$, then $I = \mathbb{R}$, $J = (-\infty, 0)$ and

$$M\left(u, v\right) = L(u, v), \quad u, v < 0, \;\; u \neq v;$$

(iii) if $a > 0$ and $b > 0$ then either $I = \left(\frac{1}{2}\log\left(\frac{a}{b}\right), \infty\right)$ or $I = \left(-\infty, \frac{1}{2}\log\left(\frac{a}{b}\right)\right)$; in both cases $J = (2\sqrt{ab}, \infty)$ and

$$M(u, v) = M_{4ab}, \quad u, v \in J, \;\; u \neq v;$$

(iv) if $a < 0$ and $b < 0$ then either $I = \left(\frac{1}{2}\log\left(\frac{a}{b}\right), \infty\right)$ or $I = \left(-\infty, \frac{1}{2}\log\left(\frac{a}{b}\right)\right)$; in both cases $J = \left(-\infty, 2\sqrt{ab}\right)$ and

$$M(u, v) = M_{4ab}, \quad u, v \in J, \;\; u \neq v;$$

(v) if $ab < 0$ then $I = \mathbb{R}$ and $M = M_{4ab}$.

Proof. Suppose that a differentiable function $f : I \to J$ and a mean $M : J^2 \to J$, continuous on the diagonal Δ, satisfy equation (1). Hence, letting $y \to x \in I$ in (1), by the reflexivity of the mean and the continuity of f, the right-hand side of (1) tends to $M\left(f(x), f(x)\right) = f(x)$. Moreover, equality (1) implies that f is twice differentiable at x, and we get

$$f''(x) = f(x), \quad x \in I,$$

whence, for some $a, b \in \mathbb{R}$,

$$f(x) = ae^{-x} + be^{x}, \quad x \in I.$$

Since, by assumption, $f'(x) \neq 0$ for $x \in I$, we get $a^2 + b^2 > 0$.

Let $I \subset \mathbb{R}$ be the maximal interval such that $f'(x) \neq 0$ for $x \in I$. Then, setting this function into equation (1), we get

$$M\left(ae^{-x}+be^{x}, ae^{-y}+be^{y}\right) = \frac{\left(-ae^{-x}+be^{x}\right)-\left(-ae^{-y}+be^{y}\right)}{x-y}, \tag{5}$$

$x, y \in I$, $x \neq y$.

To prove the "moreover" part we consider several possible cases. We begin with

Cases when $a = 0$ or $b = 0$.

Assume, for instance, that $a = 0$. Since $b \neq 0$, we have

$$f(x) = be^{x}, \qquad f'(x) = be^{x}, \quad x \in I,$$

$I = \mathbb{R}$, and from (4),

$$M\left(be^{x}, be^{y}\right) = \frac{be^{x}-be^{y}}{x-y}, \quad x, y \in I, \ x \neq y. \tag{6}$$

If $b > 0$ then, obviously, $J = (0, \infty)$ and, for arbitrary $u, v > 0$, $u \neq v$, setting here $x := \log \frac{u}{b}$, $y := \log \frac{v}{b}$, we get

$$M(u, v) = \frac{u-v}{\log u - \log v} = \frac{u-v}{\log(u/v)}.$$

If $b < 0$ then $J = (-\infty, 0)$ and, for arbitrary $u, v < 0$, $u \neq v$, setting in (5) $x := \log \frac{u}{b}$, $y := \log \frac{v}{b}$, we get

$$M(u, v) = \frac{u-v}{\log(-u) - \log(-v)} = \frac{u-v}{\log(u/v)}.$$

We omit similar reasoning in the case $b = 0$.

Case *(iii)* **when $a > 0$ and $b > 0$.**

It is easy to verify that, in this case, $f'(x_0) = -ae^{-x_0} + be^{x_0} = 0$ iff

$$x_0 = \frac{1}{2}\log\frac{a}{b},$$

the function f is strictly increasing in the interval (x_0, ∞), strictly decreasing in the interval $(-\infty, x_0)$, attains a global minimum $2\sqrt{ab}$ at the point x_0, and

$$\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = \infty.$$

It follows that either $I = (x_0, \infty)$ and $J = \left(2\sqrt{ab}, \infty\right)$, or $I = (-\infty, x_0)$ and $J = \left(2\sqrt{ab}, \infty\right)$.

Assume that $I = (x_0, \infty)$ and take arbitrary $u, v \in f(I)$, $u \neq v$. Setting

$$ae^{-x} + be^{x} = u, \qquad ae^{-y} + be^{y} = v$$

we conclude that

$$e^{x} = \frac{u + \sqrt{u^2 - 4ab}}{2b}, \qquad e^{y} = \frac{v + \sqrt{v^2 - 4ab}}{2b};$$

$$x = f^{-1}(u) = \log \frac{u + \sqrt{u^2 - 4ab}}{2b}, \qquad y = f^{-1}(v) = \log \frac{v + \sqrt{v^2 - 4ab}}{2b}.$$

Hence, applying (4), after simple calculations, we obtain

$$M(u,v) = \frac{\sqrt{u^2 - 4ab} - \sqrt{v^2 - 4ab}}{\log\left(u + \sqrt{u^2 - 4ab}\right) - \log\left(v + \sqrt{v^2 - 4ab}\right)}.$$

Assume that $I = (-\infty, x_0)$. Now, taking arbitrary $u, v \in J$, $u \neq v$, and setting

$$ae^{-x} + be^{x} = u, \qquad ae^{-y} + be^{y} = v$$

we conclude that

$$e^{x} = \frac{u - \sqrt{u^2 - 4ab}}{2b}, \qquad e^{y} = \frac{v - \sqrt{v^2 - 4ab}}{2b};$$

$$x = f^{-1}(u) = \log \frac{u - \sqrt{u^2 - 4ab}}{2b}, \qquad y = f^{-1}(v) = \log \frac{v - \sqrt{v^2 - 4ab}}{2b}.$$

Applying (4), and taking into account that

$$\log\left(u - \sqrt{u^2 - 4ab}\right) - \log\left(v - \sqrt{v^2 - 4ab}\right)$$
$$= \log \frac{u - \sqrt{u^2 - 4ab}}{v - \sqrt{v^2 - 4ab}} = \log \frac{v + \sqrt{v^2 - 4ab}}{u + \sqrt{u^2 - 4ab}},$$

after simple calculations, we obtain

$$M(u,v) = \frac{\sqrt{u^2 - 4ab} - \sqrt{v^2 - 4ab}}{\log\left(u - \sqrt{u^2 - 4ab}\right) - \log\left(v - \sqrt{v^2 - 4ab}\right)},$$

the same formula as in the case when $I = (x_0, \infty)$.

Case (iv) **when** $a < 0$ **and** $b < 0$.

Since the considerations are anologous is in the case (iii), we omit them.

Case (v) **when** $ab < 0$.

In this case $f'(x) = -ae^{-x} + be^{x} \neq 0$ for all $x \in \mathbb{R}$. Consequently, f is strictly monotonic in $I = \mathbb{R}$ and, obviously, $J = \mathbb{R}$. Simple calculations give

$$
\begin{aligned}
M(u,v) &= \frac{\left(-\frac{u-\sqrt{u^2-4ab}}{2} + \frac{u+\sqrt{u^2-4ab}}{2}\right) - \left(-\frac{v-\sqrt{v^2-4ab}}{2} + \frac{v+\sqrt{v^2-4ab}}{2}\right)}{\log\left(\frac{u+\sqrt{u^2-4ab}}{2b}\right) - \log\left(\frac{v+\sqrt{v^2-4ab}}{2b}\right)} \\
&= \frac{\sqrt{u^2-4ab} - \sqrt{v^2-4ab}}{\log\left(u+\sqrt{u^2-4ab}\right) - \log\left(v+\sqrt{v^2-4ab}\right)} = M_{-4ab}(u,v).
\end{aligned}
$$

■

Remark 2. If $a = -1$ and $b = 1$ then $f = \sinh$ and

$$
M_{\sinh} u,v) := M_4(u,v) = \frac{\sqrt{u^2+c} - \sqrt{v^2+c}}{\log \frac{u+\sqrt{u^2+c}}{v+\sqrt{v^2+c}}}, \quad \text{for } u,v \in \mathbb{R}, \ \ u \neq v.
$$

2. Equation (2)

In this section we prove the following

Theorem 3. *Let $I \subseteq \mathbb{R}$ be an interval. Suppose that $f : I \to J$ is a differentiable function, such that $f'(x) \neq 0$ for $x \in I$, and $M : I^2 \to I$ is a mean that is continuous on the diagonal $\Delta := \{(x,x) : x \in J\}$. Then*

$$
\frac{f'(x) - f'(y)}{x-y} = f\left(M\left(x,y\right)\right), \quad x,y \in I, \ \ x \neq y, \tag{2}
$$

if, and only if, there are $a, b \in \mathbb{R}$, $a^2 + b^2 > 0$, such that

$$
f(x) = ae^{-x} + be^{x}, \quad x \in I,
$$

and, for all $x, y \in I$, $x \neq y$,

$$
M\left(x,y\right) = \begin{cases} \log \frac{e^x - e^y}{x-y}, & \text{if } \ a = 0, \\ x + y - \log \frac{e^x - e^y}{x-y}, & \text{if } \ b = 0, \\ \log\left(\frac{e^x - ce^{-x} - e^y + ce^{-y}}{2(x-y)} + \sqrt{\left(\frac{e^x - ce^{-x} - e^y + ce^{-y}}{2(x-y)}\right)^2 - c}\right), & \\ & \text{if } \ ab \neq 0 \end{cases} \tag{7}
$$

where

$$
c := \frac{a}{b}.
$$

Moreover, if $c < 0$ then M in formula (6) *is a mean in $\mathbb{R}$; if $c > 0$ then M is a mean on $(0, \infty)$.*

Proof. Suppose that a differentiable function $f : I \to J$ is and a mean $M : J^2 \to J$, continuous on the diagonal Δ, satisfy equation (2). Letting $y \to x \in I$ in (2), similarly as in the previous proof, we get $f''(x) = f(x)$ for $x \in I$,whence

$$f(x) = ae^{-x} + be^{x}, \quad x \in I.$$

for some $a, b \in \mathbb{R}$. Substituting this function into (2) gives

$$ae^{-M(x,y)} + be^{M(x,y)} = \frac{-ae^{-x} + be^{x} + ae^{-y} - be^{x}}{x-y}, \quad x, y \in I, \ \ x \neq y.$$

Hence for $b = 0$, we get

$$M(x,y) = x + y - \log \frac{e^x - e^y}{x-y}, \quad x, y \in I, \ \ x \neq y.$$

Assume that $b \neq 0$. Setting $c := \frac{a}{b}$ and dividing both sides of equality (2) by b we get

$$ce^{-M(x,y)} + e^{M(x,y)} = \frac{-ce^{-x} + e^{x} + ce^{-y} - e^{y}}{x-y}, \quad x, y \in I, \ \ x \neq y.$$

Taking into account that M is reflexive, after simple calculations, we obtain

$$M(x,y) = \log\left(\frac{e^x - ce^{-x} - e^y + ce^{-y}}{2(x-y)} + \sqrt{\left(\frac{e^x - ce^{-x} - e^y + ce^{-y}}{2(x-y)}\right)^2 - c}\right)$$

for all $x, y \in I$, $x \neq y$. The converse implication is obvious. ■

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