F A S C I C U L I M A T H E M A T I C I

Nr 47 2011

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FURTHER PROPERTIES OF QUASI M-CONTINUOUS FUNCTIONS

Abstract. We obtain the further properties of quasi Mcontinuous functions which were introduced and investigated in order to establish the unified theory for several variations of quasi-continuity between bitopological spaces.

Key words: quasi-open, m_X-open, m-structure, quasi m-structure, quasi mg-closed, locally quasi mg-closed, quasi M-continuous, bitopological space.

AMS Mathematics Subject Classification: 54C08, 54E55.

1. Introduction

The notion of quasi-open sets in bitopological spaces is introduced by Datta [10] and studied in [21] and [41]. Some properties of quasi-open sets are studied in [21]. The notion of quasi-continuity between bitopological spaces is introduced and studied in [6]. Quasi-semi-open sets and quasi-semi-continuous functions are introduced in [22] and studied in [15] and [37]. Popa [36] introduced and investigated the notions of quasi preopen sets and quasi precontinuity between bitopological spaces. Thakur and Paik introduced and studied the notion of quasi-α-open sets in [43] and [44]. Moreover, Thakur and Verma [45] introduced and studied the notion of quasi semipreopen sets in bitopological spaces.

The present authors introduced the notions of minimal structures, m-spaces, M-continuity in [38] and [39]. Recently, in [7], by using these concepts we introduced the notion of quasi M-continuous functions which establish the unified theory for several variations of quasi continuity between bitopological spaces. In this paper, we obtain the further properties of quasi M-continuous functions which have been investigated in [7].

2. Preliminaries

Let (X, τ) be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively.

Definition 1. *Let* (X, τ) *be a topological space. A subset* A *of* X *is said to be* α-open *[32] (resp.* semi-open *[17],* preopen *[26],* β-open *[1] or* semi-preopen *[3]) if* $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$ (*resp.* $A \subset \mathrm{Cl}(\mathrm{Int}(A))$, $A \subset \mathrm{Int}(\mathrm{Cl}(A))$, $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$).

The family of all semi-open (resp. preopen, α-open, β-open, semi-preopen) sets in X is denoted by $\mathrm{SO}(X)$ (resp. $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$, $\mathrm{SPO}(X)$).

Definition 2. *The complement of a semi-open* (*resp. preopen,* α*-open,* β*-open , semi-preopen*) *set is said to be* semi-closed *[8]* (*resp.* preclosed *[11],* α-closed *[27],* β-closed *[1],* semi-preclosed *[3]*).

Definition 3. *The intersection of all semi-closed* (*resp. preclosed,* α*-closed,* β*-closed, semi-preclosed*) *sets of* X *containing* A *is called the* semi-closure *[8]* (*resp.* preclosure *[11],* α-closure *[27],* β-closure *[2],* semi-preclosure *[3]*) *of* A *and is denoted by sCl(A)* (*resp. pCl(A),* α*Cl(A),* $_{\beta}$*Cl(A), spCl(A)*).

Definition 4. *The union of all semi-open* (*resp. preopen,* α*-open,* β*-open, semi-preopen*) *sets of* X *contained in* A *is called the* semi-interior (*resp.* preinterior, α-interior, β-interior, semi-preinterior) *of* A *and is denoted by sInt(A)* (*resp. pInt(A),* α*Int(A),* $_{\beta}$*Int(A), spInt(A)*).

Throughout the present paper, (X, τ) and (Y, σ) always denote topological spaces and (X, τ_1, τ_2) and (Y, σ_1, σ_2) denote bitopological spaces.

3. Minimal structures

Definition 5. *A subfamily* m_X *of the power set* $\mathcal{P}(X)$ *of a nonempty set* X *is called a* minimal structure (*or briefly* m-structure) *[38], [39] on* X *if* $\emptyset \in m_X$ *and* $X \in m_X$.

By (X, m_X) (or briefly (X, m)), we denote a nonempty set X with a minimal structure m_X on X and call it an *m-space*. Each member of m_X is said to be m_X*-open* (or briefly *m-open*) and the complement of an m_X-open set is said to be m_X*-closed* (or briefly *m-closed*).

Remark 1. Let (X, τ) be a topological space. Then the families τ, $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$ and $\mathrm{SPO}(X)$ are all m-structures on X.

Definition 6. *Let* X *be a nonempty set and* m_X *an* m*-structure on* X*. For a subset* A *of* X*, the* m_X-closure *of* A *and the* m_X-interior *of* A *are defined in [25] as follows:*

(*a*) m_X*-Cl(A)* $= \cap\{F : A \subset F, X - F \in m_X\}$,

(*b*) m_X*-Int(A)* $= \cup\{U : U \subset A, U \in m_X\}$. m_X*-Cl(A) and* m_X*-Int(A) are briefly denoted by mCl(A) and mInt(A), respectively.*

Remark 2. Let (X,τ) be a topological space and A a subset of X. If $m_X = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$, $\mathrm{SPO}(X)$), then we have

(*a*) $m_X\text{-Cl}(A) = \mathrm{Cl}(A)$ (resp. $\mathrm{sCl}(A)$, $\mathrm{pCl}(A)$, $\alpha\mathrm{Cl}(A)$, ${}_\beta\mathrm{Cl}(A)$, $\mathrm{spCl}(A)$),

(*b*) $m_X\text{-Int}(A) = \mathrm{Int}(A)$ (resp. $\mathrm{sInt}(A)$, $\mathrm{pInt}(A)$, $\alpha\mathrm{Int}(A)$, ${}_\beta\mathrm{Int}(A)$, $\mathrm{spInt}(A)$).

Lemma 1 (Maki et al. [25]). *Let (X,m_X) be an m-space. For subsets A and B of X, the following properties hold:*

(*a*) $\mathrm{mCl}(X-A) = X - \mathrm{mInt}(A)$ *and* $\mathrm{mInt}(X-A) = X - \mathrm{mCl}(A)$,

(*b*) *If* $(X-A) \in m_X$, *then* $\mathrm{mCl}(A) = A$ *and if* $A \in m_X$, *then* $\mathrm{mInt}(A) = A$,

(*c*) $\mathrm{mCl}(\emptyset) = \emptyset, \mathrm{mCl}(X) = X$, $\mathrm{mInt}(\emptyset) = \emptyset$ *and* $\mathrm{mInt}(X) = X$,

(*d*) *If* $A \subset B$, *then* $\mathrm{mCl}(A) \subset \mathrm{mCl}(B)$ *and* $\mathrm{mInt}(A) \subset \mathrm{mInt}(B)$,

(*e*) $A \subset \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(A) \subset A$,

(*f*) $\mathrm{mCl}(\mathrm{mCl}(A)) = \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(\mathrm{mInt}(A)) = \mathrm{mInt}(A)$.

Lemma 2 (Popa and Noiri [38]). *Let (X,m_X) be an m-space and A a subset of X. Then $x \in m_X\text{-Cl}(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in m_X$ containing x.*

Definition 7. *A minimal structure m_X on a nonempty set X is said to have property $\mathcal{B}$ [25] if the union of any family of subsets belonging to m_X belongs to m_X.*

Remark 3. Let (X,τ) be a topological space, then τ, $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$, $\mathrm{SPO}(X)$ are all m-structures on X having property $\mathcal{B}$.

Lemma 3 (Popa and Noiri [40]). *Let (X,m_X) be an m-space and m_X satisfy property $\mathcal{B}$. Then for a subset A of X, the following properties hold:*

(*a*) $A \in m_X$ *if and only if* $m_X\text{-Int}(A) = A$,

(*b*) A *is* m_X*-closed if and only if* $m_X\text{-Cl}(A) = A$,

(*c*) $m_X\text{-Int}(A) \in m_X$ *and* $m_X\text{-Cl}(A)$ *is* m_X*-closed.*

Definition 8. *A subset A of a bitopological space (X,τ_1,τ_2) is said to be*

(*a*) quasi open *[10], [21] if* $A = B \cup C$, *where* $B \in \tau_1$ *and* $C \in \tau_2$,

(*b*) quasi semi-open *[15], [22] if* $A = B \cup C$, *where* $B \in \mathrm{SO}(X,\tau_1)$ *and* $C \in \mathrm{SO}(X,\tau_2)$,

(*c*) quasi preopen *[36] if* $A = B \cup C$, *where* $B \in \mathrm{PO}(X,\tau_1)$ *and* $C \in \mathrm{PO}(X,\tau_2)$,

(*d*) quasi semipreopen *[45] if* $A = B \cup C$, *where* $B \in \mathrm{SPO}(X,\tau_1)$ *and* $C \in \mathrm{SPO}(X,\tau_2)$,

(*e*) quasi α-open *[43] if* $A = B \cup C$, *where* $B \in \alpha(X,\tau_1)$ *and* $C \in \alpha(X,\tau_2)$.

The family of all quasi open (resp. quasi semi-open, quasi preopen, quasi semipreopen, quasi α-open) sets of (X,τ_1,τ_2) is denoted by $\mathrm{QO}(X)$ (resp. $\mathrm{QSO}(X)$, $\mathrm{QPO}(X)$, $\mathrm{QSPO}(X)$, $\mathrm{Q}\alpha(X)$).

Definition 9. *Let (X, τ_1, τ_2) be a bitopological space and m^1_X (resp. m^2_X) an m-structure on the topological space (X, τ_1) (resp. (X, τ_2)). The family*

$$qm_X = \{A \subset X : A = B \cup C, \text{ where } B \in m^1_X \text{ and } C \in m^2_X\}$$

is called a quasi m-structure *on X. Each member $A \in qm_X$ is said to be* quasi m_X-open *(or briefly quasi m-open). The complement of a quasi m_X-open set is said to be* quasi m_X-closed *(or briefly quasi m-closed).*

Remark 4. Let (X, τ_1, τ_2) be a bitopological space.

(*a*) If m^1_X and m^2_X have property $(\mathcal{B})$, then qm_X is an m-structure with property $(\mathcal{B})$.

(*b*) If $m^1_X = \tau_1$ and $m^2_X = \tau_2$ (resp. $\mathrm{SO}(X, \tau_1)$ and $\mathrm{SO}(X, \tau_2)$, $\mathrm{PO}(X, \tau_1)$ and $\mathrm{PO}(X, \tau_2)$, $\mathrm{SPO}(X, \tau_1)$ and $\mathrm{SPO}(X, \tau_2)$, $\alpha(X, \tau_1)$ and $\alpha(X, \tau_2)$), then $qm_X = \mathrm{QO}(X)$ (resp. $\mathrm{QSO}(X)$, $\mathrm{QPO}(X)$, $\mathrm{QSPO}(X)$, $\mathrm{Q}\alpha\mathrm{O}(X)$).

(*c*) Since $\mathrm{SO}(X, \tau_i)$, $\mathrm{PO}(X, \tau_i)$, $\mathrm{SPO}(X, \tau_i)$ and $\alpha(X, \tau_i)$ have property $(\mathcal{B})$, $\mathrm{QSO}(X)$, $\mathrm{QPO}(X)$, $\mathrm{QSPO}(X)$ and $\mathrm{Q}\alpha\mathrm{O}(X)$ have property $(\mathcal{B})$.

Definition 10. *Let (X, τ_1, τ_2) be a bitopological space. For a subset A of X, the* quasi m_X-closure *of A and the* quasi m_X-interior *of A are defined as follows:*

(*a*) *qmCl(A)* $= \cap\{F : A \subset F, X - F \in qm_X\}$,

(*b*) *qmInt(A)* $= \cup\{U : U \subset A, U \in qm_X\}$.

Remark 5. Let (X, τ_1, τ_2) be a bitopological space and A a subset of X. If $qm_X = \mathrm{QO}(X)$ (resp. $\mathrm{QSO}(X)$, $\mathrm{QPO}(X)$, $\mathrm{QSPO}(X)$, $\mathrm{Q}\alpha\mathrm{O}(X)$), then we have

(*a*) $\mathrm{qmCl}(A) = \mathrm{qCl}(A)$ (resp. $\mathrm{qsCl}(A)$ [15], $\mathrm{qpCl}(A)$ [36], $\mathrm{qspCl}(A)$ [45], $\mathrm{q}\alpha\mathrm{Cl}(A)$ [43]),

(*b*) $\mathrm{qmInt}(A) = \mathrm{qInt}(A)$ (resp. $\mathrm{qsInt}(A)$, $\mathrm{qpInt}(A)$, $\mathrm{qspInt}(A)$, $\mathrm{q}\alpha\mathrm{Int}(A)$).

4. Quasi mg-closed sets in bitopological spaces

Definition 11. *Let (X, τ) be a topological space. A subset A of X is said to be*

(*a*) *g-closed [18] if* $\mathrm{Cl}(A) \subset U$ *whenever* $A \subset U$ *and* $U \in \tau$,

(*b*) *gα-closed [24] if* $\alpha Cl(A) \subset U$ *whenever* $A \subset U$ *and* $U \in \alpha(X)$,

(*c*) *sg-closed [4] if* $\mathrm{sCl}(A) \subset U$ *whenever* $A \subset U$ *and* $U \in \mathrm{SO}(X)$,

(*d*) *pg-closed [34] if* $\mathrm{pCl}(A) \subset U$ *whenever* $A \subset U$ *and* $U \in \mathrm{PO}(X)$,

(*e*) *spg-closed if* $\mathrm{spCl}(A) \subset U$ *whenever* $A \subset U$ *and* $U \in \mathrm{SPO}(X)$.

Definition 12. *Let (X, m_X) be an m-space. A subset A of X is said to be mg-closed [33] if m_X-$\mathrm{Cl}(A) \subset U$ whenever $A \subset U$ and $U \in m_X$. The complement of an mg-closed set is said to be mg-open.*

The collection of all mg-open sets of (X, m_X) is denoted by $\mathrm{mGO}(X)$. Then $\mathrm{mGO}(X)$ is a new minimal structure on X.

Remark 6. Let (X, τ) be a topological space and m_X an m-structure on X. If $m_X = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\mathrm{SPO}(X)$), then, an mg-closed set is a g-closed (resp. sg-closed, pg-closed, $g\alpha$-closed, spg-closed) set.

Definition 13. *Let (X, τ_1, τ_2) be a bitopological space. A subset A of X is said to be*

(a) *quasi g-closed or* (1,2)g-closed *[16] if* $\mathrm{qCl}(A) \subset U$ *whenever* $A \subset U$ *and* $U \in \mathrm{QO}(X)$,

(b) *quasi αg-closed if $q\alpha Cl(A) \subset U$ whenever $A \subset U$ and $U \in \mathrm{Q\alpha O}(X)$,*

(c) *quasi sg-closed if* $\mathrm{qsCl}(A) \subset U$ *whenever* $A \subset U$ *and* $U \in \mathrm{QSO}(X)$,

(d) *quasi pg-closed if* $\mathrm{qpCl}(A) \subset U$ *whenever* $A \subset U$ *and* $U \in \mathrm{QPO}(X)$,

(e) *quasi spg-closed if* $\mathrm{qspCl}(A) \subset U$ *whenever* $A \subset U$ *and* $U \in \mathrm{QSPO}(X)$.

Definition 14. *Let (X, τ_1, τ_2) be a bitopological space and qm_X a quasi m-structure on X. A subset A is said to be* quasi mg-closed *in (X, τ_1, τ_2) if A is mg-closed in (X, qm_X).*

A subset A is said to be *quasi mg-open* if the complement of A is quasi mg-closed. The collection of all quasi g-open (resp. quasi sg-open, quasi pg-open, quasi αg-open, quasi spg-open) sets of (X, τ_1, τ_2) is denoted by $\mathrm{QGO}(X)$ (resp. $\mathrm{QSGO}(X)$, $\mathrm{QPGO}(X)$, $\mathrm{Q\alpha GO}(X)$, $\mathrm{QSPGO}(X)$).

Remark 7. Let (X, τ_1, τ_2) be a bitopological space.

(a) If $qm_X = \mathrm{QO}(X)$ (resp. $\mathrm{QSO}(X)$, $\mathrm{QPO}(X)$, $\mathrm{Q\alpha}(X)$, $\mathrm{QSPO}(X)$), then, a quasi mg-closed set is a quasi g-closed (resp. quasi sg-closed, quasi pg-closed, quasi αg-closed, quasi spg-closed) set.

(b) The families $\mathrm{QGO}(X)$, $\mathrm{QSGO}(X)$, $\mathrm{QPGO}(X)$, $\mathrm{Q\alpha GO}(X)$ and QSPG-O(X) are all minimal structures on X which do not have property $\mathcal{B}$ in general.

Lemma 4 (Noiri [33])**.** *Let (X, m_X) be an m-space. For subsets A, B of X, the following properties hold:*

(a) *if A is m-closed, then A is mg-closed,*

(b) *if m_X has property $\mathcal{B}$ and A is mg-closed and m-open, then A is m-closed,*

(c) *if A is mg-closed and $A \subset B \subset \mathrm{mCl}(A)$, then B is mg-closed.*

Theorem 1. *Let (X, τ_1, τ_2) be a bitopological space and qm_X a quasi m-structure on X. For subsets A, B of X, the following properties hold:*

(a) *if A is quasi m-closed, then A is quasi mg-closed,*

(b) *if qm_X has property $\mathcal{B}$ and A is quasi mg-closed and quasi m-open, then A is quasi m-closed,* *(c) if A is quasi mg-closed and $A \subset B \subset \mathrm{qmCl}(A)$, then B is quasi mg-closed.*

Proof. The proof follows from Definition 14 and Lemma 4. ∎

Corollary 1. *Let (X, τ_1, τ_2) be a bitopological space. For subsets A, B of X, the following properties hold:*
(a) if A is quasi-closed, then A is quasi g-closed,
(b) if A is quasi g-closed and quasi-open, then A is quasi-closed,
(c) if A is quasi g-closed and $A \subset B \subset \mathrm{qCl}(A)$, then B is quasi g-closed.

Lemma 5 (Noiri [33])**.** *Let (X, m_X) be an m-space. Then, for each $x \in X$, either $\{x\}$ is m-closed or $\{x\}$ is mg-open.*

Theorem 2. *Let (X, τ_1, τ_2) be a bitopological space and qm_X a quasi m-structure on X. Then, for each $x \in X$, either $\{x\}$ is quasi m-closed or $\{x\}$ is quasi mg-open.*

Proof. The proof follows from Definition 14 and Lemma 5. ∎

Corollary 2. *Let (X, τ_1, τ_2) be a bitopological space. Then, for each $x \in X$, either $\{x\}$ is quasi closed or $\{x\}$ is quasi g-open.*

Lemma 6 (Noiri [33])**.** *Let (X, m_X) be an m-space. Then, a subset A of X is mg-open if and only if $F \subset \mathrm{mInt}(A)$ whenever $F \subset A$ and F is m-closed.*

Theorem 3. *Let (X, τ_1, τ_2) be a bitopological space and qm_X a minimal structure on X. Then, a subset A of X is quasi mg-open if and only if $F \subset \mathrm{qmInt}(A)$ whenever $F \subset A$ and F is quasi m-closed.*

Proof. The proof follows from Definition 14 and Lemma 6. ∎

Corollary 3. *A subset A of (X, τ_1, τ_2) is quasi g-open if and only if $F \subset \mathrm{qInt}(A)$ whenever $F \subset A$ and F is quasi closed.*

Lemma 7 (Noiri [33])**.** *For subsets A, B of X, the following properties hold:*
(a) if A is m-open, then A is mg-open,
(b) if m_X has property $\mathcal{B}$ and A is mg-open and m-closed, then A is m-open,
(c) if A is mg-open and $\mathrm{mInt}(A) \subset B \subset A$, then B is mg-open.

Theorem 4. *Let (X, τ_1, τ_2) be a bitopological space and qm_X a quasi minimal structure on X. For subsets A, B of X, the following properties hold:*

(*a*) *if* A *is quasi m-open, then* A *is quasi mg-open,*
(*b*) *if* qm_X *has property* $\mathcal{B}$ *and* A *is quasi mg-open and quasi m-closed, then* A *is quasi m-open,*
(*c*) *if* A *is quasi mg-open and* $\mathrm{qmInt}(A) \subset B \subset A$*, then* B *is quasi mg-open.*

Proof. The proof follows from Definition 14 and Lemma 7. ■

Corollary 4. *For subsets* A*,* B *of* (X, τ_1, τ_2)*, the following properties hold:*
(*a*) *if* A *is quasi-open, then* A *is quasi g-open,*
(*b*) *if* A *is quasi g-open and quasi-closed, then* A *is quasi-open,*
(*c*) *if* A *is quasi g-open and* $\mathrm{qInt}(A) \subset B \subset A$*, then* B *is quasi g-open.*

Lemma 8 (Noiri [33]). *Let* (X, m_X) *be an m-space, where* m_X *have property* $\mathcal{B}$*. Then, for a subset* A *of* X*, the following properties are equivalent.*
(*a*) A *is mg-closed;*
(*b*) $\mathrm{mCl}(A) - A$ *does not contain any nonempty m-closed set;*
(*c*) $\mathrm{mCl}(A) - A$ *is mg-open.*

Theorem 5. *Let* (X, τ_1, τ_2) *be a bitopological space and* qm_X *a quasi minimal structure on* X *having property* $\mathcal{B}$*. Then, for a subset* A *of* X*, the following properties are equivalent:*
(*a*) A *is quasi mg-closed;*
(*b*) $\mathrm{qmCl}(A) - A$ *does not contain any nonempty quasi m-closed set;*
(*c*) $\mathrm{qmCl}(A) - A$ *is quasi mg-open.*

Proof. The proof follows from Definition 14 and Lemma 8. ■

Corollary 5. *For a subset* A *of of a bitopological space* (X, τ_1, τ_2)*, the following properties are equivalent:*
(*a*) A *is quasi g-closed;*
(*b*) $\mathrm{qCl}(A) - A$ *does not contain any nonempty quasi closed set;*
(*c*) $\mathrm{qCl}(A) - A$ *is quasi g-open.*

Lemma 9 (Noiri [33]). *Let* (X, m_X) *be an m-space. A subset* A *of* X *is mg-closed if and only if* $\mathrm{mCl}(A) \cap F = \emptyset$ *whenever* $A \cap F = \emptyset$ *and* F *is m-closed.*

Theorem 6. *Let* (X, τ_1, τ_2) *be a bitopological space and* qm_X *a quasi minimal structure on* X*. Then, a subset* A *of* X *is quasi mg-closed if and only if* $\mathrm{qmCl}(A) \cap F = \emptyset$ *whenever* $A \cap F = \emptyset$ *and* F *is quasi m-closed.*

Proof. The proof follows from Definition 14 and Lemma 9. ■

Corollary 6. *Let* (X, τ_1, τ_2) *be a bitopological space. Then, a subset* A *of* X *is quasi g-closed if and only if* $\mathrm{qCl}(A) \cap F = \emptyset$ *whenever* $A \cap F = \emptyset$ *and* F *is quasi closed.*

Lemma 10 (Noiri [33])**.** *Let* (X, m_X) *be an m-space, where* m_X *has property* $\mathcal{B}$. *A subset* A *of* X *is mg-closed if and only if* $\mathrm{mCl}(\{x\}) \cap A \neq \emptyset$ *for each* $x \in \mathrm{mCl}(A)$.

Theorem 7. *Let* (X, τ_1, τ_2) *be a bitopological space and* qm_X *a quasi minimal structure on* X *having property* $\mathcal{B}$. *Then, a subset* A *of* X *is quasi mg-closed if and only if* $\mathrm{qmCl}(\{x\}) \cap A \neq \emptyset$ *for each* $x \in \mathrm{qmCl}(A)$.

Proof. The proof follows from Definition 14 and Lemma 10. ■

Corollary 7. *Let* (X, τ_1, τ_2) *be a bitopological space. Then, a subset* A *of* X *is quasi g-closed if and only if* $\mathrm{qCl}(\{x\}) \cap A \neq \emptyset$ *for each* $x \in \mathrm{qCl}(A)$.

Definition 15. *A subset* A *of an m-space* (X, m_X) *is said to be* locally m-closed *[33] if* $A = U \cap F$, *where* $U \in m_X$ *and* F *is m-closed.*

Remark 8. Let (X, τ) be a topological space. If $m_X = \tau$ (resp. $\mathrm{SO}(X)$, $\alpha(X)$, $\mathrm{SPO}(X)$, $\mathrm{PO}(X)$), then a locally m-closed set is said to be locally closed [12] (resp. semi-locally closed [42], α-locally closed [13], β-locally closed [14], locally pre-closed).

Lemma 11 (Noiri [33])**.** *Let* (X, m_X) *be an m-space and* m_X *have property* $\mathcal{B}$. *For a subset* A *of* X, *the following properties are equivalent:*

(a) A *is locally m-closed;*
(b) $A = U \cap \mathrm{mCl}(A)$ *for some* $U \in m_X$*;*
(c) $\mathrm{mCl}(A) - A$ *is m-closed;*
(d) $A \cup (X - \mathrm{mCl}(A)) \in m_X$*;*
(e) $A \subset \mathrm{mInt}[A \cup (X - \mathrm{mCl}(A))]$.

Definition 16. *Let* (X, τ_1, τ_2) *be a bitopological space and* qm_X *a quasi minimal structure on* X. *A subset* A *of* X *is said to be* locally quasi m-closed *if* A *is locally m-closed in* (X, qm_X), *equivalently if* $A = U \cap B$, *where* $U \in qm_X$ *and* B *is quasi m-closed.*

Remark 9. Let (X, τ_1, τ_2) be a bitopological space and $qm_X = \mathrm{QO}(X)$ (resp. $\mathrm{QSO}(X)$, $\mathrm{QPO}(X)$, $\mathrm{Q}\alpha(X)$, $\mathrm{QSPO}(X)$). If a subset A of X is locally quasi m-closed, then A is locally quasi closed (resp. locally quasi semi-closed, locally quasi preclosed, locally quasi α-closed, locally quasi semi-preclosed).

Theorem 8. *Let* (X, τ_1, τ_2) *be a bitopological space and* qm_X *a quasi minimal structure on* X *having property* $\mathcal{B}$. *For a subset* A *of* X, *the following properties are equivalent:*

(a) *A is locally quasi m-closed;*
(b) $A = U \cap \mathrm{qmCl}(A)$ *for some* $U \in qm_X$*;*
(c) $\mathrm{qmCl}(A) - A$ *is quasi m-closed;*
(d) $A \cup (X - \mathrm{qmCl}(A)) \in qm_X$*;*
(e) $A \subset \mathrm{qmInt}[A \cup (X - \mathrm{qmCl}(A))]$.

Proof. The proof follows from Definition 14 and Lemma 11. ■

Corollary 8. *Let (X, τ_1, τ_2) be a bitopological space. For a subset A of X, the following properties are equivalent:*
(a) *A is locally quasi closed;*
(b) $A = U \cap \mathrm{qCl}(A)$ *for some quasi open set U of X;*
(c) $\mathrm{qCl}(A) - A$ *is quasi closed;*
(d) $A \cup (X - \mathrm{qCl}(A))$ *is quasi closed;*
(e) $A \subset \mathrm{qInt}[A \cup (X - \mathrm{qCl}(A))]$.

Lemma 12 (Noiri [33]). *Let (X, m_X) be an m-space and m_X have property $\mathcal{B}$. Then a subset A of X is m-closed if and only if A is mg-closed and locally m-closed.*

Theorem 9. *Let (X, τ_1, τ_2) be a bitopological space and qm_X a quasi minimal structure on X having property $\mathcal{B}$. Then a subset A of X is quasi m-closed if and only if A is quasi mg-closed and locally quasi m-closed.*

Proof. The proof follows from Definition 14 and Lemma 12. ■

Corollary 9. *Let (X, τ_1, τ_2) be a bitopological space and A a subset of X. Then,*
(a) *A is quasi closed if and only if it is quasi g-closed and locally quasi closed,*
(b) *A is quasi semi-closed if and only if it is quasi sg-closed and locally quasi semi-closed,*
(c) *A is quasi preclosed if and only if it is quasi pg-closed and locally quasi preclosed.*
(d) *A is quasi α-closed if and only if it is quasi αg-closed and locally quasi α-closed,*
(e) *A is quasi β-closed if and only if it is quasi spg-closed and locally quasi β-closed.*

5. Some properties of M-continuity

Definition 17. *A function $f : (X, m_X) \to (Y, m_Y)$ is said to be* M-continuous *at a point $x \in X$ [38] if for each m_Y-open sets V of Y containing $f(x)$, there exists $U \in m_X$ containing x such that $f(U) \subset V$.*

A function $f : (X, m_X) \to (Y, m_Y)$ is said to be *M-continuous* if f is M-continuous at each point x of X.

Remark 10. Let (X, τ) and (Y, σ) be topological spaces.

(a) If $m_X = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\mathrm{SPO}(X)$) and $f : (X, m_X) \to (Y, \sigma)$ is M-continuous, then f is continuous (resp. semi-continuous [17] or quasi continuous [23], precontinuous [26], α-continuous [27], semi-precontinuous [3] or β-continuous [1]).

(b) If $m_X = \mathrm{SO}(X)$ (resp. $\mathrm{PO}(X)$, $\alpha(X)$, $\mathrm{SPO}(X)$), $m_Y = \mathrm{SO}(Y)$ (resp. $\mathrm{PO}(Y)$, $\alpha(Y)$, $\mathrm{SPO}(Y)$) and $f : (X, m_X) \to (Y, m_Y)$ is M-continuous, then f is irresolute [9] (resp. preirresolute [28], α-irresolute [20], β-irresolute [29]).

(c) If $m_X = \tau$, $m_Y = \mathrm{SO}(Y)$ (resp. $\alpha(Y)$, $\mathrm{SPO}(Y)$) and f is M-continuous, then f is s-continuous [5] (resp. strongly α-irresolute [19], strongly β-irresolute [31]).

Theorem 10 (Noiri and Popa [35]). *For a function $f : (X, m_X) \to (Y, m_Y)$, the following properties are equivalent:*

(a) *f is M-continuous at $x \in X$;*

(b) *$x \in \mathrm{mInt}(f^{-1}(V))$ for every $V \in m_Y$ containing $f(x)$;*

(c) *$x \in f^{-1}(\mathrm{mCl}(f(A)))$ for every subset A of X with $x \in \mathrm{mCl}(A)$;*

(d) *$x \in f^{-1}(\mathrm{mCl}(B))$ for every subset B of Y with $x \in \mathrm{mCl}(f^{-1}(B))$;*

(e) *$x \in \mathrm{mInt}(f^{-1}(B))$ for every subset B of Y with $x \in f^{-1}(\mathrm{mInt}(B))$;*

(f) *$x \in f^{-1}(K)$ for every m_Y-closed set K of Y such that $x \in \mathrm{mCl}(f^{-1}(K))$.*

For a function $f : (X, m_X) \to (Y, m_Y)$, we define $D_M(f)$ as follows:

$$D_M(f) = \{x \in X : f \quad \text{is not } M\text{-continuous at} x\}.$$

Theorem 11 (Noiri and Popa [35]). *For a function $f : (X, m_X) \to (Y, m_Y)$, the following properties hold:*

$$\begin{aligned} D_M(f) &= \bigcup_{G \in m_Y} \{f^{-1}(G) - \mathrm{mInt}(f^{-1}(G))\} \\ &= \bigcup_{B \in \mathcal{P}(\mathcal{Y})} \{f^{-1}(\mathrm{Int}(B)) - \mathrm{mInt}(f^{-1}(B))\} \\ &= \bigcup_{B \in \mathcal{P}(\mathcal{Y})} \{\mathrm{mCl}(f^{-1}(B)) - f^{-1}(\mathrm{mCl}(B))\} \\ &= \bigcup_{A \in \mathcal{P}(\mathcal{X})} \{\mathrm{mCl}(A) - f^{-1}(\mathrm{mCl}(f(A)))\}. \end{aligned}$$

Definition 18. *Let A be a subset of an m-space (X, m_X). The* m-frontier *of A, $mFr(A)$, is defined by $mFr(A) = mCl(A) \cap \mathrm{mFr}(X - A) = mCl(A) - \mathrm{mInt}(A)$.*

Theorem 12. *Let $f : (X, m_X) \to (Y, m_Y)$ be a function. Then the set $D_M(f)$ is equal to the union of m-frontiers of the inverse images of m-open sets of Y containing $f(x)$.*

Proof. Suppose that $x \in D_M(f)$. There exists $V \in m_Y$ containing $f(x)$ such that $f(U)$ is not contained in V for every $U \in m_X$ containing x. Then $U \cap (X - f^{-1}(V)) \neq \emptyset$ for every $U \in m_X$ containing x and hence by Lemma 3.2 $x \in \mathrm{mCl}(X - f^{-1}(V))$. On the other hand, we have $x \in f^{-1}(V) \subset \mathrm{mCl}(f^{-1}(V))$ and hence $x \in \mathrm{mFr}(f^{-1}(V))$.

Conversely, suppose that f is M-continuous at $x \in X$ and let V be any m_Y-open set of Y containing $f(x)$. Then, by Theorem 5.1, we have $f^{-1}(V) \subset \mathrm{mInt}(f^{-1}(V))$. Therefore, $x \notin \mathrm{mFr}(f^{-1}(V))$ for each m_Y-open set of Y containing $f(x)$. ■

Lemma 13 (Popa and Noiri [38]). *For a function $f : (X, m_X) \to (Y, m_Y)$, the following properties are equivalent:*
(*a*) *f is M-continuous;*
(*b*) *$f^{-1}(V) = m_X\text{-Int}(f^{-1}(V))$ for every m_Y-open set V of Y;*
(*c*) *$m_X\text{-Cl}(f^{-1}(F)) = f^{-1}(F)$ for every m_Y-closed set F of Y;*
(*d*) *$m_X\text{-Cl}(f^{-1}(B)) \subset f^{-1}(m_Y\text{-Cl}(B))$ for every subset B of Y;*
(*e*) *$f(m_X\text{-Cl}(A)) \subset m_Y\text{-Cl}(f(A))$ for every subset A of X;*
(*f*) *$f^{-1}(m_Y\text{-Int}(B)) \subset m_X\text{-Int}(f^{-1}(B))$ for every subset B of Y.*

Corollary 10 (Popa and Noiri [38]). *Let (X, m_X) be an m-space and m_X satisfy property $\mathcal{B}$. For a function $f : (X, m_X) \to (Y, m_Y)$, the following properties are equivalent:*
(*a*) *f is M-continuous;*
(*b*) *$f^{-1}(V)$ is m_X-open for every m_Y-open set V of Y;*
(*c*) *$f^{-1}(F)$ is m_X–closed for every m_Y-closed set F of Y.*

Definition 19. *A function $f : (X, m_X) \to (Y, m_Y)$ is said to be* M^*-continuous *[30] if $f^{-1}(V)$ is m-open in X for each m-open set V of Y.*

Remark 11. (*a*) If $f : (X, m_X) \to (Y, m_Y)$ is M^*-continuous, then it is M-continuous. But the converse may not be true by Example 3.4 of [30].
(*b*) If m_X has property $\mathcal{B}$, then M-continuity is equivalent to M^*-continuity.

Definition 20. *A function $f : (X, m_X) \to (Y, m_Y)$ is said to be* mg-continuous *if $f : (X, \mathrm{mGO}(X)) \to (Y, m_Y)$ is M^*-continuous, equivalently if $f^{-1}(V)$ is mg-closed for each m-closed set V of Y.*

Definition 21. *A function $f : (X, m_X) \to (Y, m_Y)$ is said to be* locally mc-continuous *if $f^{-1}(K)$ is locally m-closed in X for each m-closed set K of Y.*

Theorem 13. *A function $f : (X, m_X) \to (Y, m_Y)$, where m_X has property $\mathcal{B}$, is M-continuous if and only if it is mg-continuous and locally mc-continuous.*

Proof. The proof follows from Definitions 20 and 21 and Lemma 12. ■

Definition 22. *A function* $f : (X, m_X) \to (Y, m_Y)$ *is said to be* contra M^*-continuous *if* $f^{-1}(K)$ *is* m*-closed in* X *for each* m*-open set* K *of* Y.

Theorem 14. *A function* $f : (X, m_X) \to (Y, m_Y)$, *where* m_X *has property* $\mathcal{B}$, *is mg-continuous and contra* M^**-continuous, then f is M-continuous.*

Proof. Let V be any m-open set of Y. Since f is mg-continuous, $f^{-1}(V)$ is mg-open. Since f is contra M^*-continuous, $f^{-1}(V)$ is m-closed. By Lemma 7, $f^{-1}(V)$ is m-open. Therefore, by Corollary 10 f is M-continuous. ■

6. Quasi M-continuity in bitopological spaces

Definition 23. *A function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is said to be*

(a) quasi-continuous *[6]* (*resp.* quasi semi-continuous *[22],* quasi almost-continuous *[36],* quasi semi-precontinuous *[7]*) *if* $f^{-1}(V)$ *is quasi-open* (*resp. quasi semi-open, quasi preopen, quasi semi-preopen*) *in* (X, τ_1, τ_2) *for each quasi-open set* V *of* (Y, σ_1, σ_2),

(b) quasi irresolute *[22] if* $f^{-1}(V)$ *is quasi semi-open in* (X, τ_1, τ_2) *for each quasi semi-open set* V *of* (Y, σ_1, σ_2).

Definition 24. *Let* (X, τ_1, τ_2) *and* (Y, σ_1, σ_2) *be bitopological spaces. Let* qm_X (*resp.* qm_Y) *be a quasi* m*-structure on* X (*resp.* Y). *A function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is said to be*

(a) quasi M-continuous (*resp.* quasi M-continuous *at* $x \in X$) *[7] if* $f : (X, qm_X) \to (Y, m_Y)$ *is quasi* M*-continuous* (*resp. quasi* M*-continuous at* $x \in X$),

(b) M^*-continuous *if* $f : (X, qm_X) \to (Y, m_Y)$ *is* M^**-continuous.*

Remark 12. (a) Let qm_X be a quasi structure on (X, τ_1, τ_2) having property $\mathcal{B}$. Then a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is quasi M-continuous if and only if $f^{-1}(V)$ is quasi m-open in X for each quasi m-open set V of Y.

(b) If qm_X = QO(X) (resp. QSO(X), QPO(X), QSPO(X)) and qm_Y = QO(Y), then a quasi M-continuous function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is quasi-continuous (resp. quasi semi-continuous, quasi almost continuous, quasi semi-precontinuous).

Lemma 14 (Chae, Noiri and Popa [7]). *Let* (X, τ_1, τ_2) *and* (Y, σ_1, σ_2) *be bitopological spaces. Let* qm_X (*resp.* qm_Y) *be a quasi* m*-structure on* X (*resp.* Y). *For a function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, *the following properties are equivalent:*

(a) f *is quasi* M*-continuous;*
(b) $f^{-1}(V) = \mathrm{qmInt}(f^{-1}(V))$ *for every quasi-m-open set* V *of* Y*;*
(c) $\mathrm{qmCl}(f^{-1}(F)) = f^{-1}(F)$ *for every quasi-m-closed set* F *of* Y*;*
(d) $\mathrm{qmCl}(f^{-1}(B)) \subset f^{-1}(\mathrm{qmCl}(B))$ *for every subset* B *of* Y*;*
(e) $f(\mathrm{qmCl}(A)) \subset \mathrm{qmCl}(f(A))$ *for every subset* A *of* X*;*
(f) $f^{-1}(\mathrm{qmInt}(B)) \subset \mathrm{qmInt}(f^{-1}(B))$ *for every subset* B *of* Y*.*

Corollary 11 (Chae, Noiri and Popa [7])**.** *Let* (X, τ_1, τ_2) *be a bitopological space and* qm_X *a quasi* m*-structure on* X *satisfying property* $\mathcal{B}$*. For a function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$*, the following properties are equivalent:*
(a) f *is quasi* M*-continuous;*
(b) $f^{-1}(V)$ *is quasi-m-open for every quasi-m-open set* V *of* Y*;*
(c) $f^{-1}(F)$ *is quasi-m-closed for every quasi-m-closed set* F *of* Y*.*

Theorem 15. *Let* (X, τ_1, τ_2) *and* (Y, σ_1, σ_2) *be bitopological spaces. Let* qm_X *(resp.* qm_Y*) be a quasi* m*-structure on* X *(resp.* Y*). For a function* $f : (X, m_X) \to (Y, m_Y)$*, the following properties are equivalent:*
(a) f *is quasi* M*-continuous at* $x \in X$*;*
(b) $x \in \mathrm{qmInt}(f^{-1}(V))$ *for every* $V \in qm_Y$ *containing* $f(x)$*;*
(c) $x \in f^{-1}(\mathrm{qmCl}(f(A)))$ *for every subset* A *of* X *with* $x \in \mathrm{qmCl}(A)$*;*
(d) $x \in f^{-1}(\mathrm{qmCl}(B))$ *for every subset* B *of* Y *with* $x \in \mathrm{qmCl}(f^{-1}(B))$*;*
(e) $x \in \mathrm{qmInt}(f^{-1}(B))$ *for every subset* B *of* Y *with* $x \in f^{-1}(\mathrm{qmInt}(B))$*;*
(f) $x \in f^{-1}(K)$ *for every quasi* m*-closed set* K *of* Y *such that* $x \in \mathrm{qmCl}(f^{-1}(K))$*.*

Proof. The proof follows from Definition 24 and Theorem 10. ■

For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, we define $D_{qM}(f)$ as follows:

$$D_{qM}(f) = \{x \in X : f \text{ is not quasi } M\text{-continuous at } x\}.$$

Theorem 16. *Let* (X, τ_1, τ_2) *and* (Y, σ_1, σ_2) *be bitopological spaces. Let* qm_X *(resp.* qm_Y*) be a quasi* m*-structure on* X *(resp.* Y*). Then, for a function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$*, the following equlities hold:*

$$\begin{aligned} D_{qM}(f) &= \bigcup\nolimits_{G \in qM_Y} \{f^{-1}(G) - \mathrm{qmInt}(f^{-1}(G))\} \\ &= \bigcup\nolimits_{B \in \mathcal{P}(\mathcal{Y})} \{f^{-1}(\mathrm{qmInt}(B)) - \mathrm{qmInt}(f^{-1}(B))\} \\ &= \bigcup\nolimits_{B \in \mathcal{P}(\mathcal{Y})} \{\mathrm{qmCl}(f^{-1}(B)) - f^{-1}(\mathrm{qmCl}(B))\} \\ &= \bigcup\nolimits_{A \in \mathcal{P}(\mathcal{X})} \{\mathrm{qmCl}(A) - f^{-1}(\mathrm{qmCl}(f(A)))\}. \end{aligned}$$

For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, we define $D_q(f)$ as follows:

$$D_q(f) = \{x \in X : f \text{ is not quasi continuous at } x\}.$$

Corollary 12. *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following equlities hold:*

$$\begin{aligned} D_q(f) &= \bigcup_{G \in QO(Y)} \{f^{-1}(G) - \text{qInt}(f^{-1}(G))\} \\ &= \bigcup_{B \in \mathcal{P}(\mathcal{Y})} \{f^{-1}(\text{qInt}(B)) - \text{qInt}(f^{-1}(B))\} \\ &= \bigcup_{B \in \mathcal{P}(\mathcal{Y})} \{\text{qCl}(f^{-1}(B)) - f^{-1}(\text{qCl}(B))\} \\ &= \bigcup_{A \in \mathcal{P}(\mathcal{X})} \{\text{qCl}(A) - f^{-1}(\text{qCl}(f(A)))\}. \end{aligned}$$

Definition 25. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be* quasi g-continuous *(resp.* quasi sg-continuous, quasi pg-continuous, quasi αg-continuous, quasi spg-continuous) *if $f^{-1}(K)$ is quasi g-closed (resp. quasi g-semi-closed, quasi g-preclosed, quasi g-α-closed, quasi g-semipreclosed) set in (X, τ_1, τ_2) for each quasi closed set K of (Y, σ_1, σ_2).*

Definition 26. *Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be bitopological spaces. Let qm_X (resp. qm_Y) be a quasi m-structure on X (resp. Y). A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be* quasi mg-continuous *if $f : (X, qm_X) \to (Y, qm_Y)$ is mg-continuous, equivalently if $f^{-1}(K)$ is quasi mg-closed set in X for each quasi closed set K of Y.*

Remark 13. If $qm_X = \text{QO}(X)$ (resp. $\text{QSO}(X)$, $\text{QPO}(X)$, $\text{Q}\alpha\text{O}(X)$, $\text{QSPO}(X)$)) and $qm_Y = \text{QO}(Y)$, then by Definition 26 we obtain Definition 25.

Definition 27. *Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be bitopological spaces. Let qm_X (resp. qm_Y) be a quasi m-structure on X (resp. Y). A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be* locally quasi mc-continuous *if $f : (X, qm_X) \to (Y, qm_Y)$ is locally mc-continuous, equivalently if $f^{-1}(K)$ is locally quasi mg-closed set in X for each quasi m-closed set K of Y.*

Remark 14. If $qm_X = \text{QO}(X)$ (resp. $\text{QSO}(X)$, $\text{QPO}(X)$, $\text{Q}\alpha\text{O}(X)$, $\text{QSPO}(X)$), $qm_Y = \text{QO}(Y)$ and $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is locally quasi mc-continuous, then f is locally quasi continuous (resp. locally quasi semi-continuous, locally quasi precontinuous, locally quasi α-continuous, locally quasi semi-precontinuous).

By Definitions 26 and 27 and Theorem 13, we obtain the following decomposition theorem of quasi M-continuity in bitopological spaces.

Theorem 17. *Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be bitopological spaces. Let qm_X (resp. qm_Y) be a quasi m-structure on X (resp. Y), where qm_X has property $\mathcal{B}$. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is quasi M-continuous if and only if f is quasi mg-continuous and locally quasi mc-continuous.*

Corollary 13. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a function. Then the following properties hold:*

(a) *f is quasi continuous if and only if it is quasi g-continuous and locally quasi continuous,*

(b) *f is quasi semi-continuous if and only if it is quasi sg-continuous and locally quasi semi-continuous,*

(c) *f is quasi precontinuous if and only if it is quasi pg-continuous and locally quasi precontinuous,*

(d) *f is quasi α-continuous if and only if it is quasi αg-continuous and locally quasi α-continuous,*

(e) *f is quasi semi-precontinuous if and only if it is quasi spg-continuous and locally quasi semi-precontinuous.*

Definition 28. *Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be bitopological spaces. Let qm_X (resp. qm_Y) be a quasi m-structure on X (resp. Y). A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is contra quasi M^*-continuous if $f : (X, qm_X) \to (Y, qm_Y)$ is contra M^*-continuous, equivalently if $f^{-1}(V)$ is quasi m-closed set in X for each quasi m-open set V of Y.*

Remark 15. If $qm_X = \mathrm{QO}(X)$ (resp. $\mathrm{QSO}(X)$, $\mathrm{QPO}(X)$, $\mathrm{Q}\alpha\mathrm{O}(X)$, $\mathrm{QSPO}(X)$), $qm_Y = \mathrm{QO}(Y)$ and $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is contra quasi M^*-continuous, then f is contra quasi continuous (resp. contra quasi semi-continuous, contra quasi precontinuous, contra quasi α-continuous, contra quasi semi-precontinuous).

By Definition 28 and Theorem 14, we obtain the following theorem.

Theorem 18. *Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be bitopological spaces and let qm_X (resp. qm_Y) be a quasi m-structure on X (resp. Y), where qm_X has property $\mathcal{B}$. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is quasi mg-continuous and contra quasi M^*-continuous, then f is quasi M-continuous.*

Corollary 14. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a function. Then the following properties hold:*

(a) *f is quasi continuous if it is quasi g-continuous and contra quasi continuous,*

(b) *f is quasi semi-continuous if it is quasi sg-continuous and contra quasi semi-continuous,*

(c) *f is quasi precontinuous if it is quasi pg-continuous and contra quasi precontinuous,*

(d) *f is quasi α-continuous if it is quasi αg-continuous and contra quasi α-continuous,*

(e) *f is quasi semi-precontinuous if it is quasi spg-continuous and contra quasi semi-precontinuous.*

Proof. The proof follows from Theorem 18 and $qm_Y = \mathrm{QO}(Y)$. ■

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Received on 08.11.2010 and, in revised form, on 11.01.2011.