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Nr 47 2011

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A NOTE ON SUMMABILITY FACTORS OF INFINITE SERIES

Abstract. Necessary and sufficient conditions are obtained for $\sum a_n\lambda_n$ to be $|N,p,q|_k$ −summable, $k \geq 1$, whenever $\sum a_n$ is (N,p,q)−bounded.

Key words: absolute summability, infinite series, summability factor.

AMS Mathematics Subject Classification: 40F05, 40D25.

1. Introduction

Let $\sum a_n$ be a given infinite series with sequence of partial sums (s_n). Let (T_n) denote the sequence of (N,p,q) means of (s_n). The (N,p,q) transforms of (s_n) is defined by

$$T_n = \frac{1}{R_n}\sum_{v=0}^{n} p_{n-v}q_v s_v, \tag{1}$$

where

$$R_n = \sum_{v=0}^{n} p_{n-v}q_v \neq 0, \quad \text{for any} \quad n(p_{-1} = q_{-1} = R_{-1} = 0). \tag{2}$$

Necessary and sufficient conditions for the (N,p,q) method to be regular are

(i) $\lim p_{n-v}q_v/R_n = 0$ for each v, and

(ii) $\left|\sum_{v=0}^{n} p_{n-v}q_v\right| < K\,|R_n|$, where K is a positive constant independent of n.

The series $\sum a_n$ is said to be summable $\left|\overline{N},p_n\right|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1}\left|\varphi_n - \varphi_{n-1}\right|^k < \infty, \tag{3}$$

where

$$\varphi_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v s_v. \tag{4}$$

The series $\sum a_n$ is said to be summable $|N, p_n|$, if

$$\sum_{n=1}^{\infty} |\sigma_n - \sigma_{n-1}| < \infty, \tag{5}$$

where

$$\sigma_n = \frac{1}{P_n}\sum_{v=0}^{n} p_{n-v} s_v, \tag{6}$$

and it is said to be summable $|N, p, q|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1} |T_n - T_{n-1}|^k < \infty \tag{7}$$

where T_n as defined by (1).

For $k = 1$, $|N, p, q|_k$ summability reduces to $|N, p, q|$ summability. The series $\sum a_n$ is said to be (N, p, q) bounded or $\sum a_n = O(1)(N, p, q)$ if

$$t_n = \sum_{v=1}^{n} p_{n-v} q_v s_v = O(R_n) \quad \text{as} \quad n \to \infty. \tag{8}$$

By M, we denote the set of sequences (p_n) satisfying

$$\frac{p_{n+1}}{p_n} \leq \frac{p_{n+2}}{p_{n+1}} \leq 1, \quad p_n > 0, \quad n = 0, 1, \dots \; .$$

We also assume that (p_n), (q_n) are positive sequences of numbers such that

$$P_n = p_0 + p_1 + \ldots + p_n \to \infty, \quad \text{as} \quad n \to \infty,$$

$$Q_n = q_0 + q_1 + \ldots + q_n \to \infty, \quad \text{as} \quad n \to \infty.$$

We define the sequence of constant (c_n) formally by means of the identity

$$\left(\sum_{v=0}^{n} p_n x^n\right)^{-1} = \sum_{n=0}^{\infty} c_n x^n, \quad c_{-i} = 0, \quad i \geq 1.$$

We also write $c_n^{(1)} = c_0 + c_1 + \ldots + c_n$ and $\Delta_{v,n} = \Delta_v \Delta_n$.

Das [2], in 1966, proved the following result

Theorem 1. *Let* $(p_n) \in M$, $q_n \geq 0$. *Then if* $\sum a_n$ *is* $|N, p, q|$ *−summable it is* $\left|\overline{N}, q_n\right|$ *−summable.*

Recently Singh and Sharma [3] proved the following theorem

Theorem 2. *Let* $(p_n) \in M$, $q_n > 0$ *and let* (q_n) *be a monotonic non-decreasing sequence for* $n \geq 0$. *The necessary and sufficient condition that* $\sum a_n\lambda_n$ *is* $\left|\overline{N}, q_n\right|$ *−summable whenever*

$$\sum a_n = O(1)(N, p, q),$$

$$\sum_{n=0}^{\infty} \frac{q_n}{Q_n} |\lambda_n| < \infty,$$

$$\sum_{n=0}^{\infty} |\Delta\lambda_n| < \infty,$$

$$\sum_{n=0}^{\infty} \frac{Q_{n+1}}{q_{n+1}} \left|\Delta^2\lambda_n\right| < \infty,$$

is that

$$\sum_{n=1}^{\infty} \frac{q_n}{Q_n} |s_n| |\lambda_n| < \infty.$$

2. Lemmas

The following lemmas are needed for the proof of the theorem

Lemma 1 ([1]). *Let* $p_n \in M$. *Then*

(*i*) $c_0 > 0$, $c_n \leq 0$, $n = 1, 2, \ldots,$

(*ii*) $\sum\limits_{n=0}^{\infty} c_n x^n$ *is absolutely convergent for* $|x| \leq 1$, *and*

(*iii*) $\sum\limits_{n=0}^{\infty} c_n > 0$, *except when* $\sum\limits_{n=0}^{\infty} p_n = \infty$, *in which case*

(*iv*) $\sum\limits_{n=0}^{\infty} c_n = 0$.

Lemma 2 ([2]). *If*

$$T_n = \frac{1}{R_n} \sum_{v=0}^{n} p_{n-v} q_v s_v,$$

then

$$s_n = \frac{1}{q_n} \sum_{v=0}^{n} c_{n-v} R_v T_v.$$

Lemma 3 ([2]). *We have*

$$\sum_{\mu=0}^{n} c_{n-\mu}^{(1)} R_\mu = Q_n,$$

where c_n, R_n *and* Q_n *are defined as in Section 1.*

3. Result

Our aim is to present the following new general result

Theorem 3. *Let* $(p_n) \in M$. *Then necessary and sufficient condition for* $\sum a_n \lambda_n$ *to be summable* $|N, p, q|_k$, $k \geq 1$, *whenever*

$$\sum a_n = O(1)(N, p, q), \tag{9}$$

$$\sum_{n=v+1}^{\infty} n^{k-1} Q_{n-1}^{k-1} \left| \Delta \left(\frac{1}{R_{n-1}} \right) \right|^k |\Delta_v p_{n-v}|^k = O(1), \tag{10}$$

$$\sum_{v=0}^{\infty} |\lambda_v|^k \frac{Q_v^k}{q_v^{k-1}} = O(1), \tag{11}$$

$$\sum_{n=v+1}^{\infty} n^{k-1} P_{n-1}^{k-1} \left| \Delta \left(\frac{1}{R_{n-1}} \right) \right|^k p_{n-v-1} = O(1), \tag{12}$$

$$\sum_{v=0}^{\infty} |\Delta \lambda_v|^k Q_v^k = O(1), \tag{13}$$

$$\sum_{n=v+1}^{\infty} n^{k-1} Q_{n-1}^{k-1} \frac{1}{R_{n-1}^k} |\Delta_{v,n} p_{n-v-1}|^k = O(1), \tag{14}$$

$$\sum_{n=v+1}^{\infty} n^{k-1} Q_{n-1}^{k-1} \frac{1}{R_{n-1}^k} |\Delta_n p_{n-v-2}|^k = O(1/q_v^{k-1}), \tag{15}$$

is that

$$\sum_{n=1}^{\infty} n^{k-1} \left| \Delta \left(\frac{1}{R_{n-1}} \right) \right|^k |\lambda_n|^k Q_{n-1}^k = O(1). \tag{16}$$

Proof. We have

$$\begin{aligned} T_n - T_{n-1} &= \frac{1}{R_n}\sum_{v=0}^{n} p_{n-v} q_v s_v \lambda_v - \frac{1}{R_{n-1}}\sum_{v=0}^{n-1} p_{n-v-1} q_v s_v \lambda_v \\ &= \frac{1}{R_n}\sum_{v=0}^{n} p_{n-v} q_v s_v \lambda_v - \frac{1}{R_{n-1}}\sum_{v=0}^{n} p_{n-v-1} q_v s_v \lambda_v \\ &= \left(\frac{1}{R_n} - \frac{1}{R_{n-1}}\right)\sum_{v=0}^{n} p_{n-v} q_v s_v \lambda_v \\ &\quad + \frac{1}{R_{n-1}}\sum_{v=0}^{n} \left(p_{n-v-1} - p_{n-v}\right) q_v s_v \lambda_v \\ &= -\Delta\left(\frac{1}{R_{n-1}}\right)\sum_{v=0}^{n} p_{n-v} q_v s_v \lambda_v + \frac{1}{R_{n-1}}\sum_{v=0}^{n} \Delta_n p_{n-v-1} q_v s_v \lambda_v \\ &= T_{n1} + T_{n2}. \end{aligned}$$

By Minkowski's inequality, it will sufficient to show that

$$\sum_{n=1}^{\infty} n^{k-1} \left|T_{ni}\right|^k < \infty, \quad i = 1, 2.$$

Now by Lemma 2,

$$\begin{aligned} \sum_{v=0}^{n} p_{n-v} q_v s_v \lambda_v &= \sum_{v=0}^{n} p_{n-v}\lambda_v \sum_{\mu=0}^{v} c_{v-\mu} R_\mu T_\mu \\ &= \sum_{\mu=0}^{n} R_\mu T_\mu \sum_{v=\mu}^{n} p_{n-v} c_{v-\mu} \lambda_v \\ &= \sum_{\mu=0}^{n} R_\mu T_\mu \left(\sum_{v=\mu}^{n-1} c_{v-\mu}^{(1)} \Delta_v \left(p_{n-v}\lambda_v\right) + p_0 \lambda_n c_{n-\mu-1}^{(1)}\right) \\ &= \sum_{\mu=0}^{n} R_\mu T_\mu \left(\sum_{v=\mu}^{n-1} c_{v-\mu}^{(1)} \left(\Delta_v p_{n-v}\lambda_v + p_{n-v-1}\Delta\lambda_v\right) + p_0 \lambda_n c_{n-\mu-1}^{(1)}\right) \\ &= T_{n11} + T_{n12} + T_{n13}. \end{aligned}$$

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{k-1} \left|\Delta\left(\frac{1}{R_{n-1}}\right) T_{n11}\right|^k \\ &\qquad = \sum_{n=1}^{\infty} n^{k-1} \left|\Delta\left(\frac{1}{R_{n-1}}\right)\sum_{\mu=0}^{n} R_\mu T_\mu \sum_{v=\mu}^{n-1} c_{v-\mu}^{(1)} \Delta_v p_{n-v}\lambda_v\right|^k \end{aligned}$$

$$
\begin{aligned}
&\le \sum_{n=1}^{\infty} n^{k-1}\left|\Delta\left(\frac{1}{R_{n-1}}\right)\right|^k \left(\sum_{\mu=0}^{n} R_\mu T_\mu \sum_{v=\mu}^{n-1} c_{v-\mu}^{(1)}\left|\Delta_v p_{n-v}\right|\left|\lambda_v\right|\right)^k \\
&= O(1)\sum_{n=1}^{\infty} n^{k-1}\left|\Delta\left(\frac{1}{R_{n-1}}\right)\right|^k \left(\sum_{v=0}^{n-1}\left|\Delta_v p_{n-v}\right|\left|\lambda_v\right| \sum_{\mu=0}^{v} c_{v-\mu}^{(1)} R_\mu\right)^k \\
&= O(1)\sum_{n=1}^{\infty} n^{k-1}\left|\Delta\left(\frac{1}{R_{n-1}}\right)\right|^k \left(\sum_{v=0}^{n-1}\left|\Delta_v p_{n-v}\right|\left|\lambda_v\right| Q_v\right)^k \\
&= O(1)\sum_{n=1}^{\infty} n^{k-1}\left|\Delta\left(\frac{1}{R_{n-1}}\right)\right|^k \sum_{v=0}^{n-1}\left|\Delta_v p_{n-v}\right|^k\left|\lambda_v\right|^k \frac{Q_v^k}{q_v^{k-1}}\left(\sum_{v=0}^{n-1} q_v\right)^{k-1} \\
&= O(1)\sum_{n=1}^{\infty} n^{k-1} Q_{n-1}^{k-1}\left|\Delta\left(\frac{1}{R_{n-1}}\right)\right|^k \sum_{v=0}^{n-1}\left|\Delta_v p_{n-v}\right|^k\left|\lambda_v\right|^k \frac{Q_v^k}{q_v^{k-1}} \\
&= O(1)\sum_{v=0}^{\infty}\left|\lambda_v\right|^k \frac{Q_v^k}{q_v^{k-1}} \sum_{n=v+1}^{\infty} n^{k-1} Q_{n-1}^{k-1}\left|\Delta\left(\frac{1}{R_{n-1}}\right)\right|^k\left|\Delta_v p_{n-v}\right|^k \\
&= O(1).
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{k-1}\left|\Delta\left(\frac{1}{R_{n-1}}\right) T_{n12}\right|^k \\
&\qquad = \sum_{n=1}^{\infty} n^{k-1}\left|\Delta\left(\frac{1}{R_{n-1}}\right)\sum_{\mu=0}^{n} R_\mu T_\mu \sum_{v=\mu}^{n-1} c_{v-\mu}^{(1)} p_{n-v-1}\Delta\lambda_v\right|^k \\
&\le \sum_{n=1}^{\infty} n^{k-1}\left|\Delta\left(\frac{1}{R_{n-1}}\right)\right|^k \left(\sum_{\mu=0}^{n} R_\mu T_\mu \sum_{v=\mu}^{n-1} c_{v-\mu}^{(1)} p_{n-v-1}\left|\Delta\lambda_v\right|\right)^k \\
&= O(1)\sum_{n=1}^{\infty} n^{k-1}\left|\Delta\left(\frac{1}{R_{n-1}}\right)\right|^k \left(\sum_{v=0}^{n-1} p_{n-v-1}\left|\Delta\lambda_v\right| \sum_{\mu=0}^{v} c_{v-\mu}^{(1)} R_\mu\right)^k \\
&= O(1)\sum_{n=1}^{\infty} n^{k-1}\left|\Delta\left(\frac{1}{R_{n-1}}\right)\right|^k \left(\sum_{v=0}^{n-1} p_{n-v-1}\left|\Delta\lambda_v\right| Q_v\right)^k \\
&= O(1)\sum_{n=1}^{\infty} n^{k-1} P_{n-1}^{k-1}\left|\Delta\left(\frac{1}{R_{n-1}}\right)\right|^k \sum_{v=0}^{n-1} p_{n-v-1}\left|\Delta\lambda_v\right|^k Q_v^k \left(\sum_{v=0}^{n-1} p_{n-v-1}\right)^{k-1} \\
&= O(1)\sum_{n=1}^{\infty} n^{k-1} P_{n-1}^{k-1}\left|\Delta\left(\frac{1}{R_{n-1}}\right)\right|^k \sum_{v=0}^{n-1} p_{n-v-1}\left|\Delta\lambda_v\right|^k Q_v^k
\end{aligned}
$$

$$= O(1)\sum_{v=0}^{\infty}|\Delta\lambda_v|^k Q_v^k \sum_{n=v+1}^{\infty} n^{k-1}P_{n-1}^{k-1}\left|\Delta\left(\frac{1}{R_{n-1}}\right)\right|^k p_{n-v-1}$$
$$= O(1).$$

$$\sum_{n=1}^{\infty} n^{k-1}\left|\Delta\left(\frac{1}{R_{n-1}}\right)T_{n13}\right|^k$$
$$= \sum_{n=1}^{\infty} n^{k-1}\left|\Delta\left(\frac{1}{R_{n-1}}\right)\sum_{\mu=0}^{n} R_\mu T_\mu\ p_0\lambda_n c_{n-\mu-1}^{(1)}\right|^k$$
$$= O(1)\sum_{n=1}^{\infty} n^{k-1}\left|\Delta\left(\frac{1}{R_{n-1}}\right)\right|^k |\lambda_n|^k\left(\sum_{\mu=0}^{n} R_\mu\ c_{n-\mu-1}^{(1)}\right)^k$$
$$= O(1)\sum_{n=1}^{\infty} n^{k-1}\left|\Delta\left(\frac{1}{R_{n-1}}\right)\right|^k |\lambda_n|^k Q_{n-1}^k$$
$$= O(1).$$

Again by Lemma 2,

$$\sum_{v=0}^{n}\Delta_n p_{n-v-1}q_v s_v\lambda_v = \sum_{v=0}^{n-1}\Delta_n p_{n-v-1}\lambda_v\sum_{\mu=0}^{v} c_{v-\mu}R_\mu T_\mu$$
$$= \sum_{\mu=0}^{n} R_\mu T_\mu\sum_{v=\mu}^{n}\Delta_n p_{n-v-1}c_{v-\mu}\lambda_v$$
$$= \sum_{\mu=0}^{n} R_\mu T_\mu\left(\sum_{v=\mu}^{n-1} c_{v-\mu}^{(1)}\Delta_v\left(p_{n-v}\lambda_v\right) + p_0\lambda_n c_{n-\mu-1}^{(1)}\right)$$
$$= \sum_{\mu=0}^{n} R_\mu T_\mu\left(\sum_{v=\mu}^{n-1} c_{v-\mu}^{(1)}\left(\Delta_{v,n}p_{n-v-1}\lambda_v + p_{n-v-1}\Delta\lambda_v\right) + \Delta_n\left(p_{-1}\right)\lambda_n c_{n-\mu-1}^{(1)}\right)$$
$$= \sum_{\mu=0}^{n} R_\mu T_\mu\left(\sum_{v=\mu}^{n-1} c_{v-\mu}^{(1)}\left(\Delta_{v,n}p_{n-v-1}\lambda_v + p_{n-v-1}\Delta\lambda_v\right)\right)$$
$$= T_{n21} + T_{n22}\ .$$

$$\sum_{n=1}^{\infty} n^{k-1}\left|\frac{1}{R_{n-1}}T_{n21}\right|^k$$
$$= \sum_{n=1}^{\infty} n^{k-1}\left|\frac{1}{R_{n-1}}\sum_{\mu=0}^{n} R_\mu T_\mu\ \sum_{v=\mu}^{n-1} c_{v-\mu}^{(1)}\Delta_{v,n}p_{n-v-1}\lambda_v\right|^k$$

$$
\begin{aligned}
&\leq \sum_{n=1}^{\infty} n^{k-1} \frac{1}{R_{n-1}^{k}} \left(\sum_{\mu=0}^{n} R_{\mu} T_{\mu} \sum_{v=\mu}^{n-1} c_{v-\mu}^{(1)} \left| \Delta_{v,n} p_{n-v-1} \right| \left| \lambda_{v} \right| \right)^{k} \\
&= O(1) \sum_{n=1}^{\infty} n^{k-1} \frac{1}{R_{n-1}^{k}} \left(\sum_{v=0}^{n-1} \left| \Delta_{v,n} p_{n-v-1} \right| \left| \lambda_{v} \right| \sum_{\mu=0}^{v} c_{v-\mu}^{(1)} R_{\mu} \right)^{k} \\
&= O(1) \sum_{n=1}^{\infty} n^{k-1} \frac{1}{R_{n-1}^{k}} \left(\sum_{v=0}^{n-1} \left| \Delta_{v,n} p_{n-v-1} \right| \left| \lambda_{v} \right| Q_{v} \right)^{k} \\
&= O(1) \sum_{n=1}^{\infty} n^{k-1} \frac{1}{R_{n-1}^{k}} \sum_{v=0}^{n-1} \left| \Delta_{v,n} p_{n-v-1} \right|^{k} \left| \lambda_{v} \right|^{k} \frac{Q_{v}^{k}}{q_{v}^{k-1}} \left(\sum_{v=0}^{n-1} q_{v} \right)^{k-1} \\
&= O(1) \sum_{n=1}^{\infty} n^{k-1} Q_{n-1}^{k-1} \frac{1}{R_{n-1}^{k}} \sum_{v=0}^{n-1} \left| \Delta_{v,n} p_{n-v-1} \right|^{k} \left| \lambda_{v} \right|^{k} \frac{Q_{v}^{k}}{q_{v}^{k-1}} \\
&= O(1) \sum_{v=0}^{\infty} \left| \lambda_{v} \right|^{k} \frac{Q_{v}^{k}}{q_{v}^{k-1}} \sum_{n=v+1}^{\infty} n^{k-1} Q_{n-1}^{k-1} \frac{1}{R_{n-1}^{k}} \left| \Delta_{v,n} p_{n-v-1} \right|^{k} \\
&= O(1).
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{k-1} \left| \frac{1}{R_{n-1}} T_{n22} \right|^{k} \\
&= \sum_{n=1}^{\infty} n^{k-1} \left| \frac{1}{R_{n-1}} \sum_{\mu=0}^{n} R_{\mu} T_{\mu} \sum_{v=\mu}^{n-1} c_{v-\mu}^{(1)} \Delta_{n} p_{n-v-2} \Delta \lambda_{v} \right|^{k} \\
&\leq \sum_{n=1}^{\infty} n^{k-1} \frac{1}{R_{n-1}^{k}} \left(\sum_{\mu=0}^{n} R_{\mu} T_{\mu} \sum_{v=\mu}^{n-1} c_{v-\mu}^{(1)} \left| \Delta_{n} p_{n-v-2} \right| \Delta \left| \lambda_{v} \right| \right)^{k} \\
&= O(1) \sum_{n=1}^{\infty} n^{k-1} \frac{1}{R_{n-1}^{k}} \left(\sum_{v=0}^{n-1} \left| \Delta_{n} p_{n-v-2} \right| \left| \Delta \lambda_{v} \right| \sum_{\mu=0}^{v} c_{v-\mu}^{(1)} R_{\mu} \right)^{k} \\
&= O(1) \sum_{n=1}^{\infty} n^{k-1} \frac{1}{R_{n-1}^{k}} \left(\sum_{v=0}^{n-1} \left| \Delta_{n} p_{n-v-2} \right| \left| \Delta \lambda_{v} \right| Q_{v} \right)^{k} \\
&= O(1) \sum_{n=1}^{\infty} n^{k-1} \frac{1}{R_{n-1}^{k}} \sum_{v=0}^{n-1} \left| \Delta_{n} p_{n-v-2} \right|^{k} \left| \Delta \lambda_{v} \right|^{k} \frac{Q_{v}^{k}}{q_{v}^{k-1}} \left(\sum_{v=0}^{n-1} q_{v} \right)^{k-1} \\
&= O(1) \sum_{n=1}^{\infty} n^{k-1} Q_{n-1}^{k-1} \frac{1}{R_{n-1}^{k}} \sum_{v=0}^{n-1} \left| \Delta_{n} p_{n-v-2} \right|^{k} \left| \Delta \lambda_{v} \right|^{k} \frac{Q_{v}^{k}}{q_{v}^{k-1}}
\end{aligned}
$$

$$= O(1) \sum_{v=0}^{\infty} |\Delta\lambda_v|^k \frac{Q_v^k}{q_v^{k-1}} \sum_{n=v+1}^{\infty} n^{k-1} Q_{n-1}^{k-1} \frac{1}{R_{n-1}^k} |\Delta_n p_{n-v-2}|^k$$
$$= O(1).$$

The proof is over. ■

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Received on 25.10.2010 and, in revised form, on 14.05.2011.