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FIXED POINT THEOREMS IN COMPLETE G-METRIC SPACE

Abstract. In this paper, we prove some fixed point theorems in complete G-metric space for self mapping satisfying various contractive conditions. We also discuss that these mappings are G-continuous on such a fixed point.

Key words: G-metric spaces, fixed point.

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1. Introduction

In 1984, Dhage [1] introduced the concept of D-metric space. The situation for a D-metric space is quite different from 2-metric spaces. Geometrically, a D-metric $D(x, y, z)$ represent the perimeter of the triangle with vertices x, y and z in R^2. Recently, Mustafa and Sims [2] showed that most of the results concerning Dhage's D-metric spaces are invalid. Therefore, they introduced a improved version of the generalized metric space structure, which they called it as G-metric spaces, one can refer to the papers [3]-[6].

Now, we give preliminaries and basic definitions which are used throughout the paper.

In 2004, Mustafa and Sims [3] introduced the concept of G-metric spaces as follows:

Definition 1 ([3]). *Let X be a nonempty set, and let, $G : X \times X \times X \to R^+$ be a function satisfying the following axioms:*

(G_1) $G(x, y, z) = 0$ *if* $x = y = z$,

(G_2) $0 < G(x, x, y)$, *for all* $x, y \in X$ *with* $x \neq y$,

(G_3) $G(x, x, y) \leq G(x, y, z)$, *for all* $x, y, z \in X$ *with* $z \neq y$,

(G_4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (*symmetry in all three variables*),

(G_5) $G(x, y, z) = G(x, a, a) + G(a, y, z)$ *for all* $x, y, z, a \in X$, (*rectangle inequality*) *then the function G is called a generalized metric, or, more specifically a G-metric on X and the pair (X, G) is called a G-metric space.*

Definition 2 ([5]). *Let (X, G) be a G-metric space and let $\{x_n\}$ be a sequence of points in X, a point x in X is said to be the limit of the sequence $\{x_n\}$ if $G(x, x_n, x_m) = 0$, and one says that sequence $\{x_n\}$ is G-convergent to x. Thus, if $x_n \to x$ or $x_n = x$ as $n \to \infty$, in a G-metric space (X, G), then for each $\varepsilon > 0$, there exists a positive integer N such that $G(x, x_n, x_m) < \varepsilon$ for all $m, n \in N$.*

Now, we state some results from the papers ([2]-[6]) which are helpful for proving our main results.

Proposition 1 ([5]). *Let (X, G) be a G-metric space. Then the following are equivalent:*
(i) $\{x_n\}$ *is G-convergent to x,*
(ii) $G(x_n, x_n, x) \to 0$ *as* $n \to \infty$,
(iii) $G(x_n, x, x) \to 0$ *as* $n \to \infty$,
(iv) $G(x_m, x_n, x) \to 0$ *as* $m, n \to \infty$.

Definition 3 ([4]). *Let (X, G) be a G-metric space. A sequence $\{x_n\}$ is called G-Cauchy if, for each $\varepsilon > 0$, there exists a positive integer N such that $G(x_n, x_m, x_l) < \varepsilon$, for all $n, m, l \in N$, i.e., if $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.*

Definition 4 ([4]). *If (X, G) and (X', G') be two G-metric space and let $f : (X, G) \to (X', G')$ be a function, then f is said to be G-continuous at a point $x_0 \in X$ if given $\varepsilon > 0$, there exists $\delta > 0$ such that for $x, y \in X$ and $G(x_0, x, y) < \delta$ implies $G'(f(x_0), f(x), f(y)) < \varepsilon$. A function f is G-continuous at X if and only if it is G-continuous at all $x_0 \in X$ or function f is said to be G-continuous at a point $x_0 \in X$ if and only if it is G-sequentially continuous at x_0, that is, whenever $\{x_n\}$ is G-convergent to x_0, $\{f(x_n)\}$ is G-convergent to $f(x_0)$.*

Proposition 2 ([3]). *Let (X, G) be a G-metric space. Then the function $G(x, y, z)$ is jointly continuous in all three of its variables.*

Definition 5 ([5]). *A G-metric space (X, G) is called a symmetric G-metric space if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.*

Proposition 3 ([5]). *Every G-metric space (X, G) will defines a metric space (X, d_G) by*

(i) $d_G(x, y) = G(x, y, y) + G(y, x, x)$ *for all* $x, y \in X$.

If (X, G) is a symmetric G-metric space, then

(ii) $d_G(x, y) = 2G(x, y, y)$ *for all* $x, y \in X$.

However, if (X, G) is not symmetric, then it follows from the G-metric properties that

(iii) $\frac{3}{2}G(x, y, y) \leq d_G(x, y) \leq 3G(x, y, y)$ *for all* $x, y \in X$.

Definition 6 ([4]). *A G-metric space (X, G) is said to be G-complete if every G-Cauchy sequence in (X, G) is G-convergent in X.*

Proposition 4 ([4]). *A G-metric space (X, G) is said to be G-complete if and only if (X, d_G) is a complete metric space.*

Proposition 5 ([3]). *Let (X, G) be a G-metric space. Then, for any x, y, z, a in X, it follows that:*

(i) *if* $G(x, y, z) = 0$, *then* $x = y = z$,
(ii) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$,
(iii) $G(x, y, y) \leq 2G(y, x, x)$,
(iv) $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$,
(v) $G(x, y, z) \leq \frac{2}{3}(G(x, y, a) + G(x, a, z) + G(a, y, z))$,
(vi) $G(x, y, z) \leq G(x, a, a) + G(y, a, a) + G(z, a, a)$.

2. Main result

We need the following Lemma to prove our main results:

Lemma 1. *Let (X, G) be a G-metric space and T be a self map on X satisfying*

$$G(Tx, Ty, Tz) \leq qG(x, y, z) \tag{1}$$

for all x, y, zX, where $0 \leq q < 1$, and $x_n = Tx_{n-1} = T(Tx_{n-2}) = \cdots = T^n(x_0)$, for some $x_0 \in X$, then $\{x_n\}$ is a G-Cauchy sequence in X.

Proof. Given that for some $x_0 \in X$; $T^n(x_0) = x_n$, $n = 0, 1, 2, \ldots$. From (1), we have

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &= G(Tx_{n-1}, Tx_n, Tx_n) \\ &\leq qG(x_{n-1}, x_n, x_n) \leq \ldots \leq q^n G(x_0, x_1, x_1). \end{aligned}$$

Moreover, for all $n, m \in N$, $n < m$, by G_5, ones obtain

$$\begin{aligned} G(x_n, x_m, x_m) &\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) \\ &\quad + \ldots + G(x_{m-1}, x_m, x_m) \\ &\leq (q^n + q^{n+1} + \ldots + q^{m-1})G(x_0, x_1, x_1) \\ &= \frac{q^n}{1-q} G(x_0, x_1, x_1)\,. \end{aligned}$$

Proceeding limit as $n, m \to \infty$, we have

$$G(x_n, x_m, x_m) = 0.$$

Thus, $\{x_n\}$ is a G-Cauchy sequence in X. ■

Theorem 1. *Let (X, G) be a complete G-metric space and $T : X \to X$ be the mapping satisfying the following :*

$$(2)\quad \begin{aligned} &G(T(x), T(y), T(z)) \\ &= k \max \left\{ \begin{array}{l} G(x, T(x), T(x)), G(x, T(y), T(y)), G(x, T(z), T(z)), \\ G(y, T(y), T(y)), G(y, T(x), T(x)), G(y, T(z), T(z)), \\ G(z, T(z), T(z)), G(z, T(x), T(x)), G(z, T(y), T(y)) \end{array} \right\} \end{aligned}$$

for all $x, y, z \in X$, where $0 \leq k < \frac{1}{2}$, then T has a unique fixed point and T is G-continuous at the fixed point.

Proof. Suppose T satisfy condition (2) and $x_0 \in X$ be an arbitrary point

Step 1. We inductively construct the sequence $\{x_n\}$ of point in X as:

$$\begin{aligned} x_1 &= T(x_0) \\ x_2 &= T(x_1) = T(T(x_0)) = T^2(x_0) \\ x_3 &= T(x_2) = T(T^2(x_0)) = T^3(x_0) \\ &\vdots \\ x_n &= T(x_{n-1}) = T(T^{n-1}(x_0)) = T^n(x_0) \end{aligned}$$

Clearly $\{x_n\}$ is a sequence of images of x_0, under repeated application of T.

Step 2. $\{x_n\}$ is a G-Cauchy sequence in X. Assume $x_n \neq x_{n+1}$ for all n. Since if there exist an n such that $x_n = x_{n+1}$ then, $T^n(x_0) = T(T^n(x_0))$, yields $T^n(x_0)$ is a fixed point.

Therefore, by using (2), we have

$$(3)\quad \begin{aligned} &G(x_n, x_{n+1}, x_{n+1}) \\ &\leq k \max \left\{ \begin{array}{l} G(x_{n-1}, x_n, x_n), G(x_{n-1}, x_{n+1}, x_{n+1}), G(x_{n-1}, x_{n+1}, x_{n+1}), \\ G(x_n, x_{n+1}, x_{n+1}), G(x_n, x_n, x_n), G(x_n, x_{n+1}, x_{n+1}), \\ G(x_n, x_{n+1}, x_{n+1}), G(x_n, x_n, x_n), G(x_n, x_{n+1}, x_{n+1}) \end{array} \right\} \\ &= k \max\{G(x_{n-1}, x_n, x_n) G, (x_{n-1}, x_{n+1}, x_{n+1}), G(x_n, x_{n+1}, x_{n+1})\}. \end{aligned}$$

Case 1. If

$$\begin{aligned} &\max\{G(x_{n-1}, x_n, x_n), G(x_{n-1}, x_{n+1}, x_{n+1}), G(x_n, x_{n+1}, x_{n+1})\} \\ &\quad = G(x_{n-1}, x_n, x_n) \end{aligned}$$

then, using (3), we get

$$G(x_n, x_{n+1}, x_{n+1}) \leq kG(x_{n-1}, x_n, x_n),$$

thus by Lemma 1, we have $\{x_n\}$ is a G-Cauchy sequence in X.

Case 2. If

$$\begin{aligned}&\max\{G(x_{n-1}, x_n, x_n), G(x_{n-1}, x_{n+1}, x_{n+1}), G(x_n, x_{n+1}, x_{n+1})\}\\ &\quad = G(x_{n-1}, x_{n+1}, x_{n+1})\end{aligned}$$

then, from (3) and using G_5 of Definition 1.1, ones obtain

$$\begin{aligned}G(x_n, x_{n+1}, x_{n+1}) &\leq kG(x_{n-1}, x_{n+1}, x_{n+1})\\ &\leq k\{G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1})\},\end{aligned}$$

this implies that

$$\begin{aligned}G(x_n, x_{n+1}, x_{n+1}) &\leq \frac{k}{1-k}G(x_{n-1}, x_n, x_n)\\ G(x_n, x_{n+1}, x_{n+1}) &\leq qG(x_{n-1}, x_n, x_n),\end{aligned}$$

where $q = \frac{k}{1-k}$, $q < 1$ as $0 \leq k < \frac{1}{2}$.

Thus again by Lemma 1, we have $\{x_n\}$ is a G-Cauchy sequence in X.

Case 3. Finally, if

$$\begin{aligned}&\max\{G(x_{n-1}, x_n, x_n), G(x_{n-1}, x_{n+1}, x_{n+1}), G(x_n, x_{n+1}, x_{n+1})\\ &\quad = G(x_n, x_{n+1}, x_{n+1})\\ &G(x_n, x_{n+1}, x_{n+1}) \leq kG(x_n, x_{n+1}, x_{n+1}),\end{aligned}$$

which is a contradiction, as $k < \frac{1}{2}$.

Hence in all cases the sequence $\{x_n\}$ is a G-Cauchy sequence.

Step 3. Since (X, G) is a complete G-metric space, by definition, there exists $u \in X$ such that $x_n \to u$.

Step 4. u is a fixed point of T.

Suppose, if possible, that $T(u) \neq u$, using (3), we have

$$\begin{aligned}(4)\quad &G(x_n, T(u), T(u))\\ &\leq k\max\begin{Bmatrix}G(x_{n-1}, x_n, x_n), G(x_{n-1}, T(u), T(u)), G(x_{n-1}, T(u), T(u)),\\ G(u, T(u), T(u)), G(u, x_n, x_n), G(u, T(u), T(u)),\\ G(u, T(u), T(u)), G(u, x_n, x_n), G(u, T(u), (u))\end{Bmatrix}\\ &= k\max\begin{Bmatrix}G(x_{n-1}, x_n, x_n), G(x_{n-1}, T(u), T(u)),\\ G(u, x_n, x_n), G(u, T(u), T(u))\end{Bmatrix}.\end{aligned}$$

Taking the limit as $n \to \infty$, and using the fact that function G is continuous on its variable, we obtain

$$G(u, T(u), T(u)) \le kG(u, T(u), T(u)),$$

which arises a contradiction, since, $0 \le k < \frac{1}{2}$.

Hence, $T(u) = u$, i.e., u is a fixed point of T.

Step 5. Uniqueness of fixed point u of T.

Suppose that, v ($\neq u$) is another fixed point of T, such that $T(v) = v$, then from (2), we have

$$\begin{aligned} G(u, v, v) &\le k \max \begin{Bmatrix} G(u, u, u), & G(u, v, v), & G(u, v, v), \\ G(v, v, v), & G(v, u, u), & G(v, v, v), \\ G(v, v, v), & G(v, u, u), & G(v, v, v) \end{Bmatrix} \\ &= k \max\{G(u, v, v), G(v, u, u)\} \end{aligned}$$

which reduces to,

$$G(u, v, v) \le kG(v, u, u)\,. \tag{5}$$

Again by same argument we will find

$$G(v, u, u) \le kG(u, v, v) \tag{6}$$

which, by repeated use of (5) and (6), implies

$$G(v, u, u) \le k^2 G(v, u, u) \le \cdots \le k^n G(v, u, u)\,.$$

Proceeding limit as $n \to \infty$, we have $u = v$, i.e., u is a unique fixed point of T.

Step 6. T is G-continuous at the fixed point u.

Let $\{y_n\}$ be any sequence in X, such that $\lim_{n\to\infty} y_n = u$, then, by (2), we obtain

$$\begin{aligned} &G(T(y_n), T(u), T(y_n)) \\ &\le k \max \begin{Bmatrix} G(y_n, T(y_n), T(y_n)), G(y_n, T(u), T(u)), G(y_n, T(y_n), T(y_n)), \\ G(u, T(u), T(u)), G(u, T(y_n), T(y_n)), G(u, T(y_n), T(y_n)), \\ G(y_n, T(y_n), T(y_n)), G(y_n, T(y_n), T(y_n)), G(y_n, T(u), T(u)) \end{Bmatrix} \\ &= k \max \begin{Bmatrix} G(y_n, T(y_n), T(y_n)), G(y_n, T(u), T(u)), \\ G(u, T(u), T(u)), G(u, T(y_n), T(y_n)) \end{Bmatrix}. \end{aligned}$$

This deduces to

$$\begin{aligned} &G(T(y_n), u, T(y_n)) \\ &\le k \max\{G(y_n, T(y_n), T(y_n)), G(y_n, u, u), G(u, T(y_n), T(y_n))\} \\ &= k \max\{G(y_n, T(y_n), T(y_n)), G(y_n, u, u)\}\,. \end{aligned} \tag{7}$$

Proceeding the limit as $n \to \infty$, we have, $G(u, T(y_n), T(y_n)) \to 0$, and so by definition of G-continuity of G-metric space (X, G) we have $T(y_n) \to u = T(u)$, this implies that T is G-continuous at u.

Hence completes the theorem. ■

Remark 1. If the G-metric space is bounded, i.e., for some $m > 0$, we have $G(x, y, z) \leq m$, for all $x, y, z \in X$, then an argument similar to that used above establishes the result for $0 \leq k < 1$.

Corollary 1. *Let (X, G) be a complete G-metric space and let $T : X \to X$ be the mapping which satisfy the following condition for $m \in N$ and for all $x, y, z \in X$:*

$$(8)\quad G(T^m(x), T^m(y), T^m(z)) \leq k \times \max\left\{\begin{matrix} G(x, T^m(x), T^m(x)), G(x, T^m(y), T^m(y)), G(x, T^m(z), T^m(z)), \\ G(y, T^m(y), T^m(y)), G(y, T^m(x), T^m(x)), G(y, T^m(z), T^m(z)), \\ G(z, T^m(z), T^m(z)), G(z, T^m(x), T^m(x)), G(z, T^m(y), T^m(y)) \end{matrix}\right\}$$

where $0 \leq k < \frac{1}{2}$, then T has unique fixed point (say) u and T^m is G-continuous at u.

Proof. Using Theorem 1, ones obtain, T^m has a unique fixed point (say) u, that is, $T^m(u) = u$ and T^m is G-continuous at u. But $T(u) = T(T^m(u)) = T^{m+1}(u) = T^m(T(u))$, so $T(u)$ is another fixed point of T^m by uniqueness $T(u) = u$, i.e., u is a unique fixed point of T. ■

Theorem 2. *Let (X, G) be complete G-metric space and $T : X \to X$ be the mapping satisfying the following condition:*

$$(9)\quad G(T(x), T(y), T(z)) \leq k \max\left\{\begin{matrix} G(x, T(x), T(x)) + G(y, T(y), T(y)) + G(z, T(z), T(z)), \\ G(x, T(y), T(y)) + G(y, T(x), T(x)) + G(z, T(y), T(y)), \\ G(x, T(z), T(z)) + G(y, T(z), T(z)) + G(z, T(x), T(x)) \end{matrix}\right\}$$

for all $x, y, z \in X$, where $0 \leq k < \frac{1}{4}$, then T has a unique fixed point say (u) and T is a G-continuous at u.

Proof. Suppose that T satisfy condition (9) and let x_0 be any arbitrary point of X.

Step 1. We inductively construct the sequence $\{x_n\}$ of point in X as:

$$x_1 = T(x_0)$$
$$x_2 = T(x_1) = T(T(x_0)) = T^2(x_0)$$

$$\begin{aligned} x_3 &= T(x_2) = T(T^2(x_0)) = T^3(x_0) \\ &\vdots \\ x_n &= T(x_{n-1}) = T(T^{n-1}(x_0)) = T^n(x_0). \end{aligned}$$

Clearly $\{x_n\}$ is a sequence of images of x_0, under repeated application of T.

Step 2. $\{x_n\}$ is a Cauchy sequence in X. Assume $x_n \neq x_{n+1}$ for all n. Since if there exist an n such that $x_n = x_{n+1}$ then, $T^n(x_0) = T(T^n(x_0))$, yields $T^n(\mathrm{x}_0)$ is a fixed point.

By (9), we have

$$\begin{aligned} (10)\ & G(x_n, x_{n+1}, x_{n+1}) \\ &\leq k \max \begin{Bmatrix} G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1}) + G(x_n, x_{n+1}, x_{n+1}), \\ G(x_{n-1}, x_{n+1}, x_{n+1}) + G(x_n, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1}), \\ G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1}) + G(x_n, x_n, x_n) \end{Bmatrix} \\ &= k \max \begin{Bmatrix} G(x_{n-1}, x_n, x_n) + 2G(x_n, x_{n+1}, x_{n+1}), \\ G(x_{n-1}, x_{n+1}, x_{n+1}) + G(x_n, x_{n+1}, x_{n+1}), \\ G(x_{n-1}, x_{n+1}, x_{n+1}) + G(x_n, x_{n+1}, x_{n+1}) \end{Bmatrix} \\ &= k \max \begin{Bmatrix} G(x_{n-1}, x_n, x_n) + 2G(x_n, x_{n+1}, x_{n+1}), \\ G(x_{n-1}, x_{n+1}, x_{n+1}) + G(x_n, x_{n+1}, x_{n+1}) \end{Bmatrix}. \end{aligned}$$

Case 1. If

$$\begin{aligned} \max &\begin{Bmatrix} G(x_{n-1}, x_n, x_n) + 2G(x_n, x_{n+1}, x_{n+1}), \\ G(x_{n-1}, x_{n+1}, x_{n+1}) + G(x_n, x_{n+1}, x_{n+1}) \end{Bmatrix} \\ &= G(x_{n-1}, x_n, x_n) + 2G(x_n, x_{n+1}, x_{n+1}). \end{aligned}$$

Then (10) becomes,

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &\leq k\{G(x_{n-1}, x_n, x_n) + 2G(x_n, x_{n+1}, x_{n+1})\} \\ G(x_n, x_{n+1}, x_{n+1}) &\leq \left(\frac{k}{1-2k}\right) G(x_{n-1}, x_n, x_n), \end{aligned}$$

which can be written as

$$G(x_n, x_{n+1}, x_{n+1}) \leq qG(x_{n-1}, x_n, x_n),$$

where $q = \left(\frac{k}{1-2k}\right)$, and $q < 1$, as $0 \leq k < \frac{1}{4}$.

Then by Lemma 1, we have $\{x_n\}$ is a G-Cauchy sequence in X.

Case 2. If

$$\begin{aligned} \max &\begin{Bmatrix} G(x_{n-1}, x_n, x_n) + 2G(x_n, x_{n+1}, x_{n+1}), \\ G(x_{n-1}, x_{n+1}, x_{n+1}) + G(x_n, x_{n+1}, x_{n+1}) \end{Bmatrix} \\ &= G(x_{n-1}, x_{n+1}, x_{n+1}) + G(x_n, x_{n+1}, x_{n+1}). \end{aligned}$$

Then (10) reduces to

(11) $$G(x_n, x_{n+1}, x_{n+1}) \leq k\{G(x_{n-1}, x_{n+1}, x_{n+1}) + G(x_n, x_{n+1}, x_{n+1})\}.$$

Using G_5 of Definition 1, we have

(12) $$G(x_{n-1}, x_{n+1}, x_{n+1}) \leq G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1}).$$

Now (11) becomes, $G(x_n, x_{n+1}, x_{n+1}) \leq \left(\frac{k}{1-2k}\right) G(x_{n-1}, x_n, x_n)$, which can be written as

$$G(x_n, x_{n+1}, x_{n+1}) \leq qG(x_{n-1}, x_n, x_n),$$

where $q = \left(\frac{k}{1-2k}\right)$, and $q < 1$, as $0 \leq k < \frac{1}{4}$.

Then again by using Lemma 1, we obtain $\{x_n\}$ is a G-Cauchy sequence in X.

Hence in both cases $\{x_n\}$ is a G-Cauchy sequence in X.

Step 3. Since (X, G) is a complete G-metric space, by definition, there exists a point (say) $u \in X$ such that $x_n \to u$.

Step 4. u is fixed point of T, suppose, if possible, that $T(u) \neq u$, using (9), ones obtain

$$\begin{aligned}
&G(x_n, T(u), T(u)) \\
&\leq k \max \begin{Bmatrix} G(x_{n-1}, x_n, x_n) + G(u, T(u), T(u)) + G(u, T(u), T(u)), \\ G(x_{n-1}, T(u), T(u)) + G(u, x_n, x_n) + G(u, T(u), T(u)), \\ G(x_{n-1}, T(u), T(u)) + G(u, T(u), T(u)) + G(u, x_n, x_n) \end{Bmatrix} \\
&= k \max \begin{Bmatrix} G(x_{n-1}, x_n, x_n) + 2G(u, T(u), T(u)), \\ G(x_{n-1}, T(u), T(u)) + G(u, x_n, x_n) + G(u, T(u), T(u)) \end{Bmatrix}.
\end{aligned}$$

Taking the limit as $n \to \infty$, and using the fact that function G is continuous in its variable, we get

$$G(u, T(u), T(u)) \leq k \max \begin{Bmatrix} 2G(u, T(u), T(u)), \\ 2G(u, T(u), T(u)) \end{Bmatrix} \leq 2kG(u, T(u), T(u))$$

which is a contradiction, since $0 \leq k < \frac{1}{4}$. Hence $u = T(u)$, i.e., u is a fixed point of T.

Step 5. Uniqueness of fixed point u of T.

Suppose that $v \neq u$, such that $T(v) = v$, then by (9), ones obtain

$$\begin{aligned}
G(u, v, v) &= G(T(u), T(v), T(v)) \\
&\leq k \max \begin{Bmatrix} G(u, u, u) + G(v, v, v) + G(v, v, v)), \\ G(u, v, v) + G(v, u, u) + G(v, v, v), \\ G(u, v, v) + G(v, v, v) + G(v, u, u) \end{Bmatrix} \\
&= k \max\{G(u, v, v) + G(v, u, u)\}.
\end{aligned}$$

That is,

$$G(u,v,v) \leq k\{G(u,v,v) + G(v,u,u)\}\,.$$

This implies that

$$G(u,v,v) \leq \frac{k}{1-k} G(v,u,u)\,.$$

Now, by the same argument, we have

$$G(v,u,u) \leq \frac{k}{1-k} G(u,v,v)\,.$$

Therefore, we get

$$G(u,v,v) \leq \left(\frac{k}{1-k}\right)^2 G(v,u,u),$$

but $0 \leq \frac{k}{1-k} < 1$.

Hence, we reach at the contradiction, so $u = v$, that is, the fixed point is unique.

Step 6. Finally, to prove T is G-continuous at fixed point u. For this, let us suppose that $\{y_n\}$ be a sequence in X such that $y_n \to u$ in (X, G), now using (9), we obtain

$$\begin{aligned}
(13)\quad & G(T(y_n), T(u), T(u)) \\
& \leq k \max \begin{Bmatrix} G(y_n, T(y_n), T(y_n)) + G(u, T(u), T(u)) + G(u, T(u), T(u)), \\ G(y_n, T(u), T(u)) + G(u, T(y_n), T(y_n)) + G(u, T(u), T(u)), \\ G(y_n, T(u), T(u)) + G(u, T(u), T(u)) + G(u, T(y_n), T(y_n)) \end{Bmatrix} \\
& = k \max \begin{Bmatrix} G(y_n, T(y_n), T(y_n)) + 2G(u, T(u), T(u)), \\ G(y_n, T(u), T(u)) + G(u, T(u), T(u)) + G(u, T(y_n), T(y_n)) \end{Bmatrix}.
\end{aligned}$$

Case 1. If

$$\begin{aligned}
\max & \begin{Bmatrix} G(y_n, T(y_n), T(y_n)) + 2G(u, T(u), T(u)), \\ G(y_n, T(u), T(u)) + G(u, T(u), T(u)) + G(u, T(y_n), T(y_n)) \end{Bmatrix} \\
& = \{G(y_n, T(y_n), T(y_n)) + 2G(u, T(u), T(u)\}\,.
\end{aligned}$$

Then (13) becomes,

$$G(T(y_n), T(u), T(u)) \leq k\{G(y_n, T(y_n), T(y_n)) + 2G(u, T(u), T(u))\}\,.$$

Letting limit $n \to \infty$, and using $T(u) = u$, and $y_n \to u$, we get

$$\begin{aligned}
(14)\qquad G(T(y_n), u, u) &\leq k\{G(u, T(y_n), T(y_n)) + 2G(u, u, u)\} \\
&= kG(u, T(y_n), T(y_n))\,.
\end{aligned}$$

By (*iii*) of Proposition 5, $G(u, T(y_n), T(y_n)) \leq 2G(T(y_n), u, u)$.

This implies (14) reduce to, $G(T(y_n), u, u) = 0$. But, $G(T(y_n), u, u) \geq 0$, hence, $G(T(y_n), u, u) = 0$. So, $T(y_n) \to u = T(u)$, which shows that T is G-continuous at the fixed point u.

Case 2. If

$$\max\begin{Bmatrix} G(y_n, T(y_n), T(y_n)) + 2G(u, T(u), T(u)), \\ G(y_n, T(u), T(u)) + G(u, T(u), T(u)) + G(u, T(y_n), T(y_n)) \end{Bmatrix}$$
$$= G(y_n, T(u), T(u)) + G(u, T(u), T(u)) + G(u, T(y_n), T(y_n)) \,.$$

Then (13) becomes,

$$G(T(y_n), T(u), T(u))$$
$$\leq k\{G(y_n, T(u), T(u)) + G(u, T(u), T(u)) + G(u, T(y_n), T(y_n))\} \,.$$

Letting limit as $n \to \infty$, and using $T(u) = u$, we have

$$\text{(15)} \quad \begin{aligned} G(T(y_n), u, u) &\leq k\{G(u, u, u) + G(u, u, u) + G(u, T(y_n), T(y_n))\} \\ &= kG(u, T(y_n), T(y_n)) \,. \end{aligned}$$

By (*iii*) of Proposition 5, $G(u, T(y_n), T(y_n)) \leq 2G(T(y_n), u, u)$, with this (15), reduces $G(T(y_n), u, u) \leq 0$, but $G(T(y_n), u, u) \geq 0$, hence, $G(T(y_n), u, u) = 0$.

So, $T(y_n) \to u = T(u)$, which shows that T is G-continuous at the fixed point u. Therefore in both cases T is G-continuous at point u. Hence completes the theorem. ■

Corollary 2. *Let (X, G) be a complete G-metric space and let $T : X \to X$ be the mapping which satisfy the following condition for $m \in N$ and for all $x, y, z \in X$:*

$$G(T^m(x), T^m(y), T^m(z))$$
$$\leq k \max\begin{Bmatrix} G(x, T^m(x), T^m(x)) + G(y, T^m(y), T^m(y)) + G(z, T^m(z), T^m(z)), \\ G(x, T^m(y), T^m(y)) + G(y, T^m(x), T^m(x)) + G(z, T^m(y), T^m(y)), \\ G(x, T^m(z), T^m(z)) + G(y, T^m(z), T^m(z)) + G(z, T^m(x), T^m(x)) \end{Bmatrix}.$$

Where $0 \leq k < \frac{1}{4}$, then T has unique fixed point (say) u and T^m is G-continuous at u.

Proof. Using Theorem 2, ones obtain, T^m has a unique fixed point (say) u, that is, $T^m(u) = u$ and T^m is G-continuous. But $T(u) = T(T^m(u)) = T^{m+1}(u) = T^m(T(u))$, so $T(u)$ is another fixed point of T^m by uniqueness $T(u) = u$, i.e., u is a fixed point of T. ■

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