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OPERATION–b-OPEN SETS IN TOPOLOGICAL SPACES

Abstract. In this paper we have introduced the concept of γ-b-open sets and studied some of their properties.

Key words: topological spaces, b-open set, γ-open set, γ-b-open set.

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1. Introduction

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real analysis concerns the various modified forms of continuity, separation axioms etc. by utilizing generalized open sets. Kasahara [2] defined the concept of an operation on topological spaces and introduce the concept of γ-closed graphs of a function. Ogata [5] introduced the notion γ-open sets in a topological space (X, τ). In this paper, we have introduced and studied the notion of γ-b-open sets by using operation γ on a topological space (X, τ).

2. Preliminaries

The closure and the interior of A of X are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A of X is said to be b-open [1] $A \subset \mathrm{Int}(\mathrm{Cl}(A)) \cup \mathrm{Cl}(\mathrm{Int}(A))$.

Definition 1. *Let (X, τ) be a topological space. An operation γ [2] on the topology τ is a mapping from τ into a power set $\mathcal{P}(X)$ of X such that $V \subset V^\gamma$ for each $V \in \tau$, where V^γ denotes the value of γ at V. It is denoted by $\gamma : \tau \to \mathcal{P}(X)$.*

Definition 2. *A subset A of a topological space (X, τ) is called γ-open [5] set if for each $x \in A$ there exists an open set U such that $x \in U$ and $U^\gamma \subset A$. τ_γ denotes set of all γ-open sets in (X, τ). The complement of γ-open set is called γ-closed.*

Definition 3 ([5]). *Let (X, τ) be a topological space and $A \subset X$, then* $\tau_\gamma \operatorname{Cl}(A) = \bigcap\{F : A \subset F, X \backslash F \in \tau_\gamma\}$.

Definition 4. *Let (X, τ) be a topological space. An operation γ is said to be regular if, for every open neighborhood U and V of each $x \in X$, there exists an open neighborhood W of x such that $W^\gamma \subset U^\gamma \cap V^\gamma$.*

Definition 5. *A topological space (X, τ) is said to be γ-regular,where γ is an operation on τ, if for each $x \in X$ and for each open neighborhood V of x, there exists an open neighborhood U of x such that U^γ contained in V.*

Definition 6. *Let A be any subset of X. The τ_γ-$\operatorname{Int}(A)$ is defined as* τ_γ-$\operatorname{Int}(A) = \bigcup\{U : U$ *is a γ-open set and* $U \subset A\}$.

Definition 7. *Let (X, τ) be a topological space and γ be an operation on τ. A subset A of X is said to be:*
(*i*) *γ-preopen [3] if* $A \subset \tau_\gamma - \operatorname{Int}(\tau_\gamma - \operatorname{Cl}(A))$.
(*ii*) *γ-semiopen [4] if* $A \subset \tau_\gamma - \operatorname{Cl}(\tau_\gamma - \operatorname{Int}(A))$.

The complement of γ-preopen (resp. γ-semiopen) set is called γ-preclosed (resp. γ-semiclosed).

Definition 8. *Let A be subset of a topological space (X, τ) and γ be an operation on τ. Then*
(*i*) *the τ_γ-preclosure of A is defined as intersection of all γ-preclosed sets containing A. That is,* τ_γ-$p\operatorname{Cl}(A) = \bigcap\{F : F$ *is γ-preclosed and* $A \subset F\}$.
(*ii*) *the τ_γ-preinterior of A is defined as union of all γ-preopen sets contained in A. That is,* τ_γ-$p\operatorname{Int}(A) = \bigcup\{U : U$ *is γ-preopen and* $U \subset A\}$.

The notions τ_γ-semiclosure (briefly τ_γ-$s\operatorname{Cl}(A)$) and τ_γ-semiinterior (briefly τ_γ-$s\operatorname{Int}(A)$) of a set A are similarly defined.

3. γ-b-open sets

Definition 9. *Let (X, τ) be a topological space and γ be an operation on τ. A subset A of X is said to be γ-b-open if* $A \subset \tau_\gamma - \operatorname{Cl}(\tau_\gamma - \operatorname{Int}(A)) \cup \tau_\gamma - \operatorname{Int}(\tau_\gamma - \operatorname{Cl}(A))$.

Remark 1. The set of all γ-b-open sets of a topological space (X, τ) is denoted as $\tau_\gamma - BO(X)$.

Example 1. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}\}$. Define an operation γ on τ as follows: $A^\gamma = A$ if $A = \{a\}$ and $A^\gamma = A \cup \{c\}$ if $A \neq \{a\}$. Then $\tau_\gamma = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}\}$ and $\tau_\gamma - BO(X) = \mathcal{P}(X) \backslash \{\{b\}\}$.

Theorem 1. *If A is a γ-open set in (X, τ), then it is γ-b-open set.*

Proof. Proof follows from the Definition 9 and Remark 3.8 of [4]. ■

Remark 2. The converse of the above Theorem need not be true. From the Example 1, we have $\{a, b\}$ is γ-b-open set but it is not γ-open.

Remark 3. By Theorem 1 and Remark 2, we have $\tau_\gamma \subset \tau_\gamma$-$BO(X, \tau)$.

Remark 4. The concept of b-open set and γ-b-open set are independent.

Example 2. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}\}$ define an operation $\gamma : \tau \to \mathcal{P}(X)$ as $A^\gamma = A$ if $b \in A$, $A^\gamma = \mathrm{Cl}(A)$ if $b \notin A$. Then $\{a\}$ is a b-open set but not γ-b-open. In Example 1, $\{b, c\}$ is a γ-b-open set but not b-open.

Theorem 2. *If (X, τ) is γ-regular space, then the concept of γ-b-open and b-open coincide.*

Proof. By Proposition 2.4 of [4] and Remark 3.8 of [4]. ■

Theorem 3. *Let $\gamma : \tau \to \mathcal{P}(X)$ be an operation on τ and $\{A_\alpha\}_{\alpha \in \Delta}$ be the collection of γ-b-open sets of (X, τ), then $\bigcup_{\alpha \in \Delta} A_\alpha$ is also a γ-b-open set.*

Proof. Since each A_α is γ-b-open and $A_\alpha \subset \bigcup_{\alpha \in \Delta} A_\alpha$, implies that $\bigcup_{\alpha \in \Delta} A_\alpha \subset \tau_\gamma - \mathrm{Cl}(\tau_\gamma - \mathrm{Int}(\bigcup_{\alpha \in \Delta} A_\alpha)) \cup \tau_\gamma - \mathrm{Int}(\tau_\gamma - \mathrm{Cl}(\bigcup_{\alpha \in \Delta} A_\alpha))$. Hence $\bigcup_{\alpha \in \Delta} A_\alpha$ is also a γ-b-open set in (X, τ). ■

Remark 5. If A and B are any two γ-b-open sets in (X, τ), then the Example 1, shows that $A \cap B$ need not be γ-b-open in (X, τ). In this case take $A = \{a, b\}$ and $B = \{b, c\}$, both are γ-b-open set but $A \cap B = \{b\}$ is not a γ-b-open set.

Definition 10. *Let A be subset of a topological space (X, τ) and γ be an operation on τ. Then a subset A of X is said to be γ-b-closed if and only if $X \backslash A$ is γ-b-open, equivalently a subset A of X is γ-b-closed if and only if $\tau_\gamma - \mathrm{Cl}(\tau_\gamma - \mathrm{Int}(A)) \cap \tau_\gamma - \mathrm{Int}(\tau_\gamma - \mathrm{Cl}(A)) \subset A$.*

Remark 6. The set of all γ-b-closed sets of a topological space (X, τ) is denoted as $\tau_\gamma - BC(X)$.

Definition 11. *Let A be subset of a topological space (X, τ) and γ be an operation on τ. Then*

(*i*) *the τ_γ-b-closure of A is defined as intersection of all γ-b-closed sets containing A. That is, $\tau_\gamma\text{-}b\,\mathrm{Cl}(A) = \bigcap\{F : F$ is γ-b-closed and $A \subset F\}$.*

(*ii*) *the τ_γ-b-interior of A is defined as union of all γ-b-open sets contained in A. That is, $\tau_\gamma\text{-}b\,\mathrm{Int}(A) = \bigcup\{U : U$ is γ-b-open and $U \subset A\}$.*

Theorem 4. *Let A be subset of a topological space (X, τ) and γ be an operation on τ. Then*
(*i*) *τ_γ-$b\,\mathrm{Int}(A)$ is a γ-b-open set contained in A.*
(*ii*) *τ_γ-$b\,\mathrm{Cl}(A)$ is a γ-b-closed set containing A.*
(*iii*) *A is γ-b-closed if and only if τ_γ-$b\,\mathrm{Cl}(A) = A$.*
(*iv*) *A is γ-b-open if and only if τ_γ-$b\,\mathrm{Int}(A) = A$.*

Remark 7. From the definitions, we have $A \subset \tau_\gamma$-$b\,\mathrm{Cl}(A) \subset \tau_\gamma$-$\mathrm{Cl}(A)$ for any subset A of (X, τ).

Theorem 5. *For a point $x \in X$, $x \in \tau_\gamma$-$b\,\mathrm{Cl}(A)$ if and only if for all γ-b-open set V of X containing x, $V \cap A \neq \emptyset$.*

Proof. Let F be the set of all $y \in X$ such that $V \cap A \neq \emptyset$ for every $V \in \tau_\gamma - BO(X)$ and $y \in V$. Now to prove the theorem it is enough to prove that $F = \tau_\gamma - b\,\mathrm{Cl}(A)$. Let $x \in \tau_\gamma$-$b\,\mathrm{Cl}(A)$. Let us assume $x \notin F$, then there exists a γ-b-open set U of x such that $U \cap A = \emptyset$. This implies $A \subset U^c$. Hence τ_γ-$b\,\mathrm{Cl}(A) \subset U^c$. Therefore $x \notin \tau_\gamma$-$b\,\mathrm{Cl}(A)$. This is a contradiction. Hence $\tau_\gamma - b\,\mathrm{Cl}(A) \subset F$. Conversely, let F be a set such that $A \subset F$ and $X \backslash F \in \tau_\gamma$-BO(X). Let $x \notin F$, then we have $x \in X \backslash F$ and $(X \backslash F) \cap A = \emptyset$. This implies $x \notin F$. Hence $F \subset \tau_\gamma$-$b\,\mathrm{Cl}(A)$. ■

Theorem 6 ([3])**.** *Let (X, τ) be a topological space and γ be a regular operation on τ and A be a subset of X. Then the following holds:*
(*i*) *$\tau_\gamma - p\,\mathrm{Cl}(A) = A \cup \tau_\gamma - \mathrm{Cl}(\tau_\gamma - \mathrm{Int}(A))$.*
(*ii*) *$\tau_\gamma - p\,\mathrm{Int}(A) = A \cap \tau_\gamma - \mathrm{Int}(\tau_\gamma - \mathrm{Cl}(A))$.*

Theorem 7. *Let (X, τ) be a topological space and γ be a regular operation on τ and A be a subset of X. Then the following holds good:*
(*i*) *$\tau_\gamma - s\,\mathrm{Cl}(A) = A \cup \tau_\gamma - \mathrm{Int}(\tau_\gamma - \mathrm{Cl}(A))$.*
(*ii*) *$\tau_\gamma - s\,\mathrm{Int}(A) = A \cap \tau_\gamma - \mathrm{Cl}(\tau_\gamma - \mathrm{Int}(A))$.*

Proof. The proof is similar to the Theorem 2.31 of [3]. ■

Theorem 8. *Let (X, τ) be a topological space and γ be a regular operation on τ and A be a subset of X. Then $\tau_\gamma - b\,\mathrm{Cl}(A) = \tau_\gamma - p\,\mathrm{Cl}(A) \cap \tau_\gamma - s\,\mathrm{Cl}(A)$.*

Proof. The proof follows from Theorems 6 and 7. ■

Theorem 9. *Let (X, τ) be a topological space and γ be a regular operation on τ and A be a subset of X. Then $\tau_\gamma - b\,\mathrm{Int}(A) = \tau_\gamma - p\,\mathrm{Int}(A) \cup \tau_\gamma - s\,\mathrm{Int}(A)$.*

Proof. The proof follows from Theorem 8. ■

Theorem 10. *Let (X, τ) be a topological space and γ be a regular operation on τ and A be a subset of X. Then the following hold:*

(*i*) V *is* γ*-preopen if and only if* $V \subset \tau_\gamma - p\,\mathrm{Int}(\tau_\gamma - p\,\mathrm{Cl}(V))$*;*
(*ii*) V *is* γ*-b-open if and only if* $V \subset \tau_\gamma - p\,\mathrm{Cl}(\tau_\gamma - p\,\mathrm{Int}(V))$*.*

Proof. (*i*) Let V be γ-preopen. Then $\tau_\gamma - p\,\mathrm{Int}(V) = V$ and also $V \subset \tau_\gamma - p\,\mathrm{Int}(\tau_\gamma - p\,\mathrm{Cl}(V))$. Conversely, let $V \subset \tau_\gamma - p\,\mathrm{Int}(\tau_\gamma - p\,\mathrm{Cl}(V))$. Then $V \subset \tau_\gamma - p\,\mathrm{Int}(\tau_\gamma - \mathrm{Cl}(V)) = \tau_\gamma - \mathrm{Cl}(V) \cap \tau_\gamma - \mathrm{Int}(\tau_\gamma - \mathrm{Cl}(\tau_\gamma - \mathrm{Cl}(V))) = \tau_\gamma - \mathrm{Int}(\tau_\gamma - \mathrm{Cl}(V))$. Hence, V is γ-preopen.

(*ii*) Let V be γ-b-open. Then $V \subset \tau_\gamma - \mathrm{Cl}(\tau_\gamma - \mathrm{Int}(V)) \cup \tau_\gamma - \mathrm{Int}(\tau_\gamma - \mathrm{Cl}(V)) \subset (\tau_\gamma - \mathrm{Cl}(\tau_\gamma - \mathrm{Int}(V)) \cup \tau_\gamma - \mathrm{Int}(\tau_\gamma - \mathrm{Cl}(V))) \cap V = (\tau_\gamma - \mathrm{Cl}(\tau_\gamma - \mathrm{Int}(V)) \cap V) \cup (\tau_\gamma - \mathrm{Int}(\tau_\gamma - \mathrm{Cl}(V)) \cap V) \subset \tau_\gamma - p\,\mathrm{Int}(V) \cup (\tau_\gamma - \mathrm{Int}(\tau_\gamma - \mathrm{Cl}(V)) = \tau_\gamma - p\,\mathrm{Cl}(\tau_\gamma - p\,\mathrm{Int}(V))$ by Theorem 2.33 of [3]. Conversely, suppose $V \subset \tau_\gamma - p\,\mathrm{Cl}(\tau_\gamma - p\,\mathrm{Int}(V))$. By Theorem 2.33 of [3], we have $V \subset \tau_\gamma - p\,\mathrm{Cl}(\tau_\gamma - p\,\mathrm{Int}(V)) = \tau_\gamma - p\,\mathrm{Int}(V) \cup \tau_\gamma - \mathrm{Cl}(\tau_\gamma - \mathrm{Int}(V)) = (V \cap \tau_\gamma - \mathrm{Int}(\tau_\gamma - \mathrm{Cl}(V))) \cup \tau_\gamma - \mathrm{Cl}(\tau_\gamma - \mathrm{Int}(V)) \subset \tau_\gamma - \mathrm{Cl}(\tau_\gamma - \mathrm{Int}(V)) \cup \tau_\gamma - \mathrm{Int}(\tau_\gamma - \mathrm{Cl}(V))$. Hence, V is γ-b-open. ■

Theorem 11. *Let* (X, τ) *be a topological space and* γ *be a regular operation on* τ *and* A *be a subset of* X*. Then the following hold:*

(*i*) $\tau_\gamma - b\,\mathrm{Cl}(\tau_\gamma - \mathrm{Int}(A)) = \tau_\gamma - \mathrm{Int}(\tau_\gamma - \mathrm{Cl}(\tau_\gamma - \mathrm{Int}(A)))$.
(*ii*) $\tau_\gamma - \mathrm{Int}(\tau_\gamma - b\,\mathrm{Cl}(A)) = \tau_\gamma - \mathrm{Int}(\tau_\gamma - \mathrm{Cl}(\tau_\gamma - \mathrm{Int}(A)))$.
(*iii*) $\tau_\gamma - b\,\mathrm{Int}(\tau_\gamma - \mathrm{Cl}(A)) = \tau_\gamma - \mathrm{Cl}(\tau_\gamma - b\,\mathrm{Int}(A))$
$= \tau_\gamma - \mathrm{Cl}(\tau_\gamma - \mathrm{Int}(\tau_\gamma - \mathrm{Cl}(A)))$.
(*iv*) $\tau_\gamma - b\,\mathrm{Cl}(\tau_\gamma - s\,\mathrm{Int}(A)) = \tau_\gamma - s\,\mathrm{Cl}(\tau_\gamma - s\,\mathrm{Int}(A))$.
(*v*) $\tau_\gamma - p\,\mathrm{Int}(\tau_\gamma - b\,\mathrm{Cl}(A)) = \tau_\gamma - p\,\mathrm{Int}(\tau_\gamma - p\,\mathrm{Int}(A))$.
(*vi*) $\tau_\gamma - s\,\mathrm{Int}(\tau_\gamma - b\,\mathrm{Cl}(A)) = \tau_\gamma - \mathrm{Cl}(\tau_\gamma - \mathrm{Int}(A)) \cap \tau_\gamma - s\,\mathrm{Cl}(A)$.
(*vii*) $\tau_\gamma - b\,\mathrm{Int}(\tau_\gamma - s\,\mathrm{Cl}(A)) = \tau_\gamma - s\,\mathrm{Int}(\tau_\gamma - s\,\mathrm{Cl}(A))$.
(*viii*) $\tau_\gamma - p\,\mathrm{Cl}(\tau_\gamma - b\,\mathrm{Int}(A)) = \tau_\gamma - p\,\mathrm{Cl}(\tau_\gamma - p\,\mathrm{Int}(A))$.
(*ix*) $\tau_\gamma - s\,\mathrm{Cl}(\tau_\gamma - b\,\mathrm{Int}(A)) = \tau_\gamma - \mathrm{Int}(\tau_\gamma - \mathrm{Cl}(A)) \cup \tau_\gamma - s\,\mathrm{Int}(A)$.

Proof. (*i*) By Theorem 8, we obtain $\tau_\gamma - b\,\mathrm{Cl}(\tau_\gamma - \mathrm{Int}(A)) = \tau_\gamma - p\,\mathrm{Cl}(\tau_\gamma - \mathrm{Int}(A)) \cap \tau_\gamma - s\,\mathrm{Cl}(\tau_\gamma - \mathrm{Int}(A)) = (\tau_\gamma - \mathrm{Int}(A) \cup \tau_\gamma - \mathrm{Cl}(\tau_\gamma - \mathrm{Int}(\tau_\gamma - \mathrm{Int}(A)))) \cap (\tau_\gamma - \mathrm{Int}(A) \cup \tau_\gamma - \mathrm{Int}(\tau_\gamma - \mathrm{Cl}(\tau_\gamma - \mathrm{Int}(A))))) = \tau_\gamma - \mathrm{Int}(A) \cup (\tau_\gamma - \mathrm{Int}(A) \cup \tau_\gamma - \mathrm{Cl}(\tau_\gamma - \mathrm{Int}(A))) \cap (\tau_\gamma - \mathrm{Int}(A) \cup \tau_\gamma - \mathrm{Int}(\tau_\gamma - \mathrm{Cl}(\tau_\gamma - \mathrm{Int}(A)))) = \tau_\gamma - \mathrm{Int}(A) \cup (\tau_\gamma - \mathrm{Int}(A) \cup \tau_\gamma - \mathrm{Int}(\tau_\gamma - \mathrm{Cl}(\tau_\gamma - \mathrm{Int}(A)))) = \tau_\gamma - \mathrm{Int}(\tau_\gamma - \mathrm{Cl}(\tau_\gamma - \mathrm{Int}(A))))$.

(*ii*) By Theorem 8, we obtain $\tau_\gamma - \mathrm{Int}(\tau_\gamma - b\,\mathrm{Cl}(A)) = \tau_\gamma - \mathrm{Int}(\tau_\gamma - p\,\mathrm{Cl}(A) \cap \tau_\gamma - s\,\mathrm{Cl}(A)) = \tau_\gamma - \mathrm{Int}(\tau_\gamma - p\,\mathrm{Cl}(A)) \cap \tau_\gamma - \mathrm{Int}(\tau_\gamma - s\,\mathrm{Cl}(A)) = \tau_\gamma - \mathrm{Int}(\tau_\gamma - \mathrm{Cl}(\tau_\gamma - \mathrm{Int}(A))) \cap \tau_\gamma - \mathrm{Int}(\tau_\gamma - \mathrm{Cl}(A)) = \tau_\gamma - \mathrm{Int}(\tau_\gamma - \mathrm{Cl}(\tau_\gamma - \mathrm{Int}(A)))$.

(*iii*) Follows from (*i*) and (*ii*).

(*iv*) By Theorem 8, we obtain $\tau_\gamma - b\,\mathrm{Cl}(\tau_\gamma - s\,\mathrm{Int}(A)) = \tau_\gamma - p\,\mathrm{Cl}(\tau_\gamma - s\,\mathrm{Int}(A)) \cap \tau_\gamma - s\,\mathrm{Cl}(\tau_\gamma - s\,\mathrm{Int}(A)) = \tau_\gamma - \mathrm{Cl}(\tau_\gamma - \mathrm{Int}(A)) \cap (\tau_\gamma - s\,\mathrm{Int} \cup \tau_\gamma - \mathrm{Int}(\tau_\gamma - \mathrm{Cl}(\tau_\gamma - \mathrm{Int}(A)))) = \tau_\gamma - s\,\mathrm{Int}(A)) \cup \tau_\gamma - \mathrm{Int}(\tau_\gamma - \mathrm{Cl}(\tau_\gamma - \mathrm{Int}(A))) = \tau_\gamma - s\,\mathrm{Cl}(\tau_\gamma - s\,\mathrm{Int}(A))$.

(v) By Theorem 8, we always have $\tau_\gamma - p\operatorname{Int}(\tau_\gamma - b\operatorname{Cl}(A)) \subset \tau_\gamma - p\operatorname{Int}(\tau_\gamma - p\operatorname{Cl}(A))$. Conversely, by Theorem 2.33 of [3], we obtain $\tau_\gamma - p\operatorname{Int}(\tau_\gamma - b\operatorname{Cl}(A)) = \tau_\gamma - p\operatorname{Int}(\tau_\gamma - p\operatorname{Cl}(A) \cap \tau_\gamma - s\operatorname{Cl}(A)) = \tau_\gamma - p\operatorname{Cl}(A) \cap \tau_\gamma - s\operatorname{Cl}(A) \cap \tau_\gamma - \operatorname{Int}(\tau_\gamma - \operatorname{Cl}(\tau_\gamma - p\operatorname{Cl}(A) \cap \tau_\gamma - s\operatorname{Cl}(A))) \supset \tau_\gamma - p\operatorname{Cl}(A) \cap \tau_\gamma - \operatorname{Int}(\tau_\gamma - \operatorname{Cl}(A)) \cap \tau_\gamma - s\operatorname{Cl}(A) \cap \tau_\gamma - \operatorname{Int}(\tau_\gamma - \operatorname{Cl}(\tau_\gamma - p\operatorname{Cl}(A) \cap \tau_\gamma - s\operatorname{Cl}(A))) = \tau_\gamma - p\operatorname{Cl}(A) \cap \tau_\gamma - \operatorname{Int}(\tau_\gamma - \operatorname{Cl}(A)) \cap \tau_\gamma - \operatorname{Int}(\tau_\gamma - \operatorname{Cl}(\tau_\gamma - p\operatorname{Cl}(A) \cap \tau_\gamma - s\operatorname{Cl}(A))) = \tau_\gamma - p\operatorname{Cl}(A) \cap \tau_\gamma - \operatorname{Int}(\tau_\gamma - \operatorname{Cl}(A)) = \tau_\gamma - p\operatorname{Int}(\tau_\gamma - p\operatorname{Int}(A))$.

(vi) Let A be a subset of X. By Theorem 8, $\tau_\gamma - s\operatorname{Int}(\tau_\gamma - b\operatorname{Cl}(A)) \subset \tau_\gamma - s\operatorname{Int}(\tau_\gamma - p\operatorname{Cl}(A)) = \tau_\gamma - \operatorname{Cl}(\tau_\gamma - \operatorname{Int}(A))$ and $\tau_\gamma - s\operatorname{Int}(\tau_\gamma - b\operatorname{Cl}(A)) \subset \tau_\gamma - s\operatorname{Int}(\tau_\gamma - s\operatorname{Cl}(A)) \subset \tau_\gamma - s\operatorname{Cl}(A)$. Thus, $\tau_\gamma - s\operatorname{Int}(\tau_\gamma - b\operatorname{Cl}(A)) = \tau_\gamma - \operatorname{Cl}(\tau_\gamma - \operatorname{Int}(A)) \cap \tau_\gamma - s\operatorname{Cl}(A)$. Conversely, By Theorem 8, $\tau_\gamma - s\operatorname{Int}(\tau_\gamma - b\operatorname{Cl}(A)) = \tau_\gamma - s\operatorname{Int}(\tau_\gamma - p\operatorname{Cl}(A) \cap \tau_\gamma - s\operatorname{Cl}(A)) = \tau_\gamma - p\operatorname{Cl}(A) \cap \tau_\gamma - s\operatorname{Cl}(A) \cap \tau_\gamma - \operatorname{Cl}(\tau_\gamma - \operatorname{Int}(\tau_\gamma - p\operatorname{Cl}(A) \cap \tau_\gamma - s\operatorname{Cl}(A)) \supset \tau_\gamma - p\operatorname{Cl}(A) \cap \tau_\gamma - \operatorname{Cl}(\tau_\gamma - \operatorname{Int}(A)) \cap \tau_\gamma - s\operatorname{Cl}(A) \cap \tau_\gamma - \operatorname{Cl}(\tau_\gamma - \operatorname{Int}(\tau_\gamma - p\operatorname{Cl}(A) \cap \tau_\gamma - s\operatorname{Cl}(A)) = \tau_\gamma - \operatorname{Cl}(\tau_\gamma - \operatorname{Int}(A)) \cap \tau_\gamma - s\operatorname{Cl}(A) \cap \tau_\gamma - \operatorname{Cl}(\tau_\gamma - \operatorname{Int}(\tau_\gamma - p\operatorname{Cl}(A) \cap \tau_\gamma - s\operatorname{Cl}(A)) = \tau_\gamma - s\operatorname{Cl}(A) \cap \tau_\gamma - \operatorname{Cl}(\tau_\gamma - \operatorname{Int}(A))$.

(vii), $(viii)$ and (ix) follows from (iv), (v) and (vi), respectively. ■

Definition 12. *A subset A of (X, τ) is said to be γ-b-generalized closed if τ_γ-$b\operatorname{Cl}(A) \subset U$ whenever $A \subset U$ and U is a γ-b-open set in (X, τ).*

Definition 13. *A topological space (X, τ) is said to be γ-b-$T_{1/2}$ if every γ-b-generalized closed set in (X, τ) is γ-b-closed.*

Theorem 12. *A subset A of (X, τ) is γ-b-generalized closed if and only if τ_γ-$b\operatorname{Cl}(\{x\}) \cap A \neq \emptyset$ holds for every $x \in \tau_\gamma$-$b\operatorname{Cl}(A)$.*

Proof. Let U be γ-b-open set such that $A \subset U$. Let $x \in \tau_\gamma$-$b\operatorname{Cl}(A)$. By assumption there exists $z \in \tau_\gamma$-$b\operatorname{Cl}(\{x\})$ and $z \in A \subset U$. It follows from Theorem 5 that $U \cap \{x\} \neq \emptyset$. Hence $x \in U$. This implies τ_γ-$b\operatorname{Cl}(A) \subset U$. Therefore A is γ-b-generalized closed set in (X, τ). Conversely, suppose $x \in \tau_\gamma$-$b\operatorname{Cl}(A)$ such that τ_γ-$b\operatorname{Cl}(\{x\}) \cap A = \emptyset$. Since τ_γ-$b\operatorname{Cl}(\{x\})$ is γ-b-closed set in (X, τ), $(\tau_\gamma$-$b\operatorname{Cl}(\{x\}))^c$ is a γ-b-open set of (X, τ). Since $A \subset (\tau_\gamma$-$b\operatorname{Cl}(\{x\}))^c$ and A is γ-b-generalized closed, τ_γ-$b\operatorname{Cl}(A)$ $\subset (\tau_\gamma$-$b\operatorname{Cl}(\{x\}))^c$. This implies that $x \notin \tau_\gamma$-$b\operatorname{Cl}(A)$. This is a contradiction. Hence τ_γ-$b\operatorname{Cl}(\{x\}) \cap A \neq \emptyset$. ■

Theorem 13. *A is a γ-b-generalized closed subset of a topological space (X, τ), if and only if τ_γ-$b\operatorname{Cl}(A) \backslash A$ does not contain a nonempty γ-b-closed set.*

Proof. Suppose there exists a nonempty γ-b-closed set F such that $F \subset \tau_\gamma$-$b\operatorname{Cl}(A) \backslash A$. Let $x \in F$, $x \in \tau_\gamma$-$b\operatorname{Cl}(A)$ holds. Then $F \cap A =$

τ_γ-$b\,\mathrm{Cl}(F) \cap A \supset \tau_\gamma$-$b\,\mathrm{Cl}(\{x\}) \cap A \neq \emptyset$. Hence $F \cap A \neq \emptyset$. This is a contradiction.

Conversely, suppose that τ_γ-$b\,\mathrm{Cl}(A)\backslash A$ does not contain a nonempty γ-b-closed set. Let $A \subset U$ and U a γ-b-open set in (X,τ), then $X\backslash U \subseteq X\backslash A$, follows τ_γ-$b\,\mathrm{Cl}(A) \cap (X\backslash U) \subseteq \tau_\gamma$-$b\,\mathrm{Cl}(A) \cap (X\backslash A) = \tau_\gamma$-$b\,\mathrm{Cl}(A)\backslash A)$. If we take $F = \tau_\gamma$-$b\,\mathrm{Cl}(A) \cap (X\backslash U)$, F is a γ-b-closed set and $F \subseteq \tau_\gamma$-$b\,\mathrm{Cl}(A)\backslash A$. Therefore $F = \emptyset$, in consequence, τ_γ-$b\,\mathrm{Cl}(A) \subseteq U$ and follows that A is γ-b-generalized closed set. ■

Theorem 14. *Let $\gamma : \tau \to \mathcal{P}(X)$ be an operation. Then for each $x \in X$, $\{x\}$ is γ-b-closed or $\{x\}^c$ is γ-b-generalized closed set in (X,τ).*

Proof. Suppose that $\{x\}$ is not γ-b-closed, then $X\backslash\{x\}$ is not γ-b-open. Let U be any γ-b-open set such that $\{x\}^c \subset U$. Since $U = X$, τ_γ-$b\,\mathrm{Cl}(\{x\}^c) \subset U$. Therefore, $\{x\}^c$ is γ-b-generalized closed. ■

Theorem 15. *A topological space (X,τ) is γ-b-$T_{1/2}$ space if and only if every singleton subset of X is γ-b-closed or γ-b-open in (X,τ).*

Proof. Let $x \in X$. Suppose $\{x\}$ is not γ-b-closed. Then, it follows from assumption and Theorem 14 that $\{x\}$ is γ-b-open. Conversely, Let F be a γ-b-generalized closed set in (X,τ). Let x be any point in τ_γ-$b\,\mathrm{Cl}(F)$, then by assumption $\{x\}$ is γ-b-open or γ-b-closed.

Case (i): Suppose $\{x\}$ is γ-b-open. Then by Theorem 12 we have $\{x\} \cap F \neq \emptyset$, hence $x \in F$.

Case (ii): Suppose $\{x\}$ is γ-b-closed. Assume $x \notin F$, then $x \in \tau_\gamma$-$b\,\mathrm{Cl}(F)\backslash F$. This is not possible by Theorem 13. Thus, we have $x \in F$. Therefore, τ_γ-$b\,\mathrm{Cl}(F) = F$ and hence F is γ-b-closed. ■

4. (α, β)-b-Continuous functions

Throughout this section let (X,τ) and (Y,σ) be two topological spaces and let $\gamma : \tau \to \mathcal{P}(X)$ and $\beta : \sigma \to \mathcal{P}(Y)$ be operations on τ and σ, respectively.

Definition 14. *A function $f : (X,\tau) \to (Y,\sigma)$ is said to be (α,β)-b-continuous if for each $x \in X$ and each β-b-open set V containing $f(x)$ there exists a γ-b-open set U such that $x \in U$ and $f(U) \subset V$.*

Theorem 16. *Let $f : (X,\tau) \to (Y,\sigma)$ be an (α,β)-b-continuous function. Then the following hold:*

(i) $f(\tau_\gamma$-$b\,\mathrm{Cl}(A)) \subset \sigma_\beta$-$b\,\mathrm{Cl}(f(A))$ holds for every subset A of (X,τ).

(ii) for every β-b-closed set B of (Y,σ), $f^{-1}(B)$ is γ-b-closed in (X,τ).

Proof. (i). Let $y \in f(\tau_\gamma\text{-}b\,\mathrm{Cl}(A))$ and V be any β-b-open set containing y. Then there exists $x \in X$ and γ-b-open set U such that $f(x) = y$ and $x \in U$ and $f(U) \subset V$. Since $x \in \tau_\gamma\text{-}b\,\mathrm{Cl}(A)$, we have $U \cap A \neq \emptyset$ and hence $\emptyset \neq f(U \cap A) \subset f(U) \cap f(A) \subset V \cap A$. This implies $x \in \sigma_\beta\text{-}b\,\mathrm{Cl}(f(A))$. Therefore we have $f(\tau_\gamma\text{-}b\,\mathrm{Cl}(A)) \subset \sigma_\beta\text{-}b\,\mathrm{Cl}(f(A))$. (ii). Let B be a β-b-closed set in (Y, σ). Therefore, $\sigma_\beta\text{-}b\,\mathrm{Cl}(B) = B$. By using (i) we have $f(\tau_\gamma\text{-}b\,\mathrm{Cl}(f^{-1}(B)) \subset \sigma_\beta\text{-}b\,\mathrm{Cl}(B) = B$. Therefore, we have $\tau_\gamma\text{-}b\,\mathrm{Cl}(f^{-1}(B)) = f^{-1}(B)$. Hence $f^{-1}(B)$ is γ-b-closed. ■

Definition 15. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be (α, β)-b-closed if for any γ-b-closed set A of (X, τ), $f(A)$ is a β-b-closed in Y.*

Theorem 17. *Suppose that f is (α, β)-b-continuous function and f is (α, β)-b-closed. Then,*

(i) for every γ-b-generalized closed set A of (X, τ), the image $f(A)$ is β-b-generalized closed.

(ii) for every β-b-generalized closed set B of (Y, σ), $f^{-1}(B)$ is γ-b-generalized closed.

Proof. (i) Let V be any β-b-open set in (Y, σ) such that $f(A) \subset V$. By using Theorem 16(ii), $f^{-1}(V)$ is γ-b-open in (X, τ). Since A is γ-b-generalized closed and $A \subset f^{-1}(V)$, we have $\tau_\gamma\text{-}b\,\mathrm{Cl}(A) \subset f^{-1}(V)$, and hence $f(\tau_\gamma\text{-}b\,\mathrm{Cl}(A)) \subset V$. It follows that $f(\tau_\gamma\text{-}b\,\mathrm{Cl}(A))$ is a β-b-closed set in Y. Therefore, $\sigma_\beta\text{-}b\,\mathrm{Cl}(f(A)) \subset \sigma_\beta - b\,\mathrm{Cl}(f(\tau_\gamma - b\,\mathrm{Cl}(A))) = f(\tau_\gamma - b\,\mathrm{Cl}(A)) \subset V$. This implies $f(A)$ is β-b-generalized closed.

(ii) Let U be a γ-b-open set of (X, τ) such that $f^{-1}(B) \subset U$. Put $F = \tau_\gamma\text{-}b\,\mathrm{Cl}(f^{-1}(B)) \cap U^c$. It follows that F is γ-b-closed set in (X, τ). Since f is (α, β)-b-closed, $f(F)$ is γ-b-closed in (Y, σ). Then $f(F) \subset f(\tau_\gamma\text{-}b\,\mathrm{Cl}(f^{-1}(B) \cap U^c)) \subset \sigma_\beta\text{-}b\,\mathrm{Cl}(f(f^{-1}(B)) \cap f(U^c)) \subset \tau_\gamma\text{-}b\,\mathrm{Cl}(B) \backslash B$. This implies $f(F) = \emptyset$, and hence $F = \emptyset$. Therefore, $\tau_\gamma\text{-}b\,\mathrm{Cl}(f^{-1}(B)) \subset U$. Hence $f^{-1}(B)$ is γ-b-generalized closed in (X, τ). ■

Theorem 18. *Let $f : (X, \tau) \to (Y, \sigma)$ is (α, β)-b-continuous and (α, β)-γ-b-closed. Then,*

(i) If f is injective and (Y, σ) is β-b-$T_{1/2}$, then (X, τ) is γ-b-$T_{1/2}$ space.

(ii) If f is surjective and (X, τ) is γ-b-$T_{1/2}$, then (Y, σ) is β-b-$T_{1/2}$.

Proof. Straightforward. ■

Definition 16. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be (α, β)-b-homeomorphism, if f is bijective, (α, β)-b-continuous and f^{-1} is (α, β)-b-continuous.*

Theorem 19. *Let $f : (X, \tau) \to (Y, \sigma)$ be (α, β)-b-homeomorphism. If (X, τ) is γ-b-$T_{1/2}$, then (Y, σ) is β-b-$T_{1/2}$.*

Proof. Let $\{y\}$ be a singleton set of (Y, σ). Then, there exists a point x of X such that $y = f(x)$ and by Theorem 15 that $\{x\}$ is γ-b-open or γ-b-closed. By using Theorem 17(i), then $\{y\}$ is β-b-closed or β-b-open. Now using Theorem 15, (Y, σ) is β-b-$T_{1/2}$ space. ■

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