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V. Gupta, A. Aral and M. Ozhavzali

APPROXIMATION BY q-SZÁSZ-MIRAKYAN-BASKAKOV OPERATORS *

Abstract. In the present paper we propose the q analogue of well known Szász-Mirakyan-Baskakov operators (see e.g. [14], [7]). We apply q-derivatives, and q-Beta functions to obtain the moments of the q-Szász-Mirakyan-Baskakov operators. Here we estimate some direct approximation results for these operators.

Key words: q-Szász-Mirakyan-Baskakov operators, q-binomial coefficients, q-derivatives, q-integers, q-Beta functions, q-integral.

AMS Mathematics Subject Classification: 41A25, 41A30.

1. Introduction

Recently Mahmudov [12] and Aral [2] (see also [4]) proposed the q-analogues of the well known Szász-Mirakyan operators and estimated some approximation results. The operators studied in [12] are different from those studied in [4]. King type generalization of the q-Szász operators defined in [2] can be found in [1]. Also some approximation properties of the another Szász-Mirakyan type operators were presented in [15]. The most commonly used integral modifications of the Szász-Mirakyan operators are Szász-Mirakyan-Kantorovich and Szász-Mirakyan-Durrmeyer operators. q-analogue of some Durrmeyer type operators were studied in [13] and [9]. Very recently Gupta and Aral [8] proposed q analogue of Szász-Mirakyan-Beta operators and established some approximation properties.

In the year 1983, Prasad-Agrawal-Kasana [14] proposed the integral modification of Szász-Mirakyan operators by taking the weight functions of Baskakov operators, but there were so many gaps in the results obtained in [14]. In the year 1993 Gupta [7] filled the gaps and improved the results

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of [14]. To approximate Lebesgue integrable functions on the interval $[0,\infty)$, the Szász-Mirakyan-Baskakov operators are defined as

$$G_n(f,x)=(n-1)\sum_{k=0}^{\infty}s_{n,k}(x)\int_0^{\infty}p_{n,k}(t)f(t)dt,\quad x\in[0,\infty) \tag{1}$$

where

$$s_{n,k}(x)=e^{-nx}\frac{(nx)^k}{k!},\quad p_{n,k}(t)=\left(\begin{array}{c}n+k-1\\k\end{array}\right)\frac{x^k}{(1+x)^{n+k}}.$$

First we recall some notations of q-calculus, which can also be found in [6] and [10]. Throughout the present article q be a real number satisfying the inequality $0<q<1$.

For $n\in\mathbb{N}$,

$$[n]_q:=\frac{1-q^n}{1-q},$$

$$[n]_q!:=\begin{cases}[n]_q\,[n-1]_q\cdots[1]_q\,, & n=1,\,2,...\\ 1, & n=0\end{cases}$$

and

$$(1+x)_q^n:=\begin{cases}\prod_{j=0}^{n-1}\left(1+q^jx\right), & n=1,\,2,...\\ 1, & n=0.\end{cases}$$

The q-derivative D_qf of a function f is given by

$$\left(D_qf\right)(x)=\frac{f\left(x\right)-f\left(qx\right)}{\left(1-q\right)x},\quad \text{if }\ x\neq0. \tag{2}$$

The q-improper integrals considered in the present paper are defined as (see [11])

$$\int_0^a f\left(x\right)d_qx=\left(1-q\right)a\sum_{n=0}^{\infty}f\left(aq^n\right)q^n,\quad a>0$$

and

$$\int_0^{\infty/A} f\left(x\right)d_qx=\left(1-q\right)\sum_{n=-\infty}^{\infty}f\left(\frac{q^n}{A}\right)\frac{q^n}{A},\quad A>0 \tag{3}$$

provided the sums converge absolutely.

There are two q analogues of the exponential function e^x (see [10]) as

$$e_q(z)=\sum_{k=0}^{\infty}\frac{z^k}{[k]_q!}=\frac{1}{(1-(1-q)z)_q^{\infty}},\quad |z|<\frac{1}{1-q},\quad |q|<1$$

and

$$E_q(z)=\prod_{j=0}^{\infty}(1+(1-q)q^jz)=\sum_{k=0}^{\infty}q^{k(k-1)/2}\frac{z^k}{[k]_q!}=(1+(1-q)z)_q^{\infty},$$

$|q|<1$, where $(1-x)_q^{\infty}=\prod_{j=0}^{\infty}(1-q^jx)$.

The q-Gamma integral is defined by [10]

$$\Gamma_q\left(t\right)=\int_0^{\frac{1}{1-q}}x^{t-1}E_q\left(-qx\right)d_qx,\quad t>0 \tag{4}$$

which satisfies the following functional equation:

$$\Gamma_q\left(t+1\right)=[t]_q\,\Gamma_q\left(t\right),\quad \Gamma_q\left(1\right)=1.$$

The q Beta function (see [16]) is defined as

$$B_q\left(t,s\right)=K\left(A,t\right)\int_0^{\infty/A}\frac{x^{t-1}}{(1+x)_q^{t+s}}d_qx, \tag{5}$$

where $K\left(x,t\right)=\frac{1}{x+1}x^t\left(1+\frac{1}{x}\right)_q^t(1+x)_q^{1-t}$. In particular for any positive integer n

$$K\left(x,n\right)=q^{\frac{n(n-1)}{2}},\qquad K\left(x,0\right)=1$$

and

$$B_q\left(t,s\right)=\frac{\Gamma_q\left(t\right)\Gamma_q\left(s\right)}{\Gamma_q\left(t+s\right)}. \tag{6}$$

Based on q-exponential function Mahmudov [12], introduced the following q-Szász-Mirakyan operators as

$$\begin{aligned}\mathcal{S}_{n,q}(f,x)&=\frac{1}{E_q\left([n]_qx\right)}\sum_{k=0}^{\infty}\frac{([n]_qx)^k}{[k]_q!}q^{k(k-1)/2}f\left(\frac{[k]_q}{q^{k-2}[n]_q}\right)\\&=\sum_{k=0}^{\infty}s_{n,k}^q\left(x\right)f\left(\frac{[k]_q}{q^{k-2}[n]_q}\right),\end{aligned} \tag{7}$$

$$s_{n,k}^q\left(x\right)=\frac{([n]_qx)^k}{[k]_q!}q^{k(k-1)/2}\frac{1}{E_q\left([n]_qx\right)}.$$

Lemma 1 ([12])**.** *We have*

$$\mathcal{S}_{n,q}(1,x)=1,$$
$$\mathcal{S}_{n,q}(t,x)=qx,$$
$$\mathcal{S}_{n,q}(t^2,x)=qx^2+\frac{q^2x}{[n]_q}.$$

In the present article, we introduce the q analogue of the Szász-Mirakyan -Baskakov operators, obtain its moments using q-Beta function and estimate some direct results in terms of modulus of continuity.

2. q-Operators and moments

For every $n \in \mathbb{N}$, $q \in (0,\ 1)$, the q analogue of (1) can be defined as

$$G_n^q\left(f\left(t\right),x\right) := [n-1]_q \sum_{k=0}^{\infty} s_{n,k}^q\left(x\right) q^k \int_0^{\infty/A} p_{n,k}^q\left(t\right) f\left(t\right) d_q t \tag{8}$$

where

$$s_{n,k}^q\left(x\right) = \frac{\left([n]_q x\right)^k}{[k]_q!} q^{k(k-1)/2} \frac{1}{E_q\left([n]_q x\right)}$$

and

$$p_{n,k}^q\left(t\right) := \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{k(k-1)/2} \frac{t^k}{\left(1+t\right)_q^{n+k}}$$

for $x \in [0,\infty)$ and for every real valued continuous function f on $[0,\ \infty)$. These operators satisfy linearity property. As a special case when $q = 1$ the above operators reduce to the Szász-Mirakyan-Baskakov operators (1) discussed in [14] and [7].

Remark 1. We have

$$xD_q\left(s_{n,k}^q\left(x\right)\right) = \left(\frac{[k]_q}{q^{k-2}[n]_q} - q^2 x\right) q^{k-2}[n]_q s_{n,k}^q\left(x\right),$$

and

$$\frac{t}{q}\left(1+\frac{t}{q}\right) D_q p_{n,k}^q\left(\frac{t}{q}\right) = \frac{[n]_q}{q^2} p_{n,k}^q\left(t\right)\left(\frac{[k]_q}{q^{k-1}[n]_q} - t\right).$$

Proof. Using q-derivative, we have

$$D_q\left(E_q\left([n]_q x\right)\right) = [n]_q E_q\left(q[n]_q x\right)$$

and

$$D_q\left(\frac{1}{E_q\left([n]_q x\right)}\right) = -\frac{[n]_q E_q\left(q[n]_q x\right)}{E_q\left([n]_q x\right) E_q\left(q[n]_q x\right)} = -\frac{[n]_q}{E_q\left([n]_q x\right)}.$$

Also we have

$$\begin{aligned} xD_q\left(s_{n,k}^q\left(x\right)\right) &= [n]_q [k]_q \frac{\left([n]_q x\right)^{k-1}}{[k]_q!} q^{k(k-1)/2} \frac{1}{E\left([n]_q x\right)} \\ &\quad - \frac{[n]_q}{E_q\left([n]_q x\right)} \frac{\left([n]_q q x\right)^k}{[k]_q!} q^{k(k-1)/2} \end{aligned}$$

$$= \left(\frac{[k]_q}{[n]_q} - q^k x \right) \frac{[n]_q}{E_q\left([n]_q x\right)} \frac{\left([n]_q x\right)^k}{[k]_q!} q^{k(k-1)/2}.$$

■

Remark 2. Following equality is obvious:

$$\begin{aligned} s_{n,k}^q(qx) &= \frac{\left(q[n]_q x\right)^k}{[k]_q!} q^{k(k-1)/2} \frac{1}{E_q\left(q[n]_q x\right)} \\ &= \frac{q^k\left([n]_q x\right)^k}{[k]_q!} q^{k(k-1)/2} \frac{\left(1+(1-q)[n]_q x\right)}{E_q\left([n]_q x\right)} \\ &= q^k\left(1+(1-q)[n]_q x\right) s_{n,k}^q(x). \end{aligned}$$

Therefore

$$s_{n,k}^q(qx) = q^k\left(1+(1-q)[n]_q x\right) s_{n,k}^q(x).$$

Lemma 2. *If we define the central moment as*

$$T_{n,m}(x) = G_n^q\left(t^m, x\right) := [n-1]_q \sum_{k=0}^{\infty} \mathrm{s}_{n,k}^q(x)\, q^k \int_0^{\infty/A} p_{n,k}^q(t)\, t^m d_q t$$

then we have

$$\begin{aligned} [m+1]_q T_{n,m}(qx) + q^{-1}[m+2]_q T_{n,m+1}(qx) = \big\{ & q^{-1}[n]_q T_{n,m+1}(qx) \\ & - [n]_q x T_{n,m}(qx) - \left(1+(1-q)[n]_q x\right) x D_q\left(T_{n,m}(x)\right)\big\} \end{aligned}$$

The following equalities hold:

(*i*) $T_{n,0}(x) = G_n^q(1,x) = 1$,

(*ii*) $T_{n,1}(x) = G_n^q(t,x) = \frac{[n]_q x}{q^2[n-2]_q} + \frac{1}{q[n-2]_q}$, *for* $n>1$,

(*iii*) $T_{n,2}(x) = G_n^q\left(t^2,x\right) = \frac{[n]_q^2 x^2}{q^6[n-2]_q[n-3]_q} + \frac{[n]_q x(1+q)^2}{q^5[n-2]_q[n-3]_q} + \frac{[2]_q}{q^3[n-2]_q[n-3]_q}$, *for* $n>3$,

Proof. Using Remark 1, we have

$$\begin{aligned} x D_q\left(T_{n,m}(x)\right) &= [n-1]_q \sum_{k=0}^{\infty} x D_q\left(s_{n,k}^q(x)\right) q^k \int_0^{\infty/A} p_{n,k}^q(t)\, t^m d_q t \\ &= [n]_q[n-1]_q \sum_{k=0}^{\infty} \left(\frac{[k]_q}{q^{k-2}[n]_q} - q^2 x \right) s_{n,k}^q(x)\, q^{2k-2} \\ &\quad \times \int_0^{\infty/A} p_{n,k}^q(t)\, t^m d_q t. \end{aligned}$$

Using Remark 1 and Remark 2, we have

$$\begin{aligned}
xD_q\left(T_{n,m}\left(x\right)\right) &= [n]_q[n-1]_q\sum_{k=0}^{\infty} s_{n,k}^q\left(x\right)q^{2k-2} \\
&\quad\times \int_0^{\infty/A}\left(\frac{[k]_q}{q^{k-2}[n]_q}-qt+qt-q^2x\right)p_{n,k}^q\left(t\right)t^m d_qt \\
&= [n]_q[n-1]_q\sum_{k=0}^{\infty} s_{n,k}^q\left(x\right)q^{2k-2}\int_0^{\infty/A}\left(\frac{[k]_q}{q^{k-1}[n]_q}-t\right)qp_{n,k}^q\left(t\right)t^m d_qt \\
&\quad+\ [n]_q[n-1]_q\sum_{k=0}^{\infty} s_{n,k}^q\left(x\right)q^{2k-2}\int_0^{\infty/A}\left(qt-q^2x\right)p_{n,k}^q\left(t\right)t^m d_qt \\
&= [n-1]_q\sum_{k=0}^{\infty} s_{n,k}^q\left(x\right)q^{2k-2}\int_0^{\infty/A}\left(q^2t^{m+1}+qt^{m+2}\right)D_qp_{n,k}^q\left(\frac{t}{q}\right)d_qt \\
&\quad+\ \frac{q^{-2}}{\left(1+\left(1-q\right)[n]_qx\right)}[n]_q[n-1]_q\sum_{k=0}^{\infty} s_{n,k}^q\left(qx\right)q^k \\
&\quad\times \int_0^{\infty/A}\left(qt-q^2x\right)p_{n,k}^q\left(t\right)t^m d_qt.
\end{aligned}$$

Using q-integration by parts we have

$$\begin{aligned}
xD_q\left(T_{n,m}\left(x\right)\right) &= -[n-1]_q\sum_{k=0}^{\infty} s_{n,k}^q\left(x\right)q^{2k-2} \\
&\quad\times \int_0^{\infty/A}\left(q^2[m+1]_qt^m+q[m+2]_qt^{m+1}\right)p_{n,k}^q\left(t\right)d_qt \\
&\quad+\ \frac{q^{-2}}{\left(1+\left(1-q\right)[n]_qx\right)}[n]_q[n-1]_q\sum_{k=0}^{\infty} s_{n,k}^q\left(qx\right)q^k \\
&\quad\times \int_0^{\infty/A}\left(qt-q^2x\right)p_{n,k}^q\left(t\right)t^m d_qt \\
&= -\frac{q^{-2}}{\left(1+\left(1-q\right)[n]_qx\right)}[n-1]_q\sum_{k=0}^{\infty} s_{n,k}^q\left(qx\right)q^k \\
&\quad\times \int_0^{\infty/A}\left(q^2[m+1]_qt^m+q[m+2]_qt^{m+1}\right)p_{n,k}^q\left(t\right)d_qt \\
&\quad+\ \frac{q^{-1}}{\left(1+\left(1-q\right)[n]_qx\right)}[n]_qT_{n,m+1}\left(qx\right)-\frac{1}{\left(1+\left(1-q\right)[n]_qx\right)}[n]_qxT_{n,m}\left(qx\right) \\
&= -\frac{1}{\left(1+\left(1-q\right)[n]_qx\right)}[m+1]_qT_{n,m}\left(qx\right)
\end{aligned}$$

$$- \frac{q^{-1}}{(1+(1-q)[n]_q x)}[m+2]_q T_{n,m+1}(qx)$$
$$+ \frac{q^{-1}}{(1+(1-q)[n]_q x)}[n]_q T_{n,m+1}(qx) - \frac{1}{(1+(1-q)[n]_q x)}[n]_q x T_{n,m}(qx).$$

This completes the proof of recurrence relation. The moments (i)-(iii) can be obtained easily by the above recurrence relation keeping in mind that $T_{n,0}(x) = 1$, which follows from (5) and (6) . ■

Remark 3. In case $q \to 1^-$, we get the central moments discussed in [14] and [7] as

$$G_n^1(1,x) = G_n(1,x) = 1,$$
$$G_n^1(t-x,x) = G_n(t-x,x) = \frac{1+2x}{n-2},$$
$$G_n^1\left((t-x)^2,x\right) = G_n\left((t-x)^2,x\right) = \frac{(n+6)x^2+2(n+3)x+2}{(n-2)(n-3)}.$$

3. Direct results

Let $C_B[0,\infty)$ be the space of all real-valued continuous bounded functions f on $[0,\infty)$, endowed with the norm $\|f\| = \sup_{x\in[0,\infty)} |f(x)|$. The Peetre's K-functional is defined by

$$K_2(f;\delta) = \inf_{g\in C_B^2[0,\infty)} \left\{\|f-g\| + \delta \left\|g''\right\|\right\},$$

where $C_B^2[0,\infty) := \{g \in C_B[0,\infty) : \ g', g'' \in C_B[0,\infty)\}$. By [[5], p. 177, Theorem 2.4] there exists an absolute constant $M > 0$ such that

$$K_2(f;\delta) \le M\omega_2(f;\sqrt{\delta}), \tag{9}$$

where $\delta > 0$ and the second order modulus of smoothness is defined as

$$\omega_2(f;\delta) = \sup_{0<h\le\delta} \sup_{x\in[0,\infty)} |f(x+2h) - 2f(x+h) + f(x)|,$$

where $f \in C_B[0,\infty)$ and $\delta > 0$. Also we set

$$\omega(f;\delta) = \sup_{0<h\le\delta} \sup_{x\in[0,\infty)} |f(x+h) - f(x)|, \tag{10}$$

$$\delta_n(x) = \left(\frac{[n]_q^2}{q^6[n-2]_q[n-3]_q} - 2\frac{[n]_q}{q^2[n-2]_q} + 1\right)x^2 + \left(\frac{[n]_q(1+q)^2}{q^5[n-2]_q[n-3]_q} - \frac{2}{q[n-2]_q}\right)x + \frac{[2]_q}{q^3[n-2]_q[n-3]_q},$$

$$\alpha_n(x) = \left(\frac{[n]_q}{q^2[n-2]_q} - 1\right)x + \frac{1}{q[n-2]_q}.$$

Lemma 3. *Let $f \in C_B[0,\infty)$. Then, for all $g \in C_B^2[0,\infty)$, we have*

$$\left|\widehat{G}_n^q(g;x) - g(x)\right| \leq (\delta_n(x) + \alpha_n^2(x))\left\|g''\right\|, \tag{11}$$

where

$$\widehat{G}_n^q(f;x) = G_n^q(f;x) + f(x) - f\left(\frac{[n]_q}{[n-2]_q}\left(\frac{1}{q^2}x + \frac{1}{q\,[n]_q}\right)\right). \tag{12}$$

Proof. From (12) we have

$$\begin{aligned}\widehat{G}_n^q(t-x;x) &= G_n^q(t-x;x) - \left(\frac{[n]_q}{[n-2]_q}\left(\frac{1}{q^2}x + \frac{1}{q\,[n]_q}\right) - x\right) \\ &= G_n^q(t;x) - xG_n^q(1;x) - \frac{[n]_q}{[n-2]_q}\left(\frac{1}{q^2}x + \frac{1}{q\,[n]_q}\right) + x \\ &= 0.\end{aligned} \tag{13}$$

Let $x \in [0,\infty)$ and $g \in C_B^2[0,\infty)$. Using the Taylor's formula

$$g(t) - g(x) = (t-x)g'(x) + \int_x^t (t-u)g''(u)du,$$

we can write by (13) that

$$\begin{aligned}\widehat{G}_n^q(g;x) - g(x) &= \widehat{G}_n^q((t-x)g'(x);x) + \widehat{G}_n^q\left(\int_x^t (t-u)g''(u)du;x\right) \\ &= g'(x)\widehat{G}_n^q((t-x);x) + G_n^q\left(\int_x^t (t-u)g''(u)du;x\right) \\ &\quad - \int_x^{\frac{[n]_q x}{q^2[n-2]_q} + \frac{1}{q[n-2]_q}} \left(\frac{[n]_q x}{q^2[n-2]_q} + \frac{1}{q[n-2]_q} - u\right)g''(u)du\end{aligned}$$

$$= G_n^q\left(\int_x^t (t-u)g''(u)du; x\right)$$

$$- \int_x^{\frac{[n]_q x}{q^2[n-1]_q}+\frac{1}{q[n-1]_q}} \left(\frac{[n]_q x}{q^2[n-2]_q} + \frac{1}{q[n-2]_q} - u\right) g''(u)du.$$

On the other hand, since

$$\left|\int_x^t (t-u)g''(u)du\right| \leq \left|\int_x^t |t-u|\,|g''(u)|\,du\right|$$

$$\leq \|g''\| \left|\int_x^t |t-u|\,du\right| \leq (t-x)^2\,\|g''\|$$

and

$$\left|\int_x^{\frac{[n]_q x}{q^2[n-2]_q}+\frac{1}{q[n-2]_q}} \left(\frac{[n]_q x}{q^2[n-2]_q} + \frac{1}{q[n-2]_q} - u\right) g''(u)du\right|$$

$$\leq \left(\frac{[n]_q x}{q^2[n-2]_q} + \frac{1}{q[n-2]_q} - x\right)^2 \|g''\|$$

$$= \left(\left(\frac{[n]_q}{q^2[n-2]_q} - 1\right)x + \frac{1}{q[n-2]_q}\right)^2 \|g''\| := \alpha_n^2(x)\,\|g''\|$$

we conclude that

$$\left|\widehat{G}_n^q(g;x) - g(x)\right| = \left|\widehat{G}_n^q\left(\int_x^t (t-u)g''(u)du; x\right)\right.$$

$$\left. - \int_x^{\frac{[n]_q x}{q^2[n-2]_q}+\frac{1}{q[n-2]_q}} \left(\frac{[n]_q x}{q^2[n-2]_q} + \frac{1}{q[n-2]_q} - u\right) g''(u)du\right|$$

$$\leq G_n^q((t-x)^2\,\|g''\|\,; x) + \left(\left(\frac{[n]_q}{q^2[n-2]_q} - 1\right)x + \frac{1}{q[n-2]_q}\right)^2 \|g''\|$$

$$= (\delta_n(x) + \alpha_n^2(x))\,\|g''\|\,.$$

■

Theorem 1. *Let $f \in C_B[0,\infty)$. Then, for every $x \in [0,\infty)$, there exists a constant $L > 0$ such that*

$$|G_n^q(f;x) - f(x)| \leq L\omega_2(f; \sqrt{(\delta_n(x) + \alpha_n^2(x))}) + \omega(f; \alpha_n(x)).$$

Proof. From (12), we can write that

$$\begin{aligned}
|G_n^q(f;x) - f(x)| &\leq \left|\widehat{G}_n^q(f;x) - f(x)\right| \\
&\quad + \left|f(x) - f\left(\frac{[n]_q}{[n-2]_q}\left(\frac{x}{q^2} + \frac{1}{[n]\,q}\right)\right)\right| \\
&\leq \left|\widehat{G}_n^q(f-g;x) - (f-g)(x)\right| \\
&\quad + \left|f(x) - f\left(\frac{[n]_q}{[n-2]_q}\left(\frac{x}{q^2} + \frac{1}{[n]\,q}\right)\right)\right| + \left|\widehat{G}_n^q(g;x) - g(x)\right| \\
&\leq \left|\widehat{G}_n^q(f-g;x)\right| + |(f-g)(x)| \\
&\quad + \left|f(x) - f\left(\frac{[n]_q}{[n-2]_q}\left(\frac{x}{q^2} + \frac{1}{[n]\,q}\right)\right)\right| + \left|\widehat{G}_n^q(g;x) - g(x)\right|.
\end{aligned}$$

Now, taking into account boundedness of $\widehat{G}_n^q$ and the inequality (11), we get

$$\begin{aligned}
|G_n^q(f;x) - f(x)| &\leq 4\,\|f-g\| + \left|f(x) - f\left(\frac{[n]_q}{[n-2]_q}\left(\frac{x}{q^2} + \frac{1}{q\,[n]_q}\right)\right)\right| \\
&\quad + (\delta_n(x) + \alpha_n^2(x))\,\|g''\| \\
&\leq 4\,\|f-g\| + \omega\left(f; \frac{[n]_q}{[n-2]_q}\left(\frac{1}{q^2} - 1\right)x + \frac{1}{[n-2]_q q}\right) \\
&\quad + (\delta_n(x) + \alpha_n^2(x))\,\|g''\|.
\end{aligned}$$

Now, taking infimum on the right-hand side over all $g \in C_B^2[0,\infty)$ and using (9), we get the following result

$$\begin{aligned}
|G_n^q(f;x) - f(x)| &\leq 4K_2(f; \delta_n(x) + \alpha_n^2(x)) + \omega(f; \alpha_n(x)) \\
&\leq 4M\omega_2(f; \sqrt{\delta_n(x) + \alpha_n^2(x)}) + \omega(f; \alpha_n(x)) \\
&= L\omega_2(f; \sqrt{\delta_n(x) + \alpha_n^2(x)}) + \omega(f; \alpha_n(x))
\end{aligned}$$

where $L = 4M > 0$. ■

Theorem 2. *Let $0 < \alpha \leq 1$ and $f \in C_B[0,\infty)$. Then, if $f \in Lip_M(\alpha)$, i.e. the condition*

$$|f(y) - f(x)| \leq M\,|y-x|^{\alpha}, \quad x, y \in [0,\infty), \tag{14}$$

holds, then, for each $x \in [0, \infty)$, *we have*

$$|G_n^q(f;x) - f(x)| \le M\delta_n^{\frac{\alpha}{2}}(x),$$

where δ_n *is the same as in Theorem 1,* M *is a constant depending on* α *and* f.

Proof. Let $f \in C_B[0,\infty) \cap Lip_M(\alpha)$ with $0 < \alpha \le 1$. By linearity and monotonicity of G_n^q

$$|G_n^q(f;x) - f(x)| = |G_n^q(f;x) - G_n^q(f(x);x)| \le G_n^q(|f(y) - f(x)|;x) \le MG_n^q(|y-x|;x).$$

Using the Hölder inequality with $p = \frac{2}{\alpha}$, $q = \frac{2}{2-\alpha}$ we find that

$$|G_n^q(f;x) - f(x)| \le M\left\{[G_n^q(|y-x|^{\alpha p};x)]^{\frac{1}{p}}[G_n^q(1^q;x)]^{\frac{1}{q}}\right\} = M\left[G_n^q(|y-x|^2;x)\right]^{\frac{\alpha}{2}} = M\delta_n^{\frac{\alpha}{2}}(x).$$

■

Theorem 3. *Let* f *be bounded and integrable on the interval* $[0,\infty)$, *second derivative of* f *exists at a fixed point* $x \in [0,\infty)$ *and* $q = q_n \in (0,\ 1)$ *such that* $q_n \to 1$ *as* $n \to \infty$, *then*

$$\lim_{n\to\infty}[n]_{q_n}[G_n^{q_n}(f,x) - f(x)] = (1+2x)f'(x) + \left(\frac{x^2}{2} + x\right)f''(x).$$

Proof. In order to prove this identity we use Taylor's expansion

$$f(t) - f(x) = (t-x)f'(x) + (t-x)^2\left(\frac{1}{2}f''(x) + \varepsilon(t-x)\right)$$

where ε is bounded ε is bounded and $\lim_{t\to 0}\varepsilon(t) = 0$. By applying the operator $G_n^q(f)$ to the above relation we obtain

$$\begin{aligned} G_n^{q_n}(f,x) - f(x) &= f'(x)\,G_n^{q_n}((t-x),x) + \frac{1}{2}f''(x)\,G_n^{q_n}((t-x)^2,x) \\ &\quad + G_n^{q_n}(\varepsilon(t-x)(t-x)^2,x) \\ &= f'(x)\,\alpha_n(x) + \frac{1}{2}f''(x)\,\delta_n(x) + G_n^{q_n}(\varepsilon(t-x)(t-x)^2,x), \end{aligned}$$

where $\alpha_n(x)$ and $\delta_n(x)$ defined as in (10).

Using Cauchy-Schwarz inequality we have

$$[n]_{q_n}G_n^{q_n}(\varepsilon(t-x)(t-x)^2,x) \le \left(G_n^{q_n}(\varepsilon^2(t-x))\right)^{\frac{1}{2}}\left([n]_{q_n}^2 G_n^{q_n}((t-x)^4,x)\right)^{\frac{1}{2}}.$$

Using Lemma 1, we can show that

$$\lim_{n\to\infty} [n]_{q_n}^2 \, G_n^{q_n}\left((t-x)^4, x\right) = 0$$

Also, since

$$\lim_{n\to\infty} \alpha_n(x) = 1 + 2x \quad \text{and} \quad \lim_{n\to\infty} \delta_n(x) = x^2 + 2x$$

we have desired result. ∎

4. Error estimation

The usual modulus of continuity of f on the closed interval $[0,b]$ is defined by

$$\omega_b(f,\delta) = \sup_{\substack{|t-x|\le\delta \\ x,t\in[0,b]}} |f(t)-f(x)|, \quad b>0.$$

It is well known that, for a function $f \in E$,

$$\lim_{\delta\to 0^+} \omega_b(f,\delta) = 0,$$

where

$$E := \left\{ f \in C[0,\infty) : \lim_{x\to\infty} \frac{f(x)}{1+x^2} \text{ is finite} \right\}.$$

The next theorem gives the rate of convergence of the operators $G_n^q(f;x)$ to $f(x)$, for all $f \in E$.

Theorem 4. *Let $f \in E$ and let $\omega_{b+1}(f,\delta)$ $(b>0)$ be its modulus of continuity on the finite interval $[0,b+1] \subset [0,\infty)$. Then for fixed $q \in (0,1)$, we have*

$$\left\| G_n^q(f;x) - f(x) \right\|_{C[0,b]} \le N_f \left(1+b^2\right) \delta_n(b) + 2\omega_{b+1}(f, \sqrt{\delta_n(b)}).$$

Proof. The proof is based on the following inequality

$$(15) \qquad \left| G_n^q(f;x) - f(x) \right| \le N_f \left(1+b^2\right) G_n^q((t-x)^2;x) + \left(1 + \frac{G_n^q(|t-x|;x)}{\delta}\right) \omega_{b+1}(f,\delta)$$

for all $(x,t) \in [0,b] \times [0,\infty) := S$.

To prove (15) we write

$$S = S_1 \cup S_2 := \{(x,t) : 0 \le x \le b,\ 0 \le t \le b+1\} \cup \{(x,t) : 0 \le x \le b,\ t > b+1\}.$$

If $(x,t) \in S_1$, we can write

$$|f(t) - f(x)| \leq \omega_{b+1}(f, |t-x|) \leq \left(1 + \frac{|t-x|}{\delta}\right)\omega_{b+1}(f,\delta) \tag{16}$$

where $\delta > 0$. On the other hand, if $(x,t) \in S_2$, using the fact that $t - x > 1$, we have

$$\begin{aligned} |f(t) - f(x)| &\leq M_f(1 + x^2 + t^2) \\ &\leq M_f(2 + 3x^2 + 2(t-x)^2) \\ &\leq N_f\left(1 + b^2\right)(t-x)^2 \end{aligned} \tag{17}$$

where $N_f = 6M_f$. Combining (16) and (17), we get (15).

Now from (15) it follows that

$$\begin{aligned} |G_n^q(f;x) - f(x)| &\leq N_f\left(1+b^2\right)G_n^q((t-x)^2;x) \\ &\quad + \left(1 + \frac{G_n^q(|t-x|\,;x)}{\delta}\right)\omega_{b+1}(f,\delta) \\ &\leq N_f\left(1+b^2\right)G_n^q((t-x)^2;x) \\ &\quad + \left(1 + \frac{\left[G_n^q((t-x)^2;x)\right]^{1/2}}{\delta}\right)\omega_{b+1}(f,\delta). \end{aligned}$$

By Lemma 2 we have

$$G_n^q((t-x)^2;x) \leq \delta_n(b)$$

$$|G_n^q(f;x) - f(x)| \leq N_f\left(1+b^2\right)\delta_n(b) + \left(1 + \frac{\sqrt{\delta_n(b)}}{\delta}\right)\omega_{b+1}(f,\delta).$$

Choosing $\delta = \sqrt{\delta_n(b)}$, we get the desired estimation. ■

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Vijay Gupta
School of Applied Sciences
Netaji Subhas Institute of Technology
Sector 3 Dwarka, New Delhi 110078 India
e-mail: vijaygupta2001@hotmail.com

Ali Aral and Muzeyyen Ozhavzali
Kirikalle University
Faculty of Science and Arts
Department of Mathematics
Yahşihan, Turkey
e-mail: aliaral73@yahoo.com *or* thavzalimuzeyyen@hotmail.com

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