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WEAK CONTRACTION MAPPINGS IN SAKS SPACES

Abstract. The intent of this note is to prove some fixed point and common fixed theorems in a Saks spaces by introducing a weaker inequality analogue to Albert and Delabriere [1]. We have also introduced a control functions which is certainly weaker contraction condition available in the literature of Metric Fixed Point Theory and Applications.

Key words: altering distance function, weak inequalities, fixed point, Saks space, metric space.

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1. Introduction

The Banach contraction mapping principle is widely recognized as the source of metric fixed point theory.

A mapping $T : X \to X$, where (X, d) is a metric space is said to be a contraction mapping if for all $x, y \in X$,

$$d(Tx, Ty) \leq \lambda d(x, y), \quad \text{where} \quad 0 < \lambda < 1. \tag{1}$$

According to the contraction mapping principle, any mapping T satisfying (1) in a complete metric space will have a unique fixed point. This principle has been generalised in different directions by mathematicians over the years. Also in the contemporary research it remains a heavily investigated branch of research. The works noted in [1], [2], [4], [10], [14] [16] and [22] are some examples from this line of research.

Throughout this paper, $(X_s, d) = (X, N_1, N_2)$ denotes a Saks space, and N_1 is equivalent to N_2 on X. In brief we shall define X as a Saks space. The following lemma due to Orlicz [26] is useful for the proof of our main theorem:

Lemma 1. *Let X be a Saks space. Then the following statements are equivalent:*

(i) N_1 is equivalent to N_2 on X.

(ii) (X, N_1) *is a Banach space and* $N_1 \leq N_2$ *on* X.
(iii) (X, N_2) *is a Frechet space and* $N_2 \leq N_1$ *on* X.

In [1] Alber and Guerre-Delabriere introduced the concept of weak contraction in Hilbert spaces. Rhoades [18] has shown that the result which Alber et al. had proved in [1] is also valid in complete metric spaces. We state the result of Rhoades which follows:

Definition 1 (Weakly contractive mapping). *A mapping* $T : X \to X$, *where* $(X_s, d) = (X, N_1, N_2)$ *is a Saks space is said to be weakly contractive if*

$$N_2(Tx - Ty) \leq N_2(x - y) - \phi(N_2(x - y)), \tag{2}$$

where $x, y \in X$ *and* $\phi : [0, \infty) \to [0, \infty)$ *is a continuous and nondecreasing function such that* $\phi(t) = 0$ *if and only if* $t = 0$.

If we take $\phi(t) = (1 - \lambda)t$ where $0 < \lambda < 1$, then (2) reduces to in the metric space setting.

Theorem 1 ([18]). *If* $T : X \to X$ *is a weakly contractive mapping, where* (X, d) *is a complete metric space, then* T *has a unique fixed point.*

Weak inequalities of the above type have been used to establish fixed point results in a number of subsequent works some of which are noted in [5], [6], [13], [21] and [23].

There is another important generalization of the Banach contraction principle given by Khan et al. in [15] where they used a control function, and called altering distance function.

Definition 2 (Altering distance function [15]). *A function* $\psi : [0, \infty) \to [0, \infty)$ *is called an altering distance function if the following properties are satisfied:*
(i) ψ *is monotone increasing and continuous*
(ii) $\psi(t) = 0$ *if and only if* $t = 0$.

The following generalisation of the Banach contraction mapping principle was proved in [15].

Theorem 2 ([15]). *Let* (X, d) *be a complete metric space,* ψ *be an altering distance function and* $f : X \to X$ *be a self mapping which satisfies the following inequality:*

$$\psi(d(fx, fy)) \leq \lambda\psi(d(x, y)) \tag{3}$$

for all $x, y \in X$ *and for some* $0 < \lambda < 1$, *then* f *has a unique fixed point.*

In fact Khan et al. had proved a more general theorem [15, Theorem 2] of which the above result is a corollary.

Altering distance has been used in metric fixed point theory in a number of papers, some of which are noted in [17], [19] and [20]. In [7], [8] and [3] respectively, two, three and four variable generalizations of altering distance function have been introduced and applied to fixed point problems. It has also been extended to the case of multivalued and fuzzy mappings [9]. The concept of altering distance function has also been extended to fixed point problems in Menger spaces ([10], [11] and [12]).

The purpose of this paper is to work out fixed point results for mappings in metric spaces by use of weak inequalities and the altering distance function.

2. Main result

Theorem 3. *Let* $(X_s, d) = (X, N_1, N_2)$ *is a Saks space in which* N_1 *is equivalent to* N_2 *on* X*. Let* $T : X \to X$ *be a self mapping which satisfies the following inequality:*

$$\Psi(N_2(Tx - Ty)) \leq \Psi(M(x, y)) - \Phi(N(x, y)) \tag{4}$$

where $x, y \in X$*,* $x \neq y$*,*

$$M(x,y) = \max\left\{N_2(x-y), \frac{1}{2}(N_2(x-Tx) + N_2(y-Ty)), \frac{1}{2}(N_2(y-Tx) + N_2(x-Ty))\right\}, \tag{5}$$

$$N(x,y) = \min\{N_2(x-y), \frac{1}{2}(N_2(y-Tx) + N_2(x-Ty))\}, \tag{6}$$

$\Phi : [0, \infty) \to [0, \infty)$ *is a lower semi continuous function with* $\Phi(t) > 0$ *for all* $t \in (0, \infty)$ *and* $\Phi(0) = 0$ *and* $\Psi : [0, \infty) \to [0, \infty)$ *is an altering distance function (Definition 2) which in addition is strictly monotone increasing. Then there is a unique fixed point of* T*.*

Proof. Let $x_0 \in X$. We define a sequence $\{x_n\}$ in X, such that for all $n \geq 0$,

$$x_{n+1} = Tx_n \tag{7}$$

If $x_n = x_{n+1}$, then x_n is a fixed point of T. Hence we assume for all $n \geq 0$,

$$x_n \neq x_{n+1}. \tag{8}$$

Putting $x = x_n$ and $y = x_{n+1}$ in (4), we have

$$\begin{aligned}\Psi(N_2(x_{n+1} - x_{n+2})) &= \Psi(N_2(Tx_n - Tx_{n+1})) \\ &\leq \Psi(M(x_n, x_{n+1})) - \Phi(N(x_n, x_{n+1})).\end{aligned} \tag{9}$$

Now,

$$\begin{aligned}M(x_n, x_{n+1}) = \max\Big\{&N_2(x_n - x_{n+1}), \\ &\frac{1}{2}(N_2(x_n - x_{n+1}) + N_2(x_{n+1} - x_{n+2})), \\ &\frac{1}{2}(N_2(x_{n+1} - x_{n+1}) + N_2(x_n - x_{n+2}))\Big\}\end{aligned} \tag{10}$$

and

$$\begin{aligned}N(x_n, x_{n+1}) = \min\Big\{&N_2(x_n - x_{n+1}), \\ &\frac{1}{2}(N_2(x_{n+1} - x_{n+1}) + N_2(x_n - x_{n+2}))\Big\}.\end{aligned} \tag{11}$$

If possible, let for some n, $N_2(x_n - x_{n+1}) < N_2(x_{n+1} - x_{n+2})$.

Then by the triangular inequality

$$0 < N_2(x_{n+1} - x_{n+2}) - N_2(x_n - x_{n+1}) \leq N_2(x_n - x_{n+2}).$$

Hence by virtue of (8), we have $N(x_n, x_{n+1}) > 0$. Then from (9), (10), (11) and our assumption, we have

$$\begin{aligned}\Psi(N_2(x_{n+1} - x_{n+2})) &\leq \Psi(N_2(x_{n+1} - x_{n+2})) - \Phi(N(x_n, x_{n+1})) \\ &< \Psi(N_2(x_{n+1} - x_{n+2})),\end{aligned}$$

which is a contradiction. Hence for all $n \geq 0$,

$$N_2(x_{n+1} - x_{n+2}) \leq N_2(x_n - x_{n+1}) \tag{12}$$

In view of (12), we obtain from (10) and (11), for all $n \geq 0$,

$$M(x_n, x_{n+1}) = N_2(x_n - x_{n+1}). \tag{13}$$

$$N(x_n, x_{n+1}) = \frac{1}{2}\left(N_2(x_n - x_{n+2})\right). \tag{14}$$

Putting (13) and (14) in (9), we have for all $n \geq 0$,

$$\Psi(N_2(x_{n+1} - x_{n+2})) \leq \Psi(N_2(x_n - x_{n+1})) - \Phi(\frac{1}{2}\left(N_2(x_n - x_{n+2})\right)). \tag{15}$$

Again (12) implies that the sequence $\{N_2(x_n - x_{n+1})\}$ is a monotone decreasing sequence of non-negative real numbers. Hence there exists $r \geq 0$ such that

$$\lim_{n\to\infty} N_2(x_n - x_{n+1}) = r.$$

Again

$$N_2(x_n - x_{n+2}) - 2r \leq N_2(x_n, x_{n+1}) + N_2(x_{n+1} - x_{n+2}) - 2r.$$

This implies that

$$|N_2(x_n - x_{n+2}) - 2r| \leq |N_2(x_n - x_{n+1}) - r| + |N_2(x_{n+1} - x_{n+2}) - r| \to 0$$

as $n \to \infty$. Thus

$$\lim_{n\to\infty} N_2(x_n - x_{n+2}) = 2r.$$

Making $n \to \infty$ in (15), by the continuity of Ψ-function and the lower semi continuity of Φ -function, we have $\Psi(r) \leq \Psi(r) - \Phi(\frac{1}{2}r)$, which by the properties of Ψ-function and Φ-function implies that $r = 0$. Hence we have,

$$\lim_{n\to\infty} N_2(x_n - x_{n+1}) = 0. \tag{16}$$

Next we show that $\{x_n\}$ is a Cauchy sequence. If otherwise, there exists $\epsilon > 0$ and sequences of natural numbers $\{m(k)\}$ and $\{n(k)\}$ such that for every natural number k,

$$n(k) > m(k) > k \tag{17}$$

and

$$N_2(x_{m(k)} - x_{n(k)}) \geq \epsilon. \tag{18}$$

Corresponding to $m(k)$ we can choose $n(k)$ to be the smallest integer such that (18) is satisfied. Then we have

$$N_2(x_{m(k)} - x_{n(k)-1}) < \epsilon. \tag{19}$$

Further, (18) implies $N_2(Tx_{m(k)-1} - Tx_{n(k)-1}) \neq 0$. Hence $x_{m(k)-1} \neq x_{n(k)-1}$. Putting $x = x_{m(k)-1}$ and $y = x_{n(k)-1}$ in (4), (5) and (6) respectively, we have for all k,

$$\begin{aligned} \Psi(N_2(x_{m(k)} - x_{n(k)})) &= \Psi(N_2(Tx_{m(k)-1} - Tx_{n(k)-1})) \\ &\leq \Psi(M(x_{m(k)-1}, x_{n(k)-1})) \\ &\quad - \Phi(N(x_{m(k)-1}, x_{n(k)-1})) \end{aligned} \tag{20}$$

where

$$M(x_{m(k)-1}, x_{n(k)-1} = \max\Big\{N_2(x_{m(k)-1} - x_{n(k)-1}), \frac{1}{2}(N_2(x_{m(k)-1}, x_{m(k)}) + N_2(x_{n(k)-1} - x_{n(k)})), \frac{1}{2}(N_2(x_{n(k)-1} - x_{m(k)}) + N_2(x_{m(k)-1} - x_{n(k)}))\Big\} \tag{21}$$

and

$$N(x_{m(k)-1}, x_{n(k)-1}) = \min\Big\{N_2(x_{m(k)-1} - x_{n(k)-1}), \frac{1}{2}(N_2(x_{n(k)-1} - x_{m(k)}) + N_2(x_{m(k)-1} - x_{n(k)}))\Big\}. \tag{22}$$

Then for every positive integer k we have,

$$\begin{aligned} \varepsilon &\leq N_2(x_{m(k)}, x_{n(k)}) \\ &\leq N_2(x_{m(k)} - x_{n(k)-1}) + N_2(x_{n(k)-1} - x_{n(k)}) \\ &< \varepsilon + N_2(x_{n(k)-1} - x_{n(k)}) \qquad \text{(by (2.16))} \end{aligned}$$

Making $k \to \infty$ in the above inequality, we obtain by(16),

$$\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \epsilon. \tag{23}$$

Again for all k,

$$\begin{aligned} N_2(x_{m(k)-1} - x_{n(k)-1}) \leq\ & N_2(x_{m(k)-1} - x_{m(k)}) \\ &+ N_2(x_{m(k)} - x_{n(k)}) + N_2(x_{n(k)} - x_{n(k)-1}) \end{aligned}$$

and

$$\begin{aligned} N_2(x_{m(k)} - x_{n(k)}) \leq\ & N_2(x_{m(k)} - x_{m(k)-1}) \\ &+ N_2(x_{m(k)-1} - x_{n(k)-1}) + N_2(x_{n(k)-1} - x_{n(k)}). \end{aligned}$$

Making $k \to \infty$ and using (16) and (23) in the above two inequalities, we obtain,

$$\lim_{k\to\infty} N_2(x_{m(k)-1} - x_{n(k)-1}) = \epsilon. \tag{24}$$

Again for all positive integer k,

$$N_2(x_{m(k)-1} - x_{n(k)}) \leq N_2(x_{m(k)-1} - x_{m(k)}) + N_2(x_{m(k)} - x_{n(k)})$$

and

$$N_2(x_{m(k)} - x_{n(k)}) \le N_2(x_{m(k)} - x_{m(k)-1}) + N_2(x_{m(k)-1} - x_{n(k)}).$$

Making $k \to \infty$ and using (16) and (23) in the above inequalities, we have,

$$\lim_{k\to\infty} N_2(x_{m(k)-1} - x_{n(k)}) = \epsilon. \tag{25}$$

Also for every positive integer k,

$$N_2(x_{n(k)-1} - x_{m(k)}) \le N_2(x_{n(k)-1} - x_{n(k)}) + N_2(x_{n(k)} - x_{m(k)})$$

and

$$N_2(x_{n(k)} - x_{m(k)}) \le N_2(x_{n(k)} - x_{n(k)-1}) + N_2(x_{n(k)-1} - x_{m(k)}).$$

Making $k \to \infty$ in the above inequalities, we have using (16) and (23),

$$\lim_{k\to\infty} N_2(x_{n(k)-1} - x_{m(k)}) = \epsilon. \tag{26}$$

Making $k \to \infty$ in (20) and using (16), (21) - (26), we have by continuity of Ψ-function and lower semi continuity of Φ-function,

$$\Psi(\epsilon) \le \Psi(\epsilon) - \Phi(\epsilon).$$

Then we have by virtue of a property of Φ -function that it is a contradiction with $\epsilon > 0$. Hence $\{x_n\}$ is a Cauchy sequence with respect to N_1. From Lemma 1, (X, N_1) is a Banach space, therefore and therefore $\{x_n\}$ be a convergent sequence and converges to a point z in X.

Let,

$$x_n \to z \quad \text{as} \quad n \to \infty. \tag{27}$$

By (8), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $z \ne x_{n(k)}$ for all k. Substituting $x = x_{n(k)}$ and $y = z$ in (4), (5) and (6) we obtain

$$\Psi(N_2(x_{n(k)+1} - Tz)) \le \Psi(M(x_{n(k)} - z)) - \Phi(N(x_{n(k)} - z)) \tag{28}$$

where

$$M(x_{n(k)}, z) = \max\Big\{N_2(x_{n(k)} - z), \frac{1}{2}(N_2(x_{n(k)} - x_{n(k)+1}) + N_2(z - Tz)), \frac{1}{2}(N_2(x_{n(k)} - Tz) + N_2(z - x_{n(k)+1}))\Big\}, \tag{29}$$

$$N(x_{n(k)}, z) = \min\left\{N_2(x_{n(k)} - z), \frac{1}{2}(N_2(x_{n(k)} - Tz) + N_2(z - x_{n(k)+1}))\right\}. \tag{30}$$

Making $k \to \infty$ in the above inequalities, we obtain $\Psi(N_2(z - Tz)) \leq \Psi(\frac{1}{2}(N_2(z-Tz)))$, which contradicts the strict monotone increasing property of the function ψ unless $N_2(z - Tz) = 0$, that is, $z = Tz$. Hence z is a fixed point of T.

We next establish that the fixed point is unique. Let z_1 and z_2 be two fixed points of T and $z_1 \neq z_2$, then putting $x = z_1$ and $y = z_2$ in (4), (5) and (6) respectively, we obtain,

$$\Psi(N_2(z_1 - z_2)) \leq \Psi(N_2(z_1 - z_2)) - \Phi(N_2(z_1 - z_2)),$$

which, by virtue of a property of Φ functions implies $N_2(z_1 - z_2) = 0$, that is $z_1 = z_2$.

This completes the proof of the theorem. ∎

Theorem 4. *Let $(X_s, d) = (X, N_1, N_2)$ is a Saks space in which N_1 is equivalent to N_2 on X and let $T : X \to X$ be such that for all $x, y \in X$ with $x \neq y$, the following inequality is satisfied:*

$$\Psi(N_2(Tx - Ty)) \leq \Psi(M(x, y)) - h(Q(x, y)), \tag{31}$$

where

$$M(x, y) = \max\left\{\{N_2(x - y), \frac{1}{2}(N_2(x - Tx) + N_2(y - Ty)), \frac{1}{2}(N_2(y - Tx) + N_2(x - Ty))\right\}, \tag{32}$$

$$Q(x, y) = \min\left\{N_2(x - y), \frac{1}{2}(N_2(x - Tx) + N_2(y - Ty)), \frac{1}{2}(N_2(y - Tx) + N_2(x - Ty))\right\}, \tag{33}$$

where $h : [0, \infty) \to [0, \infty)$ is such that $h(t) > 0$ and lower semi-continuous for all $t > 0$, h is discontinuous at $t = 0$ with $h(0) = 0$, and $\Psi : [0, \infty) \to [0, \infty)$ is an altering distance function. Then T has a unique fixed point.

Proof. Starting with arbitrary $x_0 \in X$, we construct the sequence $\{x_n\}$ as in (7). Further we assume (8) for all $n \geq 0$, otherwise the fixed point of T automatically exists. Putting $x = x_n$ and $y = x_{n+1}$ in (31), for all $n \geq 0$,

$$\Psi(N_2(x_{n+1}, x_{n+2})) \leq \Psi(M(x_n, x_{n+1})) - h(Q(x_n, x_{n+1})), \tag{34}$$

where,

$$(35)\quad M(x_n, x_{n+1}) = \max\Big\{N_2(x_n - x_{n+1}), \frac{1}{2}(N_2(x_n - x_{n+1}) + N_2(x_{n+1} - x_{n+2})), \frac{1}{2}(N_2(x_{n+1} - x_{n+1}) + N_2(x_n - x_{n+2}))\Big\}$$

and

$$(36)\quad Q(x_n, x_{n+1}) = \min\Big\{N_2(x_n - x_{n+1}), \frac{1}{2}(N_2(x_n - x_{n+1}) + N_2(x_{n+1} - x_{n+2})), \frac{1}{2}(N_2(x_{n+1} - x_{n+1}) + N_2(x_n - x_{n+2}))\Big\}.$$

If possible, let for some n,

$$N_2(x_n - x_{n+1}) < N_2(x_{n+1} - x_{n+2}).$$

Then by the triangular inequality $0 < N_2(x_{n+1} - x_{n+2}) - N_2(x_n - x_{n+1}) \leq N_2(x_n - x_{n+2})$. Then from (34), (35) and (36) we have by the properties of h-function

$$\Psi(N_2(x_{n+1} - x_{n+2})) < \Psi(N_2(x_{n+1} - x_{n+2})),$$

which is a contradiction. Hence for all $n \geq 0$,

$$(37)\qquad N_2(x_{n+1} - x_{n+2}) \leq N_2(x_n - x_{n+1}).$$

In view of (37), we obtain from (35) and (36) respectively, for all $n \geq 0$,

$$M(x_n, x_{n+1}) = N_2(x_n - x_{n+1}) \quad \text{and} \quad Q(x_n, x_{n+1}) = \frac{1}{2}(N_2(x_n - x_{n+2})).$$

Using the above relations we have for all $n \geq 0$,

$$(38)\qquad \Psi(N_2(x_{n+1} - x_{n+2})) \leq \Psi(N_2(x_n - x_{n+1})) - h(\frac{1}{2}N_2(x_n - x_{n+2})).$$

Again, (37) implies that the sequence $\{N_2(x_n - x_{n+1})\}$ is a monotone decreasing sequence of non-negative real numbers. Hence there exists $r \geq 0$ such that $\lim\limits_{n\to\infty} N_2(x_n - x_{n+1}) = r$.

As already observed in the proof of Theorem 3, in view of the above limit we have that

$$\lim_{n\to\infty} N_2(x_n - x_{n+2}) = 2r.$$

If possible, let $r > 0$. Making $n \to \infty$ in (38) and using the above relations, by continuity of Ψ-function and by lower semi-continuity of h-function, we have $\Psi(r) \leq \Psi(r) - h(r)$, which by a property of h-function implies a contradiction. Hence we have,

$$\lim_{n\to\infty} N_2(x_n - x_{n+1}) = 0. \tag{39}$$

Next we prove that $\{x_n\}$ is a Cauchy sequence. If otherwise, we can have some $\epsilon > 0$ and corresponding subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ such that for every natural number k, $n(k) > m(k) > k$ and

$$d(x_{m(k)}, x_{n(k)}) \geq \epsilon \tag{40}$$

and

$$N_2(x_{m(k)} - x_{n(k)-1}) < \epsilon. \tag{41}$$

From (40), $N_2(Tx_{m(k)-1} - Tx_{n(k)-1}) \neq 0$, hence $x_{m(k)-1} \neq x_{n(k)-1}$.

Further, proceeding identically as in Theorem 3, we have,

$$\lim_{k\to\infty} N_2(x_{m(k)} - x_{n(k)}) = \epsilon, \tag{42}$$

$$\lim_{k\to\infty} N_2(x_{m(k)} - x_{n(k)-1}) = \epsilon, \tag{43}$$

$$\lim_{k\to\infty} N_2(x_{m(k)-1} - x_{n(k)}) = \epsilon, \tag{44}$$

and

$$\lim_{k\to\infty} N_2(x_{m(k)-1} - x_{n(k)-1}) = \epsilon. \tag{45}$$

Now putting $x = x_{m(k)-1}$ and $y = x_{n(k)-1}$ in (31), (32) and (33) we get, for all $k \geq 0$,

$$\begin{aligned} \Psi(N_2(x_{m(k)} - x_{n(k)})) \leq\ & \Psi(M(x_{m(k)-1} - x_{n(k)-1})) \\ & - h(Q(x_{m(k)-1} - x_{n(k)-1})). \end{aligned} \tag{46}$$

Now,

$$\begin{aligned} M(x_{m(k)-1}, x_{n(k)-1}) = \max\Big\{ & N_2(x_{m(k)-1} - x_{n(k)-1}), \\ & \frac{1}{2}(N_2(x_{m(k)-1} - x_{m(k)}) + N_2(x_{n(k)-1} - x_{n(k)})), \\ & \frac{1}{2}(N_2(x_{m(k)-1} - x_{n(k)}) + N_2(x_{n(k)-1} - x_{m(k)}))\Big\} \end{aligned} \tag{47}$$

and

$$Q(x_{m(k)-1}, x_{n(k)-1}) = \min\Big\{N_2(x_{m(k)-1} - x_{n(k)-1}), \frac{1}{2}(N_2(x_{m(k)-1} - x_{m(k)}) + N_2(x_{n(k)-1} - x_{n(k)})), \frac{1}{2}(N_2(x_{m(k)-1} - x_{n(k)}) + N_2(x_{n(k)-1} - x_{m(k)}))\Big\}. \tag{48}$$

Making $k \to \infty$ and using (39), (42), (43), (44) and (45), we obtain from (47) and (48)

$$\lim_{k\to\infty} M(x_{m(k)-1} - x_{n(k)-1}) = \epsilon \tag{49}$$

$$\lim_{k\to\infty} Q(x_{m(k)-1} - x_{n(k)-1}) = 0 \tag{50}$$

Further, making $k \to \infty$ in (2.43), using (2.46) and (2.47), by continuity of Ψ we obtain,

$$\Psi(\epsilon) \leq \Psi(\epsilon) - \lim_{k\to\infty} h(Q(x_{m(k)-1} - x_{n(k)-1})). \tag{51}$$

By (48) and the fact that h has a discontinuity at $t = 0$ and $h(t) > 0$ for $t > 0$, we observe that the last term of the right hand side of the above inequality is non zero. Hence we arrive at a contradiction. Hence $\{x_n\}$ is a Cauchy sequence with respect to N_1. From Lemma 1, (X, N_1) is a Banach space, therefore and therefore $\{x_n\}$ be a convergent sequence and converges to a point z in X.

Let,

$$x_n \to z \quad as \quad n \to \infty.$$

By (8), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $z \neq x_{n(k)}$ for all $k \geq 0$. Substituting $x = x_{n(k)}$ and $y = z$ in (31), (32) and (33), we obtain,

$$\Psi(d(x_{n(k)+1}, Tz)) \leq \Psi(M(x_{n(k)}, z)) - h(Q(x_{n(k)}, z))$$

where,

$$M(x_{n(k)}, z) = \max\Big\{N_2(x_{n(k)} - z), \frac{1}{2}(N_2(x_{n(k)} - x_{n(k)+1}) + N_2(z - Tz)), \frac{1}{2}(N_2(x_{n(k)} - Tz) + N_2(z - x_{n(k)+1}))\Big\},$$

$$Q(x_{n(k)}, z) = \min\Big\{N_2(x_{n(k)} - z), \frac{1}{2}(N_2(x_{n(k)} - x_{n(k)+1}) + N_2(z - Tz)), \frac{1}{2}(N_2(x_{n(k)} - Tz) + N_2(z - x_{n(k)+1}))\Big\}.$$

Making $k \to \infty$ in the above expressions and using discontinuity of h, we obtain,

$$\Psi(N_2(z - Tz)) < \Psi(\frac{1}{2}(N_2(z - Tz))),$$

which implies that $N_2(z - Tz) = 0$, that is, $z = Tz$. Hence z is a fixed point of T.

We next establish that the fixed point is unique. Let z_1 and z_2 be two fixed points of T and $z_1 \neq z_2$. Putting $x = z_1$ and $y = z_2$ in (31), (32) and (33), we obtain,

$$\Psi(d(z_1, z_2)) \leq \Psi(N_2(z_1 - z_2)) - h(N_2(z_1 - z_2)),$$

which by virtue of a property of h function implies $N_2(z_1 - z_2) = 0$, that is $z_1 = z_2$.

This completes the proof of the Theorem 4. ■

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