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ON A NEW TYPE BERNSTEIN-STANCU OPERATORS

Abstract. In the present paper, firstly we obtain a functional differential equation corresponding to new type Bernstein-Stancu operators defined in [8]. Next we introduce some properties of these new type operators. In the end k-th order generalization of such operators is established.

Key words: functional differential equation, recurrence relation, k-th order generalization.

AMS Mathematics Subject Classification: 41A25, 41A36.

1. Introduction

For a function f defined on the interval $[0,1]$ the classical Bernstein polynomials

$$B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k} \tag{1}$$

were introduced by Bernstein [2]. In 1968 Stancu [15] proposed the following generalization of the classical Bernstein polynomials so called Bernstein-Stancu polynomials

$$B_n^{\alpha,\beta}(f;x) = \sum_{k=0}^{n} f\left(\frac{k+\alpha}{n+\beta}\right)\binom{n}{k} x^k (1-x)^{n-k} \tag{2}$$

for a function f defined on the interval $[0,1]$ and each real α and β such that $0 \le \alpha \le \beta$. There are many papers including the properties of the operators (1) and (2) and their generalizations in the literature.

Very recently, in [8] Gadjiev and Ghorbanalizadeh have constructed the Bernstein-Stancu type polynomials

$$\begin{aligned} S_{n,\alpha,\beta}(f;x) \; = \; & \left(\frac{n+\beta_2}{n}\right)^n \sum_{r=0}^{n} f\left(\frac{r+\alpha_1}{n+\beta_1}\right) \\ & \times C_n^r \left(x - \frac{\alpha_2}{n+\beta_2}\right)^r \left(\frac{n+\alpha_2}{n+\beta_2} - x\right)^{n-r}, \end{aligned} \tag{3}$$

where $C_n^r = \binom{n}{r}$, $x \in I_n := \left[\frac{\alpha_2}{n+\beta_2}, \frac{n+\alpha_2}{n+\beta_2}\right]$, α_1, α_2, β_1, β_2 are real numbers such that $0 \le \alpha_2 \le \alpha_1 \le \beta_1 \le \beta_2$, $f \in C[0,1]$ and $n \in \mathbb{N}$. The authors have proved a convergence theorem in the moved interval I_n and computed the order of approximation with the help of the usual modulus of continuity for these operators. They also have introduced two variables extension of such operators and numerical examples.

Observe that as $n \to \infty$ the interval I_n turns out to be the interval $[0,1]$ and for $\alpha_2 = \beta_2 = 0$ and $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0$ the operators $S_{n,\alpha,\beta}(f;x)$ reduce to $B_n^{\alpha,\beta}(f;x)$ and the classical Bernstein polynomials $B_n(f;x)$, respectively. From [8] the following identities hold

$$S_{n,\alpha,\beta}(1;x) = 1 \tag{4}$$

$$S_{n,\alpha,\beta}(t;x) = \left(\frac{n+\beta_2}{n+\beta_1}\right)x - \left(\frac{\alpha_2-\alpha_1}{n+\beta_1}\right) \tag{5}$$

$$\begin{aligned} S_{n,\alpha,\beta}(t^2;x) &= \left(1-\frac{1}{n}\right)\left(\frac{n+\beta_2}{n+\beta_1}\right)^2\left(x-\frac{\alpha_2}{n+\beta_2}\right)^2 \\ &\quad + (1+2\alpha_1)\frac{n+\beta_2}{(n+\beta_1)^2}\left(x-\frac{\alpha_2}{n+\beta_2}\right) + \frac{\alpha_1^2}{(n+\beta_1)^2}. \end{aligned} \tag{6}$$

In this work, as a first step we present a functional differential equation such that $S_{n,\alpha,\beta}(f;x)$ will satisfy the differential equation and then using this fact we derive a recurrence relation for the moments of the operators $S_{n,\alpha,\beta}(f;x)$. Later we show that these operators preserve some properties of the function f. Finally, we introduce a generalization of $S_{n,\alpha,\beta}(f;x)$ and investigate approximation properties of such operators.

2. Main results

As we have mentioned above, firstly we find a functional differential equation which is similar to the equations in [1], [3], [5], [6] and [12] corresponding to the operators $S_{n,\alpha,\beta}(f;x)$ defined by (3).

Theorem 1. *Let $g(t) = t$. Then the operators $S_{n,\alpha,\beta}(f;x)$ satisfy the functional differential equation*

$$\begin{aligned} &\left(x-\frac{\alpha_2}{n+\beta_2}\right)\left(\frac{n+\alpha_2}{n+\beta_2}-x\right)\frac{d}{dx}\left[S_{n,\alpha,\beta}(f;x)\right] \\ &\qquad = \frac{n(n+\beta_1)}{n+\beta_2}S_{n,\alpha,\beta}(fg;x) - n\left(\frac{\alpha_1-\alpha_2}{n+\beta_2}+x\right)S_{n,\alpha,\beta}(f;x). \end{aligned} \tag{7}$$

Proof. Differentiating both sides of (3), we have

$$\frac{d}{dx}\Big[S_{n,\alpha,\beta}(f;x)\Big] = \left(\frac{n+\beta_2}{n}\right)^n \sum_{r=0}^{n} f\left(\frac{r+\alpha_1}{n+\beta_1}\right) \times C_n^r\Bigg\{r\left(x-\frac{\alpha_2}{n+\beta_2}\right)^{r-1}\left(\frac{n+\alpha_2}{n+\beta_2}-x\right)^{n-r} - (n-r)\left(x-\frac{\alpha_2}{n+\beta_2}\right)^{r}\left(\frac{n+\alpha_2}{n+\beta_2}-x\right)^{n-r-1}\Bigg\} \tag{8}$$

which gives

$$\begin{aligned}
&\left(x-\frac{\alpha_2}{n+\beta_2}\right)\left(\frac{n+\alpha_2}{n+\beta_2}-x\right)\frac{d}{dx}\Big[S_{n,\alpha,\beta}(f;x)\Big] \\
&= \left(\frac{n+\beta_2}{n}\right)^n \sum_{r=0}^{n} f\left(\frac{r+\alpha_1}{n+\beta_1}\right) C_n^r\Bigg\{r\left(\frac{n+\alpha_2}{n+\beta_2}-x\right) - (n-r)\left(x-\frac{\alpha_2}{n+\beta_2}\right)\Bigg\} \\
&\quad\times\left(x-\frac{\alpha_2}{n+\beta_2}\right)^{r}\left(\frac{n+\alpha_2}{n+\beta_2}-x\right)^{n-r} \\
&= \frac{n}{n+\beta_2}\left(\frac{n+\beta_2}{n}\right)^n \sum_{r=1}^{n} f\left(\frac{r+\alpha_1}{n+\beta_1}\right) rC_n^r\left(x-\frac{\alpha_2}{n+\beta_2}\right)^{r}\left(\frac{n+\alpha_2}{n+\beta_2}-x\right)^{n-r} \\
&\quad - n\left(x-\frac{\alpha_2}{n+\beta_2}\right)S_{n,\alpha,\beta}(f;x) \\
&= \frac{n(n+\beta_1)}{n+\beta_2}\left(\frac{n+\beta_2}{n}\right)^n \sum_{r=0}^{n} f\left(\frac{r+\alpha_1}{n+\beta_1}\right)\frac{r+\alpha_1}{n+\beta_1} \\
&\quad\times C_n^r\left(x-\frac{\alpha_2}{n+\beta_2}\right)^{r}\left(\frac{n+\alpha_2}{n+\beta_2}-x\right)^{n-r} \\
&\quad - \frac{\alpha_1 n}{n+\beta_2}\left(\frac{n+\beta_2}{n}\right)^n \sum_{r=0}^{n} f\left(\frac{r+\alpha_1}{n+\beta_1}\right) C_n^r\left(x-\frac{\alpha_2}{n+\beta_2}\right)^{r}\left(\frac{n+\alpha_2}{n+\beta_2}-x\right)^{n-r} \\
&\quad - n\left(x-\frac{\alpha_2}{n+\beta_2}\right)S_{n,\alpha,\beta}(f;x) \\
&= \frac{n(n+\beta_1)}{n+\beta_2}\left(\frac{n+\beta_2}{n}\right)^n \sum_{r=0}^{n} f\left(\frac{r+\alpha_1}{n+\beta_1}\right)\frac{r+\alpha_1}{n+\beta_1} \\
&\quad\times C_n^r\left(x-\frac{\alpha_2}{n+\beta_2}\right)^{r}\left(\frac{n+\alpha_2}{n+\beta_2}-x\right)^{n-r} \\
&\quad - n\left(\frac{\alpha_1-\alpha_2}{n+\beta_2}+x\right)S_{n,\alpha,\beta}(f;x).
\end{aligned}$$

Since $g\left(\frac{r+\alpha_1}{n+\beta_1}\right) = \frac{r+\alpha_1}{n+\beta_1}$, from this we immediately arrive at the desired result. ■

If we take $f(t) = t^{m-1}$ for $m \in \mathbb{N}$ in (7), we can state the following fact.

Corollary 1. *Let $m \in \mathbb{N}$. Then for the moments of the operators $S_{n,\alpha,\beta}(f;x)$ we have the recurrence relation*

$$\left(x - \frac{\alpha_2}{n+\beta_2}\right)\left(\frac{n+\alpha_2}{n+\beta_2} - x\right)\frac{d}{dx}\left[S_{n,\alpha,\beta}(t^{m-1};x)\right]$$
$$= \frac{n(n+\beta_1)}{n+\beta_2} S_{n,\alpha,\beta}(t^m;x) - n\left(\frac{\alpha_1-\alpha_2}{n+\beta_2} + x\right) S_{n,\alpha,\beta}(t^{m-1};x).$$

Note that for $m = 1, 2$ the above recurrence relation yields (5) and (6), respectively.

We now show that the operators $S_{n,\alpha,\beta}(f;x)$ defined by (3) preserve some properties of the function f.

Theorem 2. *Let $f(x)$ be a non-negative function.*

(i) If $f(x)$ is non-decreasing on $[0,1]$, then the operators $S_{n,\alpha,\beta}(f;x)$ are also non-decreasing on I_n for each $n \in \mathbb{N}$.

(ii) If $x^{-1}f(x)$ is non-increasing on $(0,1]$, then $\left(x - \frac{\alpha_2}{n+\beta_2}\right)^{-1} S_{n,\alpha,\beta}(f;x)$ are also non-increasing on $\left(\frac{\alpha_2}{n+\beta_2}, \frac{n+\alpha_2}{n+\beta_2}\right]$ for each $n \in \mathbb{N}$.

Proof. (i) From (8) we can write

$$\frac{d}{dx}\left[S_{n,\alpha,\beta}(f;x)\right]$$
$$= \left(\frac{n+\beta_2}{n}\right)^n \sum_{r=0}^{n} f\left(\frac{r+\alpha_1}{n+\beta_1}\right) C_n^r \left\{ r\left(x - \frac{\alpha_2}{n+\beta_2}\right)^{r-1}\left(\frac{n+\alpha_2}{n+\beta_2} - x\right)^{n-r} - (n-r)\left(x - \frac{\alpha_2}{n+\beta_2}\right)^{r}\left(\frac{n+\alpha_2}{n+\beta_2} - x\right)^{n-r-1}\right\}$$
$$= \left(\frac{n+\beta_2}{n}\right)^n \sum_{r=1}^{n} f\left(\frac{r+\alpha_1}{n+\beta_1}\right) \frac{n!}{(r-1)!(n-r)!}\left(x - \frac{\alpha_2}{n+\beta_2}\right)^{r-1}$$
$$\times \left(\frac{n+\alpha_2}{n+\beta_2} - x\right)^{n-r} - \left(\frac{n+\beta_2}{n}\right)^n \sum_{r=0}^{n-1} f\left(\frac{r+\alpha_1}{n+\beta_1}\right) \frac{n!}{r!(n-r-1)!}$$
$$\times \left(x - \frac{\alpha_2}{n+\beta_2}\right)^{r}\left(\frac{n+\alpha_2}{n+\beta_2} - x\right)^{n-r-1}$$
$$= n\left(\frac{n+\beta_2}{n}\right)^n \sum_{r=0}^{n-1}\left[f\left(\frac{r+1+\alpha_1}{n+\beta_1}\right) - f\left(\frac{r+\alpha_1}{n+\beta_1}\right)\right] C_{n-1}^r$$
$$\times \left(x - \frac{\alpha_2}{n+\beta_2}\right)^{r}\left(\frac{n+\alpha_2}{n+\beta_2} - x\right)^{n-r-1}.$$

Since f is non-decreasing the right side of the last equality is non-negative. Thus we may conclude that the operators $S_{n,\alpha,\beta}(f;x)$ are also non-decreasing.
(ii) It is immediately seen that

$$
\begin{aligned}
&\frac{d}{dx}\left[\left(x-\frac{\alpha_2}{n+\beta_2}\right)^{-1} S_{n,\alpha,\beta}(f;x)\right] \\
&= \left(\frac{n+\beta_2}{n}\right)^n f\left(\frac{\alpha_1}{n+\beta_1}\right) \frac{d}{dx}\left[\left(x-\frac{\alpha_2}{n+\beta_2}\right)^{-1}\left(\frac{n+\alpha_2}{n+\beta_2}-x\right)^n\right] \\
&\quad + \left(\frac{n+\beta_2}{n}\right)^n \sum_{r=1}^{n} f\left(\frac{r+\alpha_1}{n+\beta_1}\right) C_n^r \frac{d}{dx}\left[\left(x-\frac{\alpha_2}{n+\beta_2}\right)^{r-1}\left(\frac{n+\alpha_2}{n+\beta_2}-x\right)^{n-r}\right] \\
&= -\left(\frac{n+\beta_2}{n}\right)^n f\left(\frac{\alpha_1}{n+\beta_1}\right)\left(x-\frac{\alpha_2}{n+\beta_2}\right)^{-2}\left(\frac{n+\alpha_2}{n+\beta_2}-x\right)^{n-1} \\
&\qquad \times\left[(n-1)\left(x-\frac{\alpha_2}{n+\beta_2}\right)+\frac{n}{n+\beta_2}\right] \\
&\quad + \left(\frac{n+\beta_2}{n}\right)^n \left\{\sum_{r=2}^{n} f\left(\frac{r+\alpha_1}{n+\beta_1}\right) C_n^r (r-1)\left(x-\frac{\alpha_2}{n+\beta_2}\right)^{r-2}\left(\frac{n+\alpha_2}{n+\beta_2}-x\right)^{n-r}\right. \\
&\quad \left. -\sum_{r=1}^{n-1} f\left(\frac{r+\alpha_1}{n+\beta_1}\right) C_n^r (n-r)\left(x-\frac{\alpha_2}{n+\beta_2}\right)^{r-1}\left(\frac{n+\alpha_2}{n+\beta_2}-x\right)^{n-r-1}\right\} \\
&= -\left(\frac{n+\beta_2}{n}\right)^n f\left(\frac{\alpha_1}{n+\beta_1}\right)\left(x-\frac{\alpha_2}{n+\beta_2}\right)^{-2}\left(\frac{n+\alpha_2}{n+\beta_2}-x\right)^{n-1} \\
&\qquad \times\left[(n-1)\left(x-\frac{\alpha_2}{n+\beta_2}\right)+\frac{n}{n+\beta_2}\right] \\
&\quad + \left(\frac{n+\beta_2}{n}\right)^n \sum_{r=1}^{n-1}\left\{f\left(\frac{r+1+\alpha_1}{n+\beta_1}\right) C_n^{r+1} r - f\left(\frac{r+\alpha_1}{n+\beta_1}\right) C_n^r (n-r)\right\} \\
&\qquad \times\left(x-\frac{\alpha_2}{n+\beta_2}\right)^{r-1}\left(\frac{n+\alpha_2}{n+\beta_2}-x\right)^{n-r-1}.
\end{aligned}
$$

Since $C_n^{r+1} r = nr\frac{1}{r+1} C_{n-1}^r$ and $C_n^r (n-r) = nr\frac{1}{r} C_{n-1}^r$, one may write

$$
\begin{aligned}
&\frac{d}{dx}\left[\left(x-\frac{\alpha_2}{n+\beta_2}\right)^{-1} S_{n,\alpha,\beta}(f;x)\right] \\
&= -\left(\frac{n+\beta_2}{n}\right)^n f\left(\frac{\alpha_1}{n+\beta_1}\right)\left(x-\frac{\alpha_2}{n+\beta_2}\right)^{-2}\left(\frac{n+\alpha_2}{n+\beta_2}-x\right)^{n-1} \\
&\qquad \times\left[(n-1)\left(x-\frac{\alpha_2}{n+\beta_2}\right)+\frac{n}{n+\beta_2}\right] \\
&\quad + n\left(\frac{n+\beta_2}{n}\right)^n \sum_{r=1}^{n-1} r\left\{f\left(\frac{r+1+\alpha_1}{n+\beta_1}\right)\frac{1}{r+1} - f\left(\frac{r+\alpha_1}{n+\beta_1}\right)\frac{1}{r}\right\} C_{n-1}^r
\end{aligned}
$$

$$
\begin{aligned}
&\times \left(x-\frac{\alpha_2}{n+\beta_2}\right)^{r-1}\left(\frac{n+\alpha_2}{n+\beta_2}-x\right)^{n-r-1} \\
&= -\left(\frac{n+\beta_2}{n}\right)^n f\left(\frac{\alpha_1}{n+\beta_1}\right)\left(x-\frac{\alpha_2}{n+\beta_2}\right)^{-2}\left(\frac{n+\alpha_2}{n+\beta_2}-x\right)^{n-1} \\
&\times\left[(n-1)\left(x-\frac{\alpha_2}{n+\beta_2}\right)+\frac{n}{n+\beta_2}\right] \\
&+\frac{n}{n+\beta_1}\left(\frac{n+\beta_2}{n}\right)^n\sum_{r=1}^{n-1} r\left\{f\left(\frac{r+1+\alpha_1}{n+\beta_1}\right)\left(\frac{r+1+\alpha_1}{n+\beta_1}\right)^{-1}\right. \\
&\times\left.\frac{r+1+\alpha_1}{r+1}-f\left(\frac{r+\alpha_1}{n+\beta_1}\right)\left(\frac{r+\alpha_1}{n+\beta_1}\right)^{-1}\frac{r+\alpha_1}{r}\right\} \\
&\times C_{n-1}^r\left(x-\frac{\alpha_2}{n+\beta_2}\right)^{r-1}\left(\frac{n+\alpha_2}{n+\beta_2}-x\right)^{n-r-1}.
\end{aligned}
$$

Using the inequality $\frac{r+1+\alpha_1}{r+1} \leq \frac{r+\alpha_1}{r}$ for $\alpha_1 \geq 0$ and $r \geq 1$ we obtain

$$
\begin{aligned}
&\frac{d}{dx}\left[\left(x-\frac{\alpha_2}{n+\beta_2}\right)^{-1} S_{n,\alpha,\beta}(f;x)\right] \\
&\leq -\left(\frac{n+\beta_2}{n}\right)^n f\left(\frac{\alpha_1}{n+\beta_1}\right)\left(x-\frac{\alpha_2}{n+\beta_2}\right)^{-2}\left(\frac{n+\alpha_2}{n+\beta_2}-x\right)^{n-1} \\
&\times\left[(n-1)\left(x-\frac{\alpha_2}{n+\beta_2}\right)+\frac{n}{n+\beta_2}\right] \\
&+\frac{n}{n+\beta_1}\left(\frac{n+\beta_2}{n}\right)^n\sum_{r=1}^{n-1}(r+\alpha_1)\left\{f\left(\frac{r+1+\alpha_1}{n+\beta_1}\right)\left(\frac{r+1+\alpha_1}{n+\beta_1}\right)^{-1}\right. \\
&-\left.f\left(\frac{r+\alpha_1}{n+\beta_1}\right)\left(\frac{r+\alpha_1}{n+\beta_1}\right)^{-1}\right\}C_{n-1}^r\left(x-\frac{\alpha_2}{n+\beta_2}\right)^{r-1}\left(\frac{n+\alpha_2}{n+\beta_2}-x\right)^{n-r-1} \\
&= -\left(\frac{n+\beta_2}{n}\right)^n f\left(\frac{\alpha_1}{n+\beta_1}\right)\left(x-\frac{\alpha_2}{n+\beta_2}\right)^{-2}\left(\frac{n+\alpha_2}{n+\beta_2}-x\right)^{n-1} \\
&\times\left[(n-1)\left(x-\frac{\alpha_2}{n+\beta_2}\right)+\frac{n}{n+\beta_2}\right] \\
&-\frac{n}{n+\beta_1}\left(\frac{n+\beta_2}{n}\right)^n\sum_{r=1}^{n-1}(r+\alpha_1)\left\{f\left(\frac{r+\alpha_1}{n+\beta_1}\right)\left(\frac{r+\alpha_1}{n+\beta_1}\right)^{-1}\right. \\
&\qquad\qquad\left.-f\left(\frac{r+1+\alpha_1}{n+\beta_1}\right)\left(\frac{r+1+\alpha_1}{n+\beta_1}\right)^{-1}\right\} \\
&\times C_{n-1}^r\left(x-\frac{\alpha_2}{n+\beta_2}\right)^{r-1}\left(\frac{n+\alpha_2}{n+\beta_2}-x\right)^{n-r-1}.
\end{aligned}
$$

Hence by the assumption we arrive at the required result. ■

We should note here that the similar result as (ii) of Theorem 2 was first proved by Li [11] for the classical Bernstein polynomials.

We now introduce a generalization of the operators $S_{n,\alpha,\beta}(f;x)$ defined by (3) via Taylor polynomial and investigate their approximation properties. For this purpose, let us recall some definitions and notations.

It is well known that if f has the k-th derivative for $k=0,1,2,\cdots$ at a point a then the polynomial

$$T_k(f,x)=\sum_{i=0}^{k}\frac{f^{(i)}(a)}{i!}(x-a)^i$$

is called the k-th degree Taylor polynomial of f at a.

Let $C^k[0,1]$, $k=0,1,2,\cdots$ denote the space of all functions f having continuous k-th derivative $f^{(k)}(x)$ on the interval $[0,1]$ with $f^{(0)}(x)=f(x)$.

We now consider the following generalization of the linear positive operators $S_{n,\alpha,\beta}(f;x)$:

$$\begin{aligned} S^{[k]}_{n,\alpha,\beta}(f;x) &= \left(\frac{n+\beta_2}{n}\right)^n \sum_{r=0}^{n}\sum_{i=0}^{k} f^{(i)}\left(\frac{r+\alpha_1}{n+\beta_1}\right)\frac{\left(x-\frac{r+\alpha_1}{n+\beta_1}\right)^i}{i!} \\ &\quad \times C_n^r\left(x-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-x\right)^{n-r} \end{aligned}$$

where $f\in C^k[0,1]$, $k=0,1,2,\cdots$, $n\in\mathbb{N}$ and $x\in I_n$.

This generalization is called the k-th order generalization of the operators $S_{n,\alpha,\beta}(f;x)$. It is obvious that $S^{[0]}_{n,\alpha,\beta}(f;x)=S_{n,\alpha,\beta}(f;x)$.

Observe that k-th order generalization of different linear positive operators studied by various authors (for instance see [3], [4], [7], [9], [10], [13], [14]).

A function $f\in C[0,1]$ belongs to $Lip_M(\gamma)$ if the inequality

$$|f(t)-f(x)|\le M|t-x|^\gamma;\quad x,t\in[0,1]$$

is satisfied for a positive constant M and $0<\gamma\le 1$.

Theorem 3. *Let $f\in C^k[0,1]$, $k=1,2,\cdots$ such that $f^{(k)}\in Lip_M(\gamma)$. Then we have*

$$\left|f(x)-S^{[k]}_{n,\alpha,\beta}(f;x)\right|\le\frac{M}{(k-1)!}\frac{\gamma}{\gamma+k}B(\gamma,k)S_{n,\alpha,\beta}\left(|t-x|^{\gamma+k};x\right),$$

where $B(\gamma,k)$ is the beta function.

Proof. From the definition of $S^{[k]}_{n,\alpha,\beta}(f;x)$, we get

$$f(x) - S^{[k]}_{n,\alpha,\beta}(f;x) = \left(\frac{n+\beta_2}{n}\right)^n \sum_{r=0}^{n} \left\{ f(x) - \sum_{i=0}^{k} f^{(i)}\left(\frac{r+\alpha_1}{n+\beta_1}\right) \times \frac{\left(x-\frac{r+\alpha_1}{n+\beta_1}\right)^i}{i!} \right\} C_n^r \left(x - \frac{\alpha_2}{n+\beta_2}\right)^r \left(\frac{n+\alpha_2}{n+\beta_2} - x\right)^{n-r}. \tag{9}$$

Then using Taylor's formula

$$\begin{aligned} f(x) &= \sum_{i=0}^{k} f^{(i)}(t) \frac{(x-t)^i}{i!} \\ &+ \frac{(x-t)^k}{(k-1)!} \int_0^1 (1-u)^{k-1} \left[f^{(k)}\big(t+u(x-t)\big) - f^{(k)}(t) \right] du \end{aligned}$$

one may write

$$\begin{aligned} & f(x) - \sum_{i=0}^{k} f^{(i)}\left(\frac{r+\alpha_1}{n+\beta_1}\right) \frac{\left(x-\frac{r+\alpha_1}{n+\beta_1}\right)^i}{i!} \\ &= \frac{\left(x-\frac{r+\alpha_1}{n+\beta_1}\right)^k}{(k-1)!} \int_0^1 (1-u)^{k-1} \\ &\times \left[f^{(k)}\left(\frac{r+\alpha_1}{n+\beta_1} + u\left(x - \frac{r+\alpha_1}{n+\beta_1}\right)\right) - f^{(k)}\left(\frac{r+\alpha_1}{n+\beta_1}\right) \right] du. \end{aligned} \tag{10}$$

On the other hand, since $f^{(k)} \in Lip_M(\gamma)$ we have

$$\left| f^{(k)}\left(\frac{r+\alpha_1}{n+\beta_1} + u\left(x - \frac{r+\alpha_1}{n+\beta_1}\right)\right) - f^{(k)}\left(\frac{r+\alpha_1}{n+\beta_1}\right) \right| \le M u^\gamma \left| x - \frac{r+\alpha_1}{n+\beta_1} \right|^\gamma.$$

Thus by means of the above inequality and the usual definition of the beta integral from (10), it follows that

$$\begin{aligned} & \left| f(x) - \sum_{i=0}^{k} f^{(i)}\left(\frac{r+\alpha_1}{n+\beta_1}\right) \frac{\left(x-\frac{r+\alpha_1}{n+\beta_1}\right)^i}{i!} \right| \\ &\qquad \le \frac{M}{(k-1)!} \frac{\gamma}{\gamma+k} B(\gamma,k) \left| x - \frac{r+\alpha_1}{n+\beta_1} \right|^{\gamma+k}. \end{aligned}$$

Hence by (9) we find

$$\left| f(x) - S^{[k]}_{n,\alpha,\beta}(f;x) \right| \le \frac{M}{(k-1)!} \frac{\gamma}{\gamma+k} B(\gamma,k) S_{n,\alpha,\beta}\left(|t-x|^{\gamma+k}; x \right)$$

which completes the proof. ■

Theorem 4. *Let $f \in C^k[0,1]$, $k = 1, 2, \cdots$ such that $f^{(k)} \in Lip_M(\gamma)$. Then we have*

$$\lim_{n\to\infty} \max_{x\in I_n} \left| S^{[k]}_{n,\alpha,\beta}\big(f;x\big) - f(x) \right| = 0.$$

Proof. Let us consider the function $g \in C[0,1]$ defined by $g(t) = |t-x|^{\gamma+k}$. Since $g(x) = 0$, from Theorem 1 in [8], we have

$$\lim_{n\to\infty} \max_{x\in I_n} S_{n,\alpha,\beta}\Big(\big|t-x\big|^{\gamma+k}; x\Big) = 0.$$

Thus by taking into consideration Theorem 3 the proof is completed. ∎

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