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R. Wituła, E. Hetmaniok, D. Słota

MEAN-VALUE THEOREMS FOR ONE-SIDED DIFFERENTIABLE FUNCTIONS

Abstract. We show that Kubik's generalizations of the classical mean-value theorems for one-sided differentiable functions are equivalent to those of Karamata and Vučkovič. Some applications of these theorems are presented.

Key words: mean value theorems, one-sided right and left derivatives, differentiable functions.

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1. Introduction

Writing our handbook [4] we have found two series of papers concerning the generalizations of Classical Mean Value Theorems (CMVT) for the one-sided differentiable functions. The first one, due to Polish mathematician L.T. Kubik [6, 7], is an effect of discussion on some problems concerning the distribution theory. The second one was presented independently by Serbian mathematicians J.Karamata [5] and V.Vučkovič [13].

We show that Kubik's generalizations of CMVT are equivalent to those proposed by Karamata and Vučkovič (cf. Propositions 1, 2, 3 in this paper). We also note that both types of the results have some nice and effective applications.

2. Kubik's theorems

In papers [6, 7] the following theorems were proved (in the paper symbols f'_- and f'_+ denote the respective one-sided derivatives of the considered function f).

Theorem 1 (Generalization of the Rolle Theorem)**.** *If $f : [a, b] \to \mathbb{R}$ is both-sided differentiable and $f(a) = f(b)$, then there exists $c \in (a, b)$ such that*

$$f'_+(c) \cdot f'_-(c) \leqslant 0. \tag{1}$$

Remark 1. If f is differentiable then (1) gives that $[f'(c)]^2 \leqslant 0$, whence $f'(c) = 0$. Thus, Theorem 1 implies the Rolle theorem.

Remark 2. Theorem 1 can be improved as follows:

If $f : [a, b] \to \mathbb{R}$ is both-sided differentiable in (a, b) and right-continuous at a, left-continuous at b and $f(a) = f(b)$, then there exists $c \in (a, b)$ such that (1) holds.

Proof of this modified version runs analogically as the proof of Theorem 1 [6, 7].

Similar modification as proposed above can be done in Theorems 2 and 3.

Theorem 2 (Generalization of the Lagrange Theorem)**.** *If $f : [a, b] \to \mathbb{R}$ is both-sided differentiable, then there exists $c \in (a, b)$ such that*

$$\left(\frac{f(b) - f(a)}{b - a} - f'_+(c)\right)\left(\frac{f(b) - f(a)}{b - a} - f'_-(c)\right) \leqslant 0. \tag{2}$$

Let us also consider the geometric interpretation of formula (2). If (2) holds then either

$$\frac{f(b) - f(a)}{b - a} \geqslant f'_+(c), \qquad \frac{f(b) - f(a)}{b - a} \leqslant f'_-(c), \tag{3}$$

or

$$\frac{f(b) - f(a)}{b - a} \leqslant f'_+(c), \qquad \frac{f(b) - f(a)}{b - a} \geqslant f'_-(c). \tag{4}$$

Formula (3) informs that the secant line led by points $(a, f(a))$ and $(b, f(b))$ is more (or the same) steep as the right-sided tangent line at the point c and less (or the same) steep as the left-sided tangent line at c. Similarly, formula (4) says that the secant line is less (or the same) steep as the right-sided tangent line at the point c and more (or the same) steep as the left-sided tangent line in c.

Let us notice that if f is differentiable then formula (2) takes the form

$$\frac{f(b) - f(a)}{b - a} = f'(c),$$

which gives the Lagrange Theorem.

From the generalized Lagrange Theorem we obtain the following obvious

Remark 3. Any function $f : [a, b] \to \mathbb{R}$ having bounded derivatives f_+ and f_- is Lipschitzian.

This remark extends the classical fact concerning differentiable functions.

Theorem 3 (Generalization of the Cauchy Theorem)**.** *If $f, h : [a, b] \to \mathbb{R}$ are both-sided differentiable and $h(a) \neq h(b)$, then there exists $c \in (a, b)$ such that*

$$\left(\frac{f(b)-f(a)}{h(b)-h(a)} h'_+(c) - f'_+(c)\right)\left(\frac{f(b)-f(a)}{h(b)-h(a)} h'_-(c) - f'_-(c)\right) \leqslant 0. \tag{5}$$

Remark 4. If the functions f and h are differentiable and if $h'(x) \neq 0$ for $x \in [a, b]$, then formula (5) takes the form

$$\frac{f(b)-f(a)}{h(b)-h(a)} = \frac{f'(c)}{h'(c)},$$

which formulates the classical Cauchy Theorem.

Kubik [6] gives an interesting application of Theorem 3 in the probability theory.

3. Generalizations according to Karamata and Vučkovič

The following two generalizations of the Rolle and Lagrange Theorems, respectively, have been presented by J.Karamata in paper [5].

Theorem 4. *If $f : [a, b] \to \mathbb{R}$ is both-sided differentiable in (a, b), right-continuous at a, left-continuous at b and $f(a) = f(b) = 0$, then there exist $c \in (a, b)$; $p, q \geqslant 0$, $p + q = 1$ such that*

$$pf'_+(c) + qf'_-(c) = 0.$$

Theorem 5. *If $f : [a, b] \to \mathbb{R}$ is both-sided differentiable in (a, b), right-continuous at a, left-continuous at b, then there exist $c \in (a, b)$; $p, q \geqslant 0$, $p + q = 1$ such that*

$$\frac{f(b)-f(a)}{b-a} = pf'_+(c) + qf'_-(c).$$

Vučkovič [13] has proved the following generalization of the Cauchy Mean Value Theorem.

Theorem 6. *If $f, h : [a, b] \to \mathbb{R}$ are both-sided differentiable, then there exist $c \in (a, b)$; $p, q \geqslant 0$, $p + q = 1$ such that*

$$\big[pf'_+(c) + qf'_-(c)\big]\big(h(b) - h(a)\big) = \big[ph'_+(c) + qh'_-(c)\big]\big(f(b) - f(a)\big). \tag{6}$$

Remark 5. In this theorem the inequalities $p, q \geqslant 0$ cannot be replaced by $p, q > 0$.

To show it let us consider the following example.

Let $a = -1$, $b = 1$ and

$$f(x) = \begin{cases} x^2 - 1, & \text{dla} \quad x \geqslant 0, \\ -x - 1, & \text{dla} \quad x < 0. \end{cases}$$

Then $\xi = 0$ and $f'_+(0) = 0$, $f'_-(0) = -1$, from where $q = 0$ (Theorems 4 and 5). If we take $g(x) = x$, $x \in [-1, 1]$, then also in Theorem 6 we have $q = 0$ (see [12]).

We show now that Theorem 1 and Theorem 4 are equivalent. More precisely, we prove the following

Proposition 1. *Suppose that $f : [a, b] \to \mathbb{R}$ is both-sided differentiable. Then the following two conditions are equivalent:*

$(KU1)$ *there exists $c \in (a, b)$ such that $f'_+(c)f'_-(c) \leqslant 0$;*

$(KA4)$ *there exist $c \in (a, b)$; $p, q \geqslant 0$, $p + q = 1$ such that*

$$p\, f'_+(c) + q\, f'_-(c) = 0.$$

Proof. $(KU1) \Rightarrow (KA4)$

Let $c \in (a, b)$ be such that

$$f'_+(c)f'_-(c) \leqslant 0.$$

If $f'_+(c) = 0$ then it is enough to take $p = 1$, $q = 0$. If $f'_-(c) = 0$ then we put $p = 0$, $q = 1$. Finally, for $f'_+(c)f'_-(c) < 0$ we can take

$$p = \frac{f'_-(c)}{f'_-(c) - f'_+(c)} \quad \text{and} \quad q = \frac{f'_+(c)}{f'_+(c) - f'_-(c)}.$$

$(KA4) \Rightarrow (KU1)$

If for $c \in (a, b)$ and $p, q \geqslant 0$ we have $p + q > 0$ and $p\, f'_+(c) + q\, f'_-(c) = 0$, then it is obvious that $f'_+(c)\, f'_-(c) \leqslant 0$.

Indeed, then in case of $f'_+(c)f'_-(c) \neq 0$ we have $pq \neq 0$, that is

$$pf'_+(c)f'_-(c) = -q(f'_-(c))^2 < 0,$$

i.e.

$$f'_+(c)f'_-(c) < 0,$$

which ends the proof. ∎

Now we show that Theorem 2 and Theorem 5 are equivalent. More precisely we prove the following

Proposition 2. *Suppose that $f : [a,b] \to \mathbb{R}$ is both-sided differentiable. Then the following two conditions are equivalent:*

$(KU2)$ *there exists $c \in (a,b)$ such that*

$$\Big(\frac{f(b)-f(a)}{b-a} - f'_+(c)\Big)\Big(\frac{f(b)-f(a)}{b-a} - f'_-(c)\Big) \leqslant 0;$$

$(KA5)$ *there exist $c \in (a,b)$; $p, q \geqslant 0$, $p+q=1$ such that*

$$\frac{f(b)-f(a)}{b-a} = p\, f'_+(c) + q\, f'_-(c).$$

Proof. $(KU2) \Rightarrow (KA5)$

Let $c \in (a,b)$ be such that (2) holds. If

$$\frac{f(b)-f(a)}{b-a} = f'_+(c),$$

then we take $p=1$ and $q=0$. If

$$\frac{f(b)-f(a)}{b-a} = f'_-(c),$$

then we put $p=0$ and $q=1$. Finally, if inequality (2) is strict then we take

$$p = \frac{\frac{f(b)-f(a)}{b-a} - f'_-(c)}{f'_+(c) - f'_-(c)} \quad \text{and} \quad q = \frac{\frac{f(b)-f(a)}{b-a} - f'_+(c)}{f'_-(c) - f'_+(c)}.$$

$(KA5) \Rightarrow (KU2)$

If $c \in (a,b)$, $p, q \geqslant 0$, $p+q=1$ and

$$\frac{f(b)-f(a)}{b-a} = pf'_+(c) + qf'_-(c),$$

then, how can be easily checked, the following equality holds

$$\left(\frac{f(b)-f(a)}{b-a} - f'_+(c)\right)\left(\frac{f(b)-f(a)}{b-a} - f'_-(c)\right) = -qp\big(f'_-(c) - f'_+(c)\big)^2.$$

In case when the above product is different than zero, the right side of this equality implies that the product is negative, which implies $(KU2)$. ■

At last we show that Theorems 3, 5 and Theorem 6 are equivalent. More precisely, we prove the following

Proposition 3. *Assume that $f, h : [a,b] \to \mathbb{R}$ are both-sided differentiable. Then the following two conditions are equivalent:*

$(KU3 + KA5)$ *there exist $c_0 \in (a,b)$; $p_0, q_0 \geqslant 0$, $p_0 + q_0 = 1$ such that*

$$\frac{f(b)-f(a)}{b-a} = p_0\, f'_+(c_0) + q_0\, f'_-(c_0).$$

Moreover, if $h(a) \neq h(b)$, then there exists $c \in (a,b)$ such that

$$\left(\frac{f(b)-f(a)}{h(b)-h(a)} h'_+(c) - f'_+(c)\right)\left(\frac{f(b)-f(a)}{h(b)-h(a)} h'_-(c) - f'_-(c)\right) \leqslant 0;$$

$(V6)$ *there exist $c \in (a,b)$; $p, q \geqslant 0$, $p + q = 1$ such that*

$$\big[p f'_+(c) + q f'_-(c)\big]\big(h(b)-h(a)\big) = \big[p h'_+(c) + q h'_-(c)\big]\big(f(b)-f(a)\big).$$

Proof. $(V6) \Rightarrow (KU3 + KA5)$

Let $c \in (a,b)$, $p, q \geqslant 0$, $p + q = 1$ be such that equality (6) holds. If $h(a) \neq h(b)$, then from (6) we receive

$$p\left(\frac{f(b)-f(a)}{h(b)-h(a)} h'_+(c) - f'_+(c)\right) = -q\left(\frac{f(b)-f(a)}{h(b)-h(a)} h'_-(c) - f'_-(c)\right),$$

from where we get

$$\begin{aligned} &p\left(\frac{f(b)-f(a)}{h(b)-h(a)} h'_+(c) - f'_+(c)\right)\left(\frac{f(b)-f(a)}{h(b)-h(a)} h'_-(c) - f'_-(c)\right) \\ &\qquad = -q\left(\frac{f(b)-f(a)}{h(b)-h(a)} h'_-(c) - f'_-(c)\right)^2. \end{aligned} \tag{7}$$

If the product

$$\left(\frac{f(b)-f(a)}{h(b)-h(a)} h'_+(c) - f'_+(c)\right)\left(\frac{f(b)-f(a)}{h(b)-h(a)} h'_-(c) - f'_-(c)\right) \neq 0,$$

then, in view of the condition $p + q = 1$, also the condition $pq \neq 0$ holds. From this, because of (7), inequality (5) results.

Furthermore, by taking $h(x) := x$, $x \in [a,b]$, in formula (6) we obtain the equality

$$\frac{f(b)-f(a)}{b-a} = p f'_+(c) + q f'_-(c),$$

which ends the proof.

$(KU3 + KA5) \Rightarrow (V6)$

Let $c \in (a,b)$ be such that inequality (5) holds. Suppose $h(b) \neq h(a)$. If

$$\frac{f(b)-f(a)}{h(b)-h(a)} h'_+(c) = f'_+(c),$$

then we take $p = 1$, $q = 0$. If

$$\frac{f(b) - f(a)}{h(b) - h(a)} h'_-(c) = f'_-(c),$$

then we put $p = 0$, $q = 1$. Finally, if inequality (5) is strict we can take

$$p = \frac{s}{s-t} \quad \text{and} \quad q = \frac{t}{t-s},$$

where

$$\begin{aligned} s &:= \frac{f(b) - f(a)}{h(b) - h(a)} h'_-(c) - f'_-(c), \\ t &:= \frac{f(b) - f(a)}{h(b) - h(a)} h'_+(c) - f'_+(c). \end{aligned}$$

The case when $h(b) = h(a)$ still left. Then, by assumption $(KA5)$ for the function h we get that there exist $c_0 \in (a, b)$ and $p_0, q_0 \geqslant 0$, $p_0 + q_0 = 1$ such that

$$p_0 h'_+(c_0) + q_0 h'_-(c_0) = 0.$$

Substituting those values for c, p, q, respectively, we obtain equality (6). ■

Remark 6. It follows from Theorem 5, there exists $c \in (a, b)$ such that

$$\min\left\{f'_-(c), f'_+(c)\right\} \leqslant \frac{f(b) - f(a)}{b - a} \leqslant \max\left\{f'_-(c), f'_+(c)\right\}.$$

In note [9], the following theorem has been proved, which is, in an interesting way, connected with the above inequality.

Theorem 7. *Let us assume that $f : [a, b] \to \mathbb{R}$ is a continuous function and it possess the right derivative $f'_+(x)$ at every point $x \in (a, b)$. Then, there exist $p, q \in (a, b)$ such that*

$$f'_+(q) \leqslant \frac{f(b) - f(a)}{b - a} \leqslant f'_+(p). \tag{8}$$

Remark 7. Theorem 7 remains true also in the case of replacing the right derivative of function f with the left derivative of this function.

Remark 8. Theorem 7 has been applied in [10] for characterization of the convex functions defined in the open intervals. In particular, it has been proved that if $f : (a, b) \to \mathbb{R}$ is a convex function then for every $x \in (a, b)$ we have (see also Remark 10 below):

$$f'_-(x) \leqslant f'_+(x).$$

It seems that, in particular case, after eliminating an affine functions from the consideration, Theorem 7 can be significantly strengthen. The following theorem holds.

Theorem 8. *If $f : [a,b] \to \mathbb{R}$ is a continuous function, differentiable in (a,b) and, additionally, it is not an affine function in $[a,b]$ then there exist $p, q \in (a,b)$ such that*

$$f'(p) < \frac{f(b)-f(a)}{b-a} < f'(q).$$

Proof. Since f is not an affine function, so there exists $c \in (a,b)$ such that

$$\frac{f(c)-f(a)}{c-a} \neq \frac{f(b)-f(c)}{b-c}. \tag{9}$$

Of course, the Lagrange Mean Value Theorem implies that there exist $p \in (a,c)$ and $q \in (c,b)$ such that

$$\frac{f(c)-f(a)}{c-a} = f'(p) \qquad \text{and} \qquad \frac{f(b)-f(c)}{b-c} = f'(q).$$

Taking (9) into account we have $f'(p) \neq f'(q)$. Without violating the generality of our consideration let us assume that $f'(p) < f'(q)$. Let $\lambda := \frac{c-a}{b-a}$. Thus we get

$$\begin{aligned}\frac{f(b)-f(a)}{b-a} &= \lambda\frac{f(c)-f(a)}{c-a} + (1-\lambda)\frac{f(b)-f(c)}{b-c} = \\ &= \lambda f'(p) + (1-\lambda)f'(q) < \lambda f'(q) + (1-\lambda)f'(q) = f'(q)\end{aligned}$$

and

$$\frac{f(b)-f(a)}{b-a} > \lambda f'(p) + (1-\lambda)f'(p) = f'(p),$$

which ends the proof. ■

Remark 9. There exists $r \in (a,b)$ such that

$$\left|\frac{f(b)-f(a)}{b-a}\right| < f'(r).$$

Remark 10. Let us notice that in general case of the function $f : I \to \mathbb{R}$, where I is an interval, like it results from the Sierpiński-Young Theorem (see [8]), each of the following inequalities is true in I (with the exception of at most some countable set):

$$D_- f(x) \leqslant D^+ f(x)$$

and

$$D_+ f(x) \leqslant D^- f(x),$$

where the Dini derivatives from the function f appear. Dini derivatives are defined as follows

$$D^+ f(x_0) := \limsup_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0}, \quad D_+ f(x_0) := \liminf_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0},$$
$$D^- f(x_0) := \limsup_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0}, \quad D_- f(x_0) := \liminf_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0}.$$

D. B. Goodner in paper [2] has generalized Theorems 4-6 for the case of Dini derivatives.

Remark 11. D.H. Trahan in [11] has received the results similar to Kubik's Theorems (Theorems 1, 2, 3), but for the Flett's type Mean Value Theorem and only for derivatives.

Whereas, D.B. Goodner in paper [3] has obtained the Karamata's version of the Flett's Theorem for the one-sided derivatives.

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Roman Wituła
Institute of Mathematics
Silesian University of Technology
Kaszubska 23, Gliwice 44-100, Poland
e-mail: Roman.Witula@polsl.pl

Edyta Hetmaniok
Institute of Mathematics
Silesian University of Technology
Kaszubska 23, Gliwice 44-100, Poland
e-mail: Edyta.Hetmaniok@polsl.pl

Damian Słota
Institute of Mathematics
Silesian University of Technology
Kaszubska 23, Gliwice 44-100, Poland
e-mail: Damian.Slota@polsl.pl

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