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COMMON FIXED POINTS FOR FOUR MAPS IN ORDERED PARTIAL METRIC SPACES

Abstract. In this paper, we present some common fixed point theorems for four maps in partially ordered partial metric spacess. Our results generalize the main theorems of Abbas, Nazir and Radenović [1].

Key words: common fixed point, partial-compatibility, weak compatibility, ordered set, partial metric space.

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1. Introduction and preliminaries

The Banach contraction principle [16], which is the most famous metrical fixed point theorem, plays a very important role in nonlinear analysis. Basically, it asserts that, if (X, d) is a complete metric space and $T : X \to X$ is a contraction, i.e., there exists a constant $c \in [0, 1)$ such that

$$d(Tx, Ty) \leq cd(x, y), \quad \text{for all } x, y \in X,$$

then T has a unique fixed point $u \in X$, i.e., $Tu = u$. The Banach contraction principle has been generalized in several directions, see for example [19] and [38] for recent surveys. The existence of fixed points in partially ordered metric spaces was investigated in 2004 by Ran and Reurings [34], and then by Nieto and López [28]. Further results in this direction were proved, e.g., in [3, 17, 32]. Results on weakly contractive mappings in such spaces, together with applications to differential equations, were obtained by Harjani and Sadarangani in [23], for other results, we can refer to ([1, 7, 8, 20, 24, 26, 27, 28, 29]).

For instance, Abbas, Nazir and Radenović [1] proved a common fixed point for four maps in partially ordered metric spaces. In this paper we extend their result to the class of partially ordered partial metric spaces.

The concept of a partial metric space was introduced by Matthews [25] in 1994. After that, fixed point results in partial metric spaces have been studied, see for example [2, 4, 5, 6, 9, 10, 11, 12, 13, 14, 15, 18, 25, 30, 35, 36, 37, 40, 41, 42, 43].

Throughout this paper, the letters $\mathbb{R}_+$ and $\mathbb{N}$ will denote the set of all non-negative real numbers and the set of all non-negative integer numbers, respectively. First, we start by recalling some known definitions and properties of partial metric spaces.

Definition 1 ([25]). *A partial metric on a nonempty set X is a function $p : X \times X \to [0, +\infty)$ such that for all $x, y, z \in X$:*
($p1$) $x = y \iff p(x,x) = p(x,y) = p(y,y)$,
($p2$) $p(x,x) \leq p(x,y)$,
($p3$) $p(x,y) = p(y,x)$,
($p4$) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.
A partial metric space is a pair (X,p) such that X is a nonempty set and p is a partial metric on X.

It is clear that, if $p(x,y) = 0$, then from ($p1$) and ($p2$), $x = y$. But if $x = y$, $p(x,y)$ may not be 0. A basic example of a partial metric space is the pair $(\mathbb{R}_+, p)$, where $p(x,y) = \max\{x,y\}$ for all $x, y \in \mathbb{R}_+$.

Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p-balls $\{B_p(x,\varepsilon),\ x \in X, \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

If p is a partial metric on X, then the function $p^s : X \times X \to \mathbb{R}_+$ given by

$$p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y), \tag{1}$$

is a metric on X.

Let $\{x_n\}$ be a sequence in X. Then

(i) $\{x_n\}$ converges to a point $x \in X$ if and only if $p(x,x) = \lim_{n\to+\infty} p(x,x_n)$,

(ii) $\{x_n\}$ is called a Cauchy sequence if there exists (and is finite) $\lim_{n,m\to+\infty} p(x_n,x_m)$.

(iii) (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$, such that $p(x,x) = \lim_{n,m\to+\infty} p(x_n,x_m)$.

Lemma 1. *Let (X,p) be a partial metric space.*

(a) *$\{x_n\}$ is a Cauchy sequence in (X,p) if and only if it is a Cauchy sequence in the metric space (X,p^s).*

(b) *A partial metric space (X,p) is complete if and only if the metric space (X,p^s) is complete. Furthermore, $\lim_{n\to+\infty} p^s(x_n,x) = 0$ if and only if*

$$p(x,x) = \lim_{n\to+\infty} p(x_n,x) = \lim_{n,m\to+\infty} p(x_n,x_m).$$

Definition 2 ([5]). *Let (X, p) be a partial metric space, $F : X \to X$ be a given mapping. We say that F is continuous at $x_0 \in X$, if for every $\varepsilon > 0$, there exists $\eta > 0$ such that $F(B_p(x_0, \eta)) \subseteq B_p(Fx_0, \varepsilon)$.*

It is easy to chech that:

Lemma 2. *Let (X, p) be a partial metric space, $F : X \to X$ be a given mapping. Suppose that F is continuous at $x_0 \in X$. Then, for all sequence $\{x_n\}$ in X such that $x_n \to x_0$, we have $Fx_n \to Fx_0$.*

On the other hand, Abbas et al. [1] introduced the following definitions.

Definition 3 ([1]). *Let $(X, \leq)$ be a partially ordered set and f and g be two self maps on X. An ordered pair (f, g) is said to be partially weakly increasing if $fx \leq gfx$ for all $x \in X$.*

Definition 4 ([1]). *Let $(X, \leq)$ be a partially ordered set. A mapping f is a called weak annihilator of g if $fgx \leq x$ for all $x \in X$.*

Definition 5 ([1]). *Let $(X, \leq)$ be a partially ordered set. A mapping f is a called dominating if $x \leq fx$ for all $x \in X$.*

Also, some examples illustrating above definitions are given in [1].

In the sequel, let ψ and φ be as follows (as in [21]):

(*i*) $\psi : [0, +\infty) \to [0, +\infty)$ is a continuous non-decreasing function with $\psi(t) = 0$ if and only if $t = 0$,

(*ii*) $\varphi : [0, +\infty) \to [0, +\infty)$ is a lower semi-continuous function with $\varphi(t) = 0$ if and only if $t = 0$.

Such ψ and φ are called control functions.

Definition 6. *Let X be a non-empty set, N is a natural number such that $N \geq 2$ and $T_1, T_2, \cdots, T_N : X \to X$ are given self-mappings on X. If $w = T_1x = T_2x = \cdots = T_Nx$ for some $x \in X$, then x is called a coincidence point of $T_1, T_2, \cdots, T_{N-1}$ and T_N. If $w = x$, then x is called a common fixed point of $T_1, T_2, \cdots, T_{N-1}$ and T_N.*

A subset W of a partially ordered set X is said to be well ordered if every two elements of W are comparable. The main theorem given in [1] is

Theorem 1 ([1]). *Let $(X, \leq, d)$ be a partially ordered complete metric space. Let f, g, S and T be self maps on X, (T, f) and (S, g) be partially weakly increasing with $fX \subseteq TX$ and $gX \subseteq SX$, dominating maps f and g are weak annihilators of T and S, respectively. Suppose that there exists control functions ψ and φ such that for every two comparable elements $x, y \in X$,*

$$\psi(d(fx, gy)) \leq \psi(\theta(x, y)) - \varphi(\theta(x, y)), \tag{2}$$

is satisfied where

$$\theta(x,y) = \max\{d(Sx,Ty), d(fx,Sx), d(gy,Ty), \frac{d(Sx,gy)+d(fx,Ty)}{2}\}.$$

If for a non-decreasing sequence $\{x_n\}$ *with* $x_n \leq y_n$ *for all* n *and* $y_n \to u$ *implies that* $x_n \leq u$ *and either*

(a) $\{f, S\}$ *are compatible,* f *or* S *is continuous and* $\{g, T\}$ *are weakly compatible or*

(b) $\{g, T\}$ *are compatible,* g *or* T *is continuous and* $\{f, S\}$ *are weakly compatible,*

then f, g, S *and* T *have a common fixed point. Moreover, the set of common fixed points of* f, g, S *and* T *is well ordered if and only if* f, g, S *and* T *have one and only one common fixed point.*

The aim of this paper is to extend Theorem 1 to ordered partial metric spaces. For this, we recall the following definition of partial-compatibility introduced by Samet et al. [40].

Definition 7 ([40]). *Let* (X, p) *be a partial metric space and* $f, g : X \to X$ *are mappings of* X *into itself. We say that the pair* $\{f, g\}$ *is partial-compatible if the following conditions hold:*

($b1$) $p(x, x) = 0$ *implies that* $p(gx, gx) = 0$,

($b2$) $\lim_{n\to+\infty} p(fgx_n, gfx_n) = 0$, *whenever* $\{x_n\}$ *is a sequence in* X *such that* $fx_n \to t$ *and* $gx_n \to t$ *for some* $t \in X$.

Note that Definition 7 extends and generalizes the notion of compatibility introduced by Jungck [22].

2. Main results

Our first result is the following.

Theorem 2. *Let* $(X, \leq, p)$ *be a partially ordered complete partial metric space. Let* f, g, S *and* T *be self maps on* X, (T, f) *and* (S, g) *be partially weakly increasing with* $fX \subseteq TX$ *and* $gX \subseteq SX$, *dominating maps* f *and* g *are weak annihilators of* T *and* S, *respectively. Suppose that there exist control functions* ψ *and* φ *such that for every two comparable elements* $x, y \in X$,

$$\psi(p(fx,gy)) \leq \psi(M(x,y)) - \varphi(M(x,y)), \tag{3}$$

is satisfied where

$$M(x,y) = \max\{p(Sx,Ty), p(fx,Sx), p(gy,Ty), \frac{p(Sx,gy)+p(fx,Ty)}{2}\}.$$

If for a non-decreasing sequence $\{x_n\}$ *with* $x_n \leq y_n$ *for all* n *and* $y_n \to u$ *implies that* $x_n \leq u$ *and either*

(a) $\{f, S\}$ *are partial-compatible,* f *or* S *is continuous and* $\{g, T\}$ *are weakly compatible or*

(b) $\{g, T\}$ *are partial-compatible,* g *or* T *is continuous and* $\{f, S\}$ *are weakly compatible,*

then f*,* g*,* S *and* T *have a common fixed point. Moreover, the set of common fixed points of* f*,* g*,* S *and* T *is well ordered if and only if* f*,* g*,* S *and* T *have one and only one common fixed point.*

Proof. Let x_0 be an arbitrary point in X. Since $fX \subseteq TX$, there exists $x_1 \in X$ such that $Tx_1 = fx_0$. Also, since $gX \subseteq SX$, there exists $x_2 \in X$ such that $Sx_2 = gx_1$. Continuing this process, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X defined by

$$y_{2n-1} = Tx_{2n-1} = fx_{2n-2}, \quad y_{2n} = Sx_{2n} = gx_{2n-1} \quad \forall\, n \in \mathbb{N}^*. \tag{4}$$

By given assumptions

$$x_{2n-2} \leq fx_{2n-2} = Tx_{2n-1} \leq fTx_{2n-1} \leq x_{2n-1},$$

and

$$x_{2n-1} \leq gx_{2n-1} = Sx_{2n} \leq Sgx_{2n} \leq x_{2n}.$$

Thus, for all $n \in \mathbb{N}$, we have $x_n \leq x_{n+1}$.

Suppose for some n, $p(y_{2n}, y_{2n+1}) = 0$, then $y_{2n} = y_{2n+1}$. We have

$$\begin{aligned} M(x_{2n}, x_{2n+1}) &= \max\{p(Sx_{2n}, Tx_{2n+1}), p(fx_{2n}, Sx_{2n}), p(gx_{2n+1}, Tx_{2n+1}), \\ &\quad \frac{p(Sx_{2n}, gx_{2n+1}) + p(fx_{2n}, Tx_{2n+1})}{2}\} \\ &= \max\{p(y_{2n}, y_{2n+1}), p(y_{2n+1}, y_{2n}), p(y_{2n+2}, y_{2n+1}), \\ &\quad \frac{p(y_{2n}, y_{2n+2}) + p(y_{2n+1}, y_{2n+1})}{2}\} \\ &= \max\{0, 0, p(y_{2n+2}, y_{2n+1}), \frac{p(y_{2n}, y_{2n+2}) + p(y_{2n+1}, y_{2n+1})}{2}\} \\ &= p(y_{2n+1}, y_{2n+2}), \end{aligned}$$

because $p(y_{2n}, y_{2n+2}) + p(y_{2n+1}, y_{2n+1}) \leq p(y_{2n}, y_{2n+1}) + p(y_{2n+1}, y_{2n+2}) = p(y_{2n+1}, y_{2n+2})$. Since x_{2n} and x_{2n+1} are comparable, then by (3), we get

$$\begin{aligned} \psi(d(y_{2n+1}, y_{2n+2})) &= \psi(p(fx_{2n}, gx_{2n+1})) \\ &\leq \psi(M(x_{2n}, x_{2n+1})) - \varphi(M(x_{2n}, x_{2n+1})) \\ &= \psi(p(y_{2n+1}, y_{2n+2})) - \varphi(p(y_{2n+1}, y_{2n+2})), \end{aligned}$$

which implies that $\varphi(p(y_{2n+1}, y_{2n+2})) = 0$. By the fact that $\varphi(t) = 0$ if and only if $t = 0$, so $p(y_{2n+1}, y_{2n+2}) = 0$, that is, $y_{2n+1} = y_{2n+2}$. Following the similar arguments, we obtain $y_{2n+2} = y_{2n+3}$ and so on. Thus $\{y_n\}$ becomes a constant sequence and y_{2n} is the common fixed point of f, g, S and T.

From now on, assume that $p(y_n, y_{n+1}) > 0$ for all $n \in \mathbb{N}$. By (3), we have

$$\begin{aligned}\psi(p(y_{2n+1}, y_{2n+2})) &= \psi(p(fx_{2n}, gx_{2n+1}))\\ &\leq \psi(M(x_{2n}, x_{2n+1})) - \varphi(M(x_{2n}, x_{2n+1}))\\ &\leq \psi(M(x_{2n}, x_{2n+1})).\end{aligned}$$

Therefore, since ψ is non-decreasing, we have

$$p(y_{2n+1}, y_{2n+2}) \leq M(x_{2n}, x_{2n+1}), \tag{5}$$

where

$$\begin{aligned}M(x_{2n}, x_{2n+1}) &= \max\{p(Sx_{2n}, Tx_{2n+1}), p(fx_{2n}, Sx_{2n}), p(gx_{2n+1}, Tx_{2n+1}),\\ &\qquad \frac{p(Sx_{2n}, gx_{2n+1}) + p(fx_{2n}, Tx_{2n+1})}{2}\}\\ &= \max\{p(y_{2n}, y_{2n+1}), p(y_{2n+1}, y_{2n}), p(y_{2n+2}, y_{2n+1}),\\ &\qquad \frac{p(y_{2n}, y_{2n+2}) + p(y_{2n+1}, y_{2n+1})}{2}\}\\ &= \max\{p(y_{2n}, y_{2n+1}), p(y_{2n+2}, y_{2n+1})\},\end{aligned}$$

since

$$p(y_{2n}, y_{2n+2}) + p(y_{2n+1}, y_{2n+1}) \leq p(y_{2n}, y_{2n+1}) + p(y_{2n+1}, y_{2n+2}).$$

If for some n, $\max\{p(y_{2n}, y_{2n+1}), p(y_{2n+2}, y_{2n+1})\} = p(y_{2n+2}, y_{2n+1})$, then by (5),

$$M(x_{2n}, x_{2n+1}) = p(y_{2n+1}, y_{2n+2}),$$

and $\psi(p(y_{2n+1}, y_{2n+2})) \leq \psi(p(y_{2n+1}, y_{2n+2})) - \varphi(p(y_{2n+1}, y_{2n+2}))$, so $\varphi(p(y_{2n+1}, y_{2n+2})) = 0$, that is a contradiction with respect to $p(y_{2n+1}, y_{2n+2}) > 0$. Thus,

$$p(y_{2n+1}, y_{2n+2}) \leq M(x_{2n}, x_{2n+1}) = p(y_{2n}, y_{2n+1}) \quad \text{for each } n \in \mathbb{N}.$$

By the same way, we may find

$$p(y_{2n+2}, y_{2n+3}) \leq M(x_{2n+1}, x_{2n+2}) = p(y_{2n+1}, y_{2n+2}) \quad \text{for each } n \in \mathbb{N}.$$

The two above inequalities yield that

$$p(y_{n+1}, y_{n+2}) \leq M(x_n, x_{n+1}) = p(y_n, y_{n+1}) \quad \text{for each } n \in \mathbb{N}. \tag{6}$$

Thus, the sequence $\{p(y_n, y_{n+1})\}$ is non-increasing and so there exists $\delta \geq 0$ such that

$$\lim_{n\to+\infty} p(y_n, y_{n+1}) = \delta.$$

By (6), we have

$$\lim_{n\to+\infty} M(x_n, x_{n+1}) = \delta,$$

Suppose that $\delta > 0$. Since

$$\psi(p(y_{2n+1}, y_{2n+2})) \leq \psi(M(x_{2n}, x_{2n+1})) - \varphi(M(x_{2n}, x_{2n+1})),$$

so taking $\limsup_{n\to+\infty}$ in above inequality

$$\begin{aligned}\limsup_{n\to+\infty} \psi(p(y_{2n+1}, y_{2n+2})) &\leq \limsup_{n\to+\infty} \psi(M(x_{2n}, x_{2n+1})) \\ &\quad - \liminf_{n\to+\infty} \varphi(M(x_{2n}, x_{2n+1})).\end{aligned}$$

By continuity of ψ and lower semi-continuity of φ, we get $\psi(\delta) \leq \psi(\delta) - \varphi(\delta)$, so $\varphi(\delta) = 0$, i.e, $\delta = 0$, a contradiction. We conclude that

$$\lim_{n\to+\infty} p(y_n, y_{n+1}) = \lim_{n\to+\infty} M(x_n, x_{n+1}) = 0. \tag{7}$$

By definition of p^s, we have $p^s(x,y) \leq 2p(x,y)$ for each $x, y \in X$, so (7) gives us

$$\lim_{n\to+\infty} p^s(y_n, y_{n+1}) = 0. \tag{8}$$

We shall show that $\{y_n\}$ is a Cauchy sequence in the partial metric space (X, p). From Lemma 1, we need to prove that $\{y_n\}$ is a Cauchy sequence in the metric space (X, p^s). For this, it is sufficient to prove that $\{y_{2n}\}$ is a Cauchy in (X, p^s). Suppose to the contrary. Then, there is a $\varepsilon > 0$ such that for an integer k there exist integers $2m(k) > 2n(k) > k$ such that

$$p^s(y_{2n(k)}, y_{2m(k)}) > \varepsilon. \tag{9}$$

For every integer k, let $m(k)$ be the least positive integer exceeding $n(k)$ satisfying (9) and such that

$$p^s(y_{2n(k)}, y_{2m(k)-2}) \leq \varepsilon. \tag{10}$$

Now, using (9), (10) and the triangular inequality

$$\begin{aligned}\varepsilon < p^s(y_{2n(k)}, y_{2m(k)}) &\leq p^s(y_{2n(k)}, y_{2m(k)-2}) + p^s(y_{2m(k)-2}, y_{2m(k)-1}) \\ &\quad + p^s(y_{2m(k)-1}, y_{2m(k)}) \\ &\leq \varepsilon + p^s(y_{2m(k)-2}, y_{2m(k)-1}) + p^s(y_{2m(k)-1}, y_{2m(k)}).\end{aligned}$$

Then by (8) it follows that

$$\lim_{k\to+\infty} p^s(y_{2n(k)}, y_{2m(k)}) = \varepsilon. \tag{11}$$

Also, by the triangle inequality, we have

$$|p^s(y_{2n(k)}, y_{2m(k)-1}) - p^s(y_{2n(k)}, y_{2m(k)})| \leq p^s(y_{2m(k)-1}, y_{2m(k)}).$$

From (8)-(11) we get

$$\lim_{k\to+\infty} p^s(y_{2n(k)}, y_{2m(k)-1}) = \varepsilon. \tag{12}$$

Similarly, we have

$$\lim_{k\to+\infty} p^s(y_{2n(k)+1}, y_{2m(k)}) = \varepsilon. \tag{13}$$

On the other hand, by definition of p^s,

$$p^s(y_{2n(k)}, y_{2m(k)}) = 2p(y_{2n(k)}, y_{2m(k)}) - p(y_{2n(k)}, y_{2n(k)}) - p(y_{2m(k)}, y_{2m(k)}),$$

$$\begin{aligned} p^s(y_{2n(k)}, y_{2m(k)-1}) &= 2p(y_{2n(k)}, x_{2m(k)-1}) - p(y_{2n(k)}, y_{2n(k)}) \\ &\quad - p(y_{2m(k)-1}, y_{2m(k)-1}), \end{aligned}$$

hence letting $k \to +\infty$, by (11), (12), the condition $(p3)$ in Definition 1 and from (7), we have

$$\lim_{k\to+\infty} p(y_{2n(k)}, y_{2m(k)}) = \frac{\varepsilon}{2}. \tag{14}$$

$$\lim_{k\to+\infty} p(y_{2n(k)}, y_{2m(k)-1}) = \frac{\varepsilon}{2}. \tag{15}$$

Similarly, from (13), we have

$$\lim_{k\to+\infty} p(y_{2n(k)+1}, y_{2m(k)}) = \frac{\varepsilon}{2}. \tag{16}$$

We have

$$\begin{aligned} M(x_{2n(k)}, x_{2m(k)-1}) &= \max\{p(Sx_{2n(k)}, Tx_{2m(k)-1}), p(fx_{2n(k)}, Sx_{2n(k)}), \\ &\qquad p(gx_{2m(k)-1}, Tx_{2m(k)-1}), \\ &\qquad \frac{p(Sx_{2n(k)}, gx_{2m(k)-1}) + p(fx_{2n(k)}, Tx_{2n(k)})}{2}\} \\ &= \max\{p(y_{2n(k)}, y_{2m(k)-1}), p(y_{2n(k)+1}, y_{2n(k)}), \\ &\qquad p(y_{2m(k)}, y_{2m(k)-1}), \\ &\qquad \frac{p(y_{2n(k)}, y_{2m(k)}) + p(y_{2n(k)+1}, y_{2n(k)})}{2}\}, \end{aligned}$$

thus, from (7), (14) and (15), we get

$$\lim_{k\to+\infty} M(x_{2n(k)}, x_{2m(k)-1}) = \max\{\frac{\varepsilon}{2}, 0, 0, \frac{\varepsilon}{4}\} = \frac{\varepsilon}{2}.$$

From (3), we have

$$\begin{aligned}\psi(p(y_{2n(k)+1}, y_{2m(k)})) &= \psi(p(fx_{2n(k)}, gx_{2m(k)-1})) \\ &\leq \psi(M(x_{2n(k)}, x_{2m(k)-1})) - \varphi(M(x_{2n(k)}, x_{2m(k)-1})).\end{aligned}$$

Letting $k \to +\infty$ and referring to (16), we obtain $\psi(\frac{\varepsilon}{2}) \leq \psi(\frac{\varepsilon}{2}) - \varphi(\frac{\varepsilon}{2})$, $\psi(\frac{\varepsilon}{2}) = 0$, it is a contradiction as $\varepsilon > 0$. Thus we proved that $\{y_n\}$ is a Cauchy sequence in the metric space (X, p^s).

Since (X, p) is complete, then from Lemma 1, (X, p^s) is a complete metric space. Therefore, the sequence $\{y_n\}$ converges to some $z \in X$, that is, $\lim_{n\to+\infty} p^s(y_n, z) = 0$. Again, from Lemma 1,

$$p(z, z) = \lim_{n\to+\infty} p(y_n, z) = \lim_{n\to+\infty} p(y_n, y_n).$$

On the other hand, thanks to (7) and the condition ($p2$) from Definition 1

$$\lim_{n\to+\infty} p(y_n, y_n) = 0,$$

so it follows that

$$p(z, z) = \lim_{n\to+\infty} p(y_n, z) = \lim_{n\to+\infty} p(y_n, y_n) = 0. \tag{17}$$

This implies that

$$\lim_{n\to+\infty} p(y_{2n}, z) = \lim_{n\to+\infty} p(y_{2n+1}, z) = 0. \tag{18}$$

Thus, from (4) and (17) we have

$$\lim_{n\to+\infty} p(fx_{2n}, z) = \lim_{n\to+\infty} p(Tx_{2n+1}, z) = p(z, z) = 0 \tag{19}$$

and

$$\lim_{n\to+\infty} p(gx_{2n-1}, z) = \lim_{n\to+\infty} p(Sx_{2n}, z) = p(z, z) = 0. \tag{20}$$

Assume that (a) holds. Using the partial-compatibility of the pair $\{f, S\}$, (19) and (20), we get

$$\lim_{n\to+\infty} p(fSx_{2n}, Sfx_{2n}) = 0, \tag{21}$$

and since $p(z,z)=0$, then again the partial-compatibility of the pair $\{f,S\}$ gives that

$$p(Sz,Sz)=0.$$

Assume that S is continuous, then since $\{y_n\}$ converges to z in (X,p), hence

$$\lim_{n\to+\infty} p(Sy_{2n},Sz)=p(Sz,Sz)=0. \tag{22}$$

By triangular inequality (still holds for partial metric spaces)

$$\begin{aligned} p(fSx_{2n+2},Sy_{2n+2}) &\le p(fSx_{2n+2},Sfx_{2n+2})+p(Sfx_{2n+2},Sz)\\ &\quad + p(Sz,Sy_{2n+2})\\ &= p(fSx_{2n+2},Sfx_{2n+2})+p(Sy_{2n+3},Sz)\\ &\quad + p(Sz,Sy_{2n+2}). \end{aligned}$$

Letting $n\to+\infty$ and having in mind (21) and (22)

$$\lim_{n\to+\infty} p(fSx_{2n},Sy_{2n+2})=0. \tag{23}$$

Moreover, using triangular inequality, (17) and (22), it is clear that

$$\lim_{n\to+\infty} p(Sy_{2n+2},y_{2n+2})=\lim_{n\to+\infty} p(Sy_{2n+2},y_{2n+1})=p(Sz,z). \tag{24}$$

On the other hand, since

$$\begin{aligned} p(fSx_{2n+2},y_{2n+1}) &\le p(fSx_{2n+2},Sfx_{2n+2})+p(Sfx_{2n+2},Sz)\\ &\quad + p(Sz,z)+p(z,y_{2n+2})\\ &= p(fSx_{2n+2},Sfx_{2n+2})+p(Sy_{2n+3},Sz)\\ &\quad + p(Sz,z)+p(z,y_{2n+2}), \end{aligned}$$

and

$$\begin{aligned} p(Sz,z) &\le p(Sz,Sy_{2n+3})+p(Sfx_{2n+2},fSx_{2n+2})\\ &\quad + p(fSx_{2n+2},y_{2n+1})+p(y_{2n+1},z), \end{aligned}$$

then, letting $n\to+\infty$ and from (17), (21) and (22)

$$\lim_{n\to+\infty} p(fSx_{2n+2},y_{2n+1})=p(Sz,z). \tag{25}$$

Now, we have

$$\begin{aligned} M(Sx_{2n+2},x_{2n+1}) &= \max\{p(SSx_{2n+2},Tx_{2n+1}),p(fSx_{2n+2},SSx_{2n+2}),\\ &\qquad p(gx_{2n+1},Tx_{2n+1}),\\ &\qquad \frac{p(SSx_{2n+2},gx_{2n+1})+p(fSx_{2n+2},Tx_{2n+1})}{2}\}\\ &= \max\{p(Sy_{2n+2},y_{2n+1}),p(fSx_{2n+2},Sy_{2n+2}),\\ &\qquad p(y_{2n+2},y_{2n+1}),\\ &\qquad \frac{p(Sy_{2n+2},y_{2n+2})+p(fSx_{2n+2},y_{2n+1})}{2}\}. \end{aligned}$$

By (7), (23)-(25), we get

$$\lim_{n\to+\infty} M(Sx_{2n+2}, x_{2n+1}) = p(Sz, z). \tag{26}$$

Also, $x_{2n+1} \le gx_{2n+1} = Sx_{2n+2}$. From (3), we have

$$\begin{aligned}\psi(p(fSx_{2n+2}, y_{2n+2})) &= \psi(p(fSx_{2n+2}, gx_{2n+1})) \\ &\le \psi(M(Sx_{2n+2}, x_{2n+1})) - \varphi(M(Sx_{2n+2}, x_{2n+1})),\end{aligned} \tag{27}$$

Taking $n \to +\infty$, we get using (25), (26), the continuity of ψ and the lower semi-continuity of φ, we obtain

$$\psi(p(Sz, z)) \le \psi(p(Sz, z)) - \varphi(p(Sz, z)),$$

that is $p(Sz, z) = 0$, so $Sz = z$.

Now, $x_{2n+1} \le gx_{2n+1}$ and $gx_{2n+1} \to z$ as $n \to +\infty$, $x_{2n+1} \le z$ and (3) becomes

$$\begin{aligned}\psi(p(fz, y_{2n+2})) &= \psi(p(fz, gx_{2n+1})) \\ &\le \psi(M(z, x_{2n+1})) - \varphi(M(z, x_{2n+1})),\end{aligned} \tag{28}$$

where

$$\begin{aligned}M(z, x_{2n+1}) &= \max\{p(Sz, Tx_{2n+1}), p(fz, Sz), p(gx_{2n+1}, Tx_{2n+1}), \\ &\qquad \frac{p(Sz, gx_{2n+1}) + p(fz, Tx_{2n+1})}{2}\} \\ &= \max\{p(z, y_{2n+1}), p(fz, z), p(y_{2n+2}, y_{2n+1}), \\ &\qquad \frac{p(z, y_{2n+2}) + p(fz, y_{2n+1})}{2}\} \\ &\to \max\{0, p(fz, z), 0, \frac{p(fz, z)}{2}\} = p(fz, z) \quad \text{as } n \to +\infty.\end{aligned}$$

Taking $n \to +\infty$ in (28), we have $\psi(p(fz, z)) \le \psi(p(fz, z)) - \varphi(p(fz, z))$, and so $p(fz, z) = 0$, then $fz = z$.

Since $fX \subseteq TX$, there exists a point $w \in X$ such that $fz = Tw$. Suppose that $p(gw, Tw) \ne 0$. Since $z = fz = Tw \le fTw \le w$ implies $z \le w$. From (3), we obtain

$$\psi(p(Tw, gw)) = \psi(p(fz, gw)) \le \psi(M(z, w)) - \varphi(M(z, w)), \tag{29}$$

where

$$\begin{aligned}M(z, w) &= \max\{p(Sz, Tw), p(fz, Sz), p(gw, Tw), \\ &\qquad \frac{p(Sz, gw) + p(fz, Tw)}{2}\} \\ &= \max\{p(Tw, Tw), p(Tw, Tw), p(gw, Tw), \frac{p(Tw, gw) + p(z, z)}{2}\} \\ &= p(Tw, gw).\end{aligned}$$

Thus, (29) becomes $\psi(p(Tw,gw)) \leq \psi(p(Tw,gw)) - \varphi(p(Tw,gw))$, that is, $p(Tw,gw) = 0$, so $Tw = gw$. Since g and T are weakly compatible, hence $gz = gfz = gTw = Tgw = Tfz = Tz$. We deduce that z is a coincidence point of g and T.

Also, since $x_{2n} \leq fx_{2n}$ and $fx_{2n} \to z$ as $n \to +\infty$, so $x_{2n} \leq z$ and then from (3)

$$\psi(p(y_{2n+1},gz)) = \psi(p(fx_{2n},gz)) \leq \psi(M(x_{2n},z)) - \varphi(M(x_{2n},z)),$$

where

$$\begin{aligned} M(x_{2n},z) &= \max\{p(Sx_{2n},Tz), p(fx_{2n},Sx_{2n}), p(gz,Tz), \\ &\qquad \frac{p(Sx_{2n},gz)+p(fx_{2n},Tz)}{2}\} \\ &= \max\{p(y_{2n},gz), p(y_{2n+1},y_{2n}), p(gz,gz), \\ &\qquad \frac{p(y_{2n},gz)+p(y_{2n+1},gz)}{2}\} \\ &\quad \to \max\{p(z,gz), 0, p(gz,gz), \frac{p(z,gz)+p(z,gz)}{2}\} \\ &= p(z,gz) \quad \text{as } n \to +\infty, \end{aligned}$$

since $p(gz,gz) \leq p(z,gz)$ because of property $(p3)$ in Definition 1. On taking limit as $n \to +\infty$, we have $\psi(p(z,gz)) - \psi(p(z,gz)) - \varphi(p(z,gz))$, so $p(z,gz) = 0$, hence $z = gz$. Therefore, $fz = gz = Sz = Tz = z$. The proof is similar when f is continuous.

Similarly, the result follows when (b) holds.

Now suppose that set of common fixed points of f, g, S and T is well ordered. We claim that common fixed point of f, g, S and T is unique. Assume on contrary that, $fu = gu = Su = Tu = u$ and $fv = gv = Sv = Tv = v$ but $u \neq v$, so $p(u,v) \neq 0$. By supposition, we can replace x by u and y by v in (3) to obtain

$$\psi(p(u,v)) = \psi(p(fu,gv)) \leq \psi(M(u,v)) - \varphi(M(u,v)),$$

where

$$\begin{aligned} M(u,v) &= \max\{p(Su,Tv), p(fu,Su), p(gv,Tv), \frac{p(Su,gv)+p(fu,Tv)}{2}\} \\ &= \{p(u,v), p(u,u), p(v,v), \frac{p(u,v)+p(u,v)}{2}\} = p(u,v). \end{aligned}$$

This yields that $\psi(p(u,v)) \leq \psi(p(u,v)) - \varphi(p(u,v))$, then $\varphi(p(u,v)) = 0$ so $p(u,v) = 0$, it is contradiction. Hence $u = v$. Conversely, if f, g, S and T have only one common fixed point then the set of common fixed point of

f, g, S and T being singleton is well ordered. This completes the proof of Theorem 2. ■

Now, we state some corollaries.

Corollary 1. *Let $(X, \leq, p)$ be a partially ordered complete partial metric space. Let f, S and T be self maps on X, (T, f) and (S, f) be partially weakly increasing with $fX \subseteq TX$ and $fX \subseteq SX$, and dominating map f is weak annihilator of T and S. Suppose that there exist control functions ψ and φ such that for every two comparable elements $x, y \in X$,*

$$\begin{aligned}\psi(p(fx, fy)) \ \leq \ & \psi(\max\{p(Sx, Ty), p(fx, Sx), p(fy, Ty), \\ & \frac{p(Sx, fy) + p(fx, Ty)}{2}\}) \\ & - \varphi(\max\{p(Sx, Ty), p(fx, Sx), p(fy, Ty), \\ & \frac{p(Sx, fy) + p(fx, Ty)}{2}\}),\end{aligned}$$

is satisfied. If for a non-decreasing sequence $\{x_n\}$ with $x_n \leq y_n$ for all n and $y_n \to u$ implies that $x_n \leq u$ and either

(a) $\{f, S\}$ are compatible, f or S is continuous and $\{f, T\}$ are weakly compatible or

(b) $\{f, T\}$ are compatible, f or T is continuous and $\{f, S\}$ are weakly compatible,

then f, S and T have a common fixed point. Moreover, the set of common fixed points of f, S and T is well ordered if and only if f, S and T have one and only one common fixed point.

Proof. It follows by taking $g = f$ in Theorem 2. ■

Corollary 2. *Let $(X, \leq, p)$ be a partially ordered complete partial metric space. Let f, g and T be self maps on X, (T, f) and (T, g) be partially weakly increasing with $fX \subseteq TX$ and $gX \subseteq TX$, and dominating maps f and g are weak annihilators of T. Suppose that there exist control functions ψ and φ such that for every two comparable elements $x, y \in X$,*

$$\begin{aligned}\psi(p(fx, gy)) \ \leq \ & \psi(\max\{p(Tx, Ty), p(fx, Tx), p(gy, Ty), \\ & \frac{p(Tx, gy) + p(fx, Ty)}{2}\}) \\ & - \varphi(\max\{p(Tx, Ty), p(fx, Tx), p(gy, Ty), \\ & \frac{p(Tx, gy) + p(fx, Ty)}{2}\}),\end{aligned}$$

is satisfied. If for a non-decreasing sequence $\{x_n\}$ with $x_n \leq y_n$ for all n and $y_n \to u$ implies that $x_n \leq u$ and either (*a*) $\{f,T\}$ *are compatible, f or T is continuous and $\{f,T\}$ are weakly compatible or*

(*b*) $\{g,T\}$ *are compatible, g or T is continuous and $\{g,T\}$ are weakly compatible,*

then f, g and T have a common fixed point. Moreover, the set of common fixed points of f, g and T is well ordered if and only if f, g and T have one and only one common fixed point.

Proof. It follows by taking $S = T$ in Theorem 2. ■

Corollary 3. *Let $(X, \leq, p)$ be a partially ordered complete partial metric space. Let f and T be self maps on X, (T, f) be partially weakly increasing with $fX \subseteq TX$, and dominating map f is weak annihilator of T. Suppose that there exist control functions ψ and φ such that for every two comparable elements $x, y \in X$,*

$$\begin{aligned}\psi(p(fx,fy)) \;\leq\; & \psi(\max\{p(Tx,Ty), p(fx,Tx), p(fy,Ty), \\ & \frac{p(Tx,fy)+p(fx,Ty)}{2}\}) \\ & - \varphi(\max\{p(Tx,Ty), p(fx,Tx), p(fy,Ty), \\ & \frac{p(Tx,fy)+p(fx,Ty)}{2}\}),\end{aligned}$$

is satisfied. If for a non-decreasing sequence $\{x_n\}$ with $x_n \leq y_n$ for all n and $y_n \to u$ implies that $x_n \leq u$. Assume that $\{f,T\}$ are compatible, f or T is continuous and $\{f,T\}$ are weakly compatible, then f and T have a common fixed point. Moreover, the set of common fixed points of f and T is well ordered if and only if f and T have one and only one common fixed point.

Proof. It follows by taking $g = f$ in Corollary 2. ■

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