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Huzoor H. Khan and Rifaqat Ali

GROWTH AND APPROXIMATION ERRORS OF ENTIRE SERIES HAVING INDEX-PAIR (p, q)

Abstract. In this paper we Studied the polynomial coefficients and approximation errors for functions of the form $f(z) = \sum_{k=1}^{\infty} q_k(z)[\gamma(z)]^{k-1}$ belong to $L^s(B)$, the class of all functions holomorphic on Caratheödory domain B. The lower (p, q)-order and generalized lower (p, q)-type with respect to proximate order have been characterized in terms of these polynomial coefficients and approximation errors.

Key words: index-pair, lower (p, q)-order, generalized lower (p, q)-type, lemniscate, polynomial coefficients and approximation errors.

AMS Mathematics Subject Classification: 30E10, 41A15.

1. Introduction

The growth of an entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$, is studied with the help of growth parameters ρ, λ, T and t, known as order, lower order, type and lower type respectively and defined as follows:

$$\begin{matrix} \rho \\ \lambda \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log\log M(r,f)}{\log r}$$

and

$$\begin{matrix} T \\ t \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log M(r,f)}{r^\rho}, \quad (0 < \rho < \infty)$$

where $M(r, f) = \max_{|B|=r} |f(z)|$. The coefficients equivalents of order ρ and type T are known [1, p. 9-11]. Thus,

$$\rho = \limsup_{n\to\infty} \frac{n \log n}{\log |a_n|^{-1}} \tag{1}$$

$$e\rho T = \limsup_{n\to\infty} n|a_n|^{\rho/n}. \tag{2}$$

A coefficient formula analogous to (1) and (2) does not hold always for lower order and lower type. Juneja [2] and Juneja and Kapoor [3] obtained formula for the lower order and Shah [13] for lower type, involving coefficients which hold for every entire function. Thus, if $f(z)$ be an entire function of lower order λ and lower type t, then

$$\lambda = \max_{\{m_k\}} \left[\liminf_{k\to\infty} \frac{m_k \log m_{k-1}}{\log |a_{m_k}|^{-1}} \right] \tag{3}$$

$$\lambda = \max_{\{m_k\}} \left[\liminf_{k\to\infty} \frac{(m_k - m_{k-1}) \log m_{k-1}}{\log |a_{m_{k-1}}/a_{m_k}|} \right] \tag{4}$$

and

$$te\rho = \max_{\{m_k\}} \left\{ \liminf_{k\to\infty} \; m_{k-1}(|a_{m_k}|^{\rho/m_k}) \right\}. \tag{5}$$

Rice [11] studied the results (1) and (2) by considering the polynomial expansion of $f(z)$ of the form

$$f(z) = \sum_{k=1}^{\infty} q_k(z)[\gamma(z)]^{k-1}, \tag{6}$$

where $\gamma(z)$ is a polynomial of degree ζ and $q_k z$ is a uniquely determined polynomial of degree $\zeta - 1$ or less. He showed that $f(z)$ given by (6) is an entire function of order ρ, if and only if,

$$\limsup_{R\to\infty} \frac{\log\log M(\Gamma_R, f)}{\log R} = \rho/\zeta = \limsup_{n\to\infty} \frac{n \log n}{(\log \|q_n(z)\|_{\Gamma_\alpha})^{-1}}, \tag{7}$$

and $f(z)$ is of order $\rho > 0$ and type T $(0 < T < \infty)$, if and only if,

$$\limsup_{R\to\infty} \frac{\log M(\Gamma_R, f)}{R^{\rho/\zeta}} = T = \frac{\zeta}{e\rho} \limsup_{n\to\infty} n(\|q_n(z)\|_{\Gamma_\alpha})^{\rho/\zeta n}, \tag{8}$$

where Γ_R be the lemniscate $\Gamma_R = \{z : |\gamma(z)|\} = R$, $\|\Gamma_R\|$ be the length of Γ_R and $M(\Gamma_R, f) = \|f(z)\|_{\Gamma_R} = \max_{z\in\Gamma_R} |f(z)|$.

(7) and (8) characterize the influence of the rate of decrease of $\|q_n(z)\|_{\Gamma_\alpha}$ on the growth of $f(z)$. But, as in the power series, these results do not give any precise information about the growth of the function $f(z)$ if it is of $\rho = 0$ and $\rho = \infty$. For further classification of entire function, in the present paper, we have picked a concept of index- pair (p, q) which was introduced by Juneja et al ([5], [6]). Also, to compare the growth of entire functions given by (6) which are of the same order and of infinite type, we used the

concept of proximate order with (p, q)-growth which is extended by Nandan et al [10].

Moreover, we obtained formula for the lower (p, q)-order and generalized lower (p, q)-type in terms of polynomial coefficients and approximation errors in $L^s - norm$, $1 \leq s \leq \infty$, of $f(z)$ given by (6). Recently, Kumar [8] studied the results for (α, α)-orders. Kumar and kaur [9] obtained the results for (α, β)-orders for analytic functions. However, they have to study separately the entire functions of slow and fast growth. *That's* why in our studied the (p, q) -growth has been preferred to the (α, α) and (α, β)-growths. Our results extend and improve some results of Sato [12], Rice [11], Juneja [2], and Juneja and Kapoor [4].

Let B^* be a component of the complement of the closure of the Carattheö-dory domain B that contains the points ∞. Set $B_R = \{z : |\bar{\phi}(z)| = R\}$, $R > 1$ where the function $w^* = \bar{\phi}(z)$ maps B^* Conformally onto $|w^*| > 1$ such that $\bar{\phi}(\infty) = \infty$ and $\bar{\phi}'(\infty) > 0$. Here B_R is the largest equipotential curve of the modulus of the mapping function associated with the domain B, B_1 correspond to the boundary of B.

Given $\varepsilon > 0$ there is a lemniscate $\Gamma_\alpha = \{z : |\gamma(z)| = \alpha\}$ so that Γ_α is interior to $B_{1+\varepsilon}$ and exterior to B_1.

$L^s(B)$, $1 \leq s \leq \infty$ be the class of all functions f holomorphic on B and satisfying

$$\|f\|_{B,s} = \left[\frac{1}{A^*} \int \int_B |f(z)|^s dx\ dy\right]^{1/s} < \infty,$$

where the last inequality is understood to be $\sup_{z \in B} |f(z)| < \infty$ for $s = \infty$. Then $\|, \|_{B,s}$ is called the $L^s - norm$ on $L^s(B)$. For $f \in L^s(B)$ we define $E^s_\zeta(f)$, the error in approximating the function f by polynomial of degree at most $\zeta = nk$ in L^s-norm as

$$E^s_\zeta(f) = E^s_\zeta(f, B) = \inf_{t \in \pi_s} \|f - t\|_{B,s}\ , \quad n = 0, 1, 2, ...\ .$$

where π_ζ consists of all polynomials of degree at most $\zeta = nk$.

2. Definitions and notations

First we introduce with the concept of (p, q)-scale, $p \geq q \geq 1$, and certain notations which will be frequently used in the tex:

$$\begin{aligned}\exp^{[m]} x &= \log^{[-m]} x \\ &= \exp(\exp^{[m-1]} x) = \log(\log^{[-m-1]} x), \quad m = \pm 1, \pm 2, \ldots,\end{aligned}$$

$$\Lambda_{[r]}(x) = \Pi^r_{i=0} \log^{[i]} x \ \text{ for } \ r = 0, 1, \ldots,$$

$$P_\chi(L(p,q)) = \begin{cases} L(p,q) & \text{if} \quad q < p < \infty, \\ \chi + L(p,q) & \text{if} \quad p = q = 2, \\ \max(1, L(p,q)) & \text{if} \quad 3 \le p \le q, \\ \infty & \text{if} \quad p = q = \infty, \end{cases}$$

$$M(p,q) = \begin{cases} \frac{(\rho(2,2)-1)^{\rho(2,2)-1}}{(\rho(2,2))^{\rho(2,2)}} & \text{if} \quad (p,q) = (2,2), \\ \frac{1}{e\rho(2,1)} & \text{if} \quad (p,q) = (2,1), \\ 1 & \text{otherwise.} \end{cases}$$

Definition 1. *An entire function $f(z)$ is said to be of (p,q)-order $\rho(p,q)$ and lower (p,q)-order $\lambda(p,q)$ if it is of index-pair (p,q) such that*

$$\lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p]} M(r,f)}{\log^{[q]} r} = \begin{matrix} \rho(p,q) \\ \lambda(p,q) \end{matrix},$$

and the function $f(z)$ having (p,q)-order $\rho(p,q)$ $(b < \rho(p,q) < \infty)$ is said to be of (p,q)-type $T(p,q)$ and lower (p,q)-type $t(p,q)$ if

$$\lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p-1]} M(r,f)}{(\log^{[q-1]} r)^{\rho}} = \begin{matrix} T(p,q) \\ t(p,q) \end{matrix}, \quad 0 \le t(p,q) \le T(p,q) \le \infty,$$

where $b = 1$ if $p = q$, $b = 0$ if $p > q$.

Definition 2. *A positive function $\rho_{p,q}(r)$ defined on $[r_0, \infty)$, $r_0 > \exp^{[q-1]} 1$, is said to be a proximate order of an entire function with index-pair (p,q) if*

(i) $\rho_{p,q}(r) \longrightarrow \rho(p,q)$ as $r \to \infty$, $(b < \rho(p,q) < \infty)$,

(ii) $\Delta_{[q]}(r)\ \rho'_{p,q}(r) \longrightarrow 0$ as $r \to \infty$; $\rho'_{p,q}(r)$ are denote the derivative of $\rho_{p,q}(r)$.

It is known that [10, Thm. 4] that $(\log^{[q-1]} r)^{\rho_{p,q}(r)-A}$ is a monotonically increasing function of r for $r > r_0$, Hence we can define the function $\phi(x)$ for to be the unique solution of equation,

$$x = (\log^{[q-1]} r)^{\rho_{p,q}(r)-A} \Leftrightarrow \phi(x) = \log^{[q-1]} r.$$

where $A = 1$ of $(p,q) = (2,2)$ and $A = 0$ otherwise.

Definition 3. *Let $f(z)$ be an entire function of (p,q)- order $\rho(p,q)$ $(b < \rho(p,q) < \infty)$ such that*

$$\lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p-1]} M(r,f)}{(\log^{[q-1]} r)^{\rho_{p,q}(r)}} = \begin{matrix} T^*(p,q) \\ t^*(p,q) \end{matrix}, \quad 0 \le t^*(p,q) \le T^*(p,q) \le \infty.$$

If the quantity $t^*(p,q)$ is different from zero and infinite then $\rho_{p,q}(r)$ said to be the lower proximate order and $t^*(p,q)$ is generalized (p,q)- type of a given entire function.

Definition 4. *An entire function with index-pair (p,q) is said to be of regular (p,q)-growth if $b < \lambda(p,q) = \rho(p,q) < \infty$ and further, it is of perfectly regular (p,q)- with respect to a proximate order $\rho_{p,q}(r)$ if $0 \le t^*(p,q) \le T^*(p,q) \le \infty$.*

3. Auxiliary results

In this section we discuss some auxiliary results which will be used in the sequel.

Lemma 1. *If $f(z) = \sum_{k=1}^{\infty} q_k(z)[\gamma(z)]^{k-1}$ is an entire function of (p,q)-order $\rho(p,q,)$ and lower (p,q)-order $\lambda(p,q)$, then*

$$\lim_{R\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p]} M(\Gamma_R, f)}{\log^{[q]} R} = \begin{matrix} \rho(p,q)/\zeta \\ \lambda(p,q)/\zeta \end{matrix},$$

and for $\rho(p,q)$ $(b < \rho(p,q) < \infty)$, $T^*(p,q)$ and $t^*(p,q)$ are given by

$$\lim_{R\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p-1]} M(\Gamma_R, f)}{(\log^{[q-1]} R)^{\rho_{p,q}(R)/\zeta}} = \begin{matrix} T^*(p,q) \\ t^*(p,q) \end{matrix}, \quad 0 \le t^*(p,q) \le T^*(p,q) \le \infty.$$

Proof. The lemma can be proved easily following the lines of Rice [11, Lemma 3] so we omit the proof. ■

Lemma 2. *Let $f(z) = \sum_{k=1}^{\infty} q_z(z)[\gamma(z)]^{k-1}$ be an entire function of (p,q)-order $\rho(p,q)$ and lower (p,q)-order $\lambda(p,q)$, then*

$$\lim_{R\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p]} \bar{M}(\Gamma_R, f)}{\log^{[q]} R} = \begin{matrix} \rho(p,q)/\zeta \\ \lambda(p,q)/\zeta \end{matrix},$$

and for $\rho(p,q)$ $(b < \rho(p,q) < \infty)$, $T^*(p,q)$ and $t^*(p,q)$ are given by

$$\lim_{R\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p-1]} \bar{M}(\Gamma_R, f)}{(\log^{[q-1]} R)^{\rho_{p,q}(R)/\zeta}} = \begin{matrix} T^*(p,q) \\ t^*(p,q) \end{matrix}, \quad 0 \le t^*(p,q) \le T^*(p,q) \le \infty,$$

where $\bar{M}(\Gamma_R, f) = \max_{z\in B_R} |f(z)|$.

Proof. Let z_0 be a fixed point of the set B and $R > 1$. Then from [16], we get

$$R - 2|B| - |z_0| \le |z| \le R + |B| + |z_0|, \quad z \in B_R.$$

For $p \geq q \geq 1$, $\beta < 1$ and $\eta > 1$, using $\log^{[q]} Kx \simeq \log^{[q]} x$ as $x \to \infty$, $0 < K < \infty$,

$$\frac{\log^{[p]} M(\Gamma_{\beta R}, f)}{\log^{[q]} R} \leq \frac{\log^{[p]} \bar{M}(\Gamma_R, f)}{\log^{[q]} R} \leq \frac{\log^{[p]} M(\Gamma_{\eta R}, f)}{\log^{[q]} R}.$$

and, for $\rho(p,q)$ $(b < \rho(p,q) < \infty)$, using $\log^{[q]}(K+x) \simeq \log^{[q]} x$ as $x \to \infty$,

$$\frac{\log^{[p-1]} M(\Gamma_{R-a}, f)}{(\log^{[q-1]})^{\rho_{p,q}(R)/\zeta}} \leq \frac{\log^{[p-1]} \bar{M}(\Gamma_R, f)}{(\log^{[q-1]})^{\rho_{p,q}(R)/\zeta}} \leq \frac{\log^{[p-1]} M(\Gamma_{R+c}, f)}{(\log^{[q-1]})^{\rho_{p,q}(R)/\zeta}},$$

where $a = 2|B| + z_0$, $c = |B| + |z_0|$.

Now by virtue of Lemma 1, the proof is complete. ■

4. Main results

In this section we shall prove our main results.

Consider the function

$$H_\alpha(w) = \sum_{k=1}^{\infty} \|q_k(z)\|_{\Gamma_\alpha} w^k, \quad \alpha < R,$$

where $\|q_k(z)\|_{\Gamma_\alpha} = \max_{z \in \Gamma_R}\{|q_k(z)|\}$ as $k \to \infty$. It is known [10, Lemma 2] that if $f(z)$ is analytic in Γ_R, then there exists a polynomial $Q(z)$ of degree $\zeta - 1$ independent of k and R such that for $\alpha < R$ and $k = 1, 2, \ldots$.

$$\|q_k(z)\|_{\Gamma_\alpha} \leq \frac{\|\Gamma_R\| M(\Gamma_R, f)}{2\pi R^k} \|Q(z)\|_{\Gamma_R}. \tag{9}$$

In view of (9) we can seen that $H_\alpha(w)$ is entire if and only if

$$\left[\|q_k(z)\|_{\Gamma_\alpha}\right]^{1/k} = 0. \tag{10}$$

Moreover, $H_\alpha(w) = \sum_{k=1}^{\infty} \|q_k(z)\|_{\Gamma_\alpha} w^k$, holds in the whole complex plane.

Now we prove

Theorem 1. *If $f \in L^s(B)$, $1 \leq s \leq \infty$, can be extended to an entire function with index- pair (p,q), lower (p,q)-order $\lambda(p,q)$ $(b < \lambda(p,q) < \infty)$ and generalized lower (p,q)-type $t^*(p,q)$, then for every $\|q_k(z)\|_{\Gamma_\alpha}$, there exists an entire function $H_\alpha(w) = \sum_{k=1}^{\infty} \|q_k(z)\|_{\Gamma_\alpha} w^k$, such that*

$$\lambda(p,q,f)/\zeta = \lambda(p,q,H_\alpha) \quad \text{and} \quad t^*(p,q,f) = t^*(p,q,H_\alpha). \tag{11}$$

Proof. H_α is entire function by (10). For $R > \alpha$ from [15, p. 77] we have

$$\|q_k(z)\|_{\Gamma_R} \leq \|q_k(z)\|_{\Gamma_\alpha} R^{\zeta-1},$$

for $z \in \Gamma_R$

$$|f(z)| = \sum_{k=1}^{\infty} \|q_k(z)\|_{\Gamma_R} \; \|\gamma(z)\|_{\Gamma_R}^{k-1}$$

or

$$\begin{aligned} \bar{M}(\Gamma_R, f) \; &\leq \sum_{k=1}^{\infty} \|q_k(z)\|_{\Gamma_\alpha} R^{k+\zeta-2}, \quad z \in B_k \\ &= R^{\zeta-2} \sum_{k=1}^{\infty} \|q_k(z)\|_{\Gamma_\alpha} R^k \\ &= R^{\zeta-2} \; H_\alpha(R), \quad R > 1. \end{aligned}$$

Now using Lemma 2, we obtain that for all index- pair (p,q),

$$\lambda(p,q,f)/\zeta \leq \lambda(p,q,H_\alpha) \quad \text{and} \quad t^*(p,q,f) \leq t^*(p,q,H_\alpha). \tag{12}$$

In view of Lemma 1, Lemma 2 and (10) for every $\varepsilon > 0$, the power series expansion of $H_\alpha(w)$ yields

$$\begin{aligned} H_\alpha(R/e^\varepsilon) \; &= \sum_{k=1}^{\infty} \; \|q_k(z)\|_{\Gamma_\alpha} (R/e^\varepsilon)^k \\ &\leq \sum_{k=1}^{\infty} \frac{\bar{M}(\Gamma_R, f) \; \|\Gamma_R\| \; \|Q\|_{\Gamma_\alpha} (R/e^\varepsilon)^k}{2\pi R^k} \\ &= \bar{M}(\Gamma_R, f) R^{1/\zeta} (1 + O(1)) \; \|Q\|_{\Gamma_\alpha} \sum_{k=1}^{\infty} \frac{1}{e^{k\varepsilon}} \\ &= \bar{M}(\Gamma_R, f) R^{1/\zeta} (1 + O(1)) \; \|Q\|_{\Gamma_\alpha} \frac{1}{e^{\varepsilon - 1}}. \end{aligned}$$

Again in view of above inequality and Lemma 1, lemma 2 for $p \geq 2$ and $q \geq 1$,

$$\lambda(p,q,H_\alpha) \leq \lambda(p,q,f)/\zeta \quad \text{and} \quad t^*(p,q,H_\alpha) \leq t^*(p,q,f). \tag{13}$$

Combining (13) and (14), the proof is complete for $s = \infty$. The result can be obtained for $1 \leq s < \infty$ after a simple manipulation. ∎

Theorem 2. *If $f \in L^s(B)$, $1 \le s \le \infty$, can be extended to be an entire function with index- pair (p,q), lower (p,q)-order $\lambda(p,q)$ $(b < \lambda(p,q) < \infty)$ and generalized lower (p,q)-type $t^*(p,q)$, then for every $E^s_\zeta(f)$, there exists an entire function $\tilde{H}(t) = \sum\limits_{k=1}^{\infty} E^s_\zeta(f)t^k$, such that*

$$\lambda(p,q,f)/\zeta = \lambda(p,q,\tilde{H}) \quad \textit{and} \quad t^*(p,q,f) = t^*(p,q,\tilde{H}). \tag{14}$$

Proof. From [14], we have

$$E^s_\zeta(f) \le \sum_{k=n}^{\infty} \|q_k(z)\|_{\Gamma_\alpha}\ \alpha^{k-1}.$$

Using (9) we get

$$E^s_\zeta(f) \le \sum_{k=n}^{\infty} \frac{\|\Gamma_R\|\bar{M}(\Gamma_R,f)}{2\pi R^k}\|Q(z)\|_{\Gamma_R}\ \alpha^{k-1}.$$

For $\alpha > 1$ be fixed constant and $R > \alpha$, we obtain

$$E^s_\zeta(f) \le \gamma^* \bar{M}(\Gamma_R,f)\Big(\frac{\alpha}{R}\Big)^n\Big(\frac{1}{1-\alpha/R}\Big)R^{1/\zeta}(1+O(1))\ , \tag{15}$$

for sufficiently large R and

$$\mu(R,\alpha\tilde{H}) \le \gamma^* \bar{M}(\Gamma_R,f)\Big(\frac{R^{1+1/\zeta}}{R-\alpha}\Big)(1+O(1)). \tag{16}$$

Now using Lemma 1, Lemma 2 and (16) we set for $p \ge 2$ and $q \ge 1$,

$$\lambda(p,q,f)/\zeta \ge \lambda(p,q,\tilde{H}) \quad \textit{and} \quad t^*(p,q,f) \ge t^*(p,q,\tilde{H}). \tag{17}$$

In order to prove reverse inequalities, consider the function

$$\tilde{f}(z) = \sum_{k=0}^{\infty}(P_{k+1}(z) - P_k(z)), \tag{18}$$

since

$$|P_{k+1}(z) - P_k(z)| \le \|P_{k+1}(z) - P_k(z)\| \le 2\ \|f - P_k(z)\|, \quad z \in B.$$

Using Walsh inequality, [15, p. 77], we have get

$$|P_{k+1}(z) - P_k(z)| \le 2\ \|f - P_k(z)\|^2_{B,1}R'^k, \quad z \in B_{k'},\ \ R' > 1.$$

Now Applying Holder inequality, we get

$$\|P_{k+1}(z)-P_k(z)\|/R'^k \le 2\ A^{*^{q'}}\|f-P_k(z)\|_{B_{R',q'}},$$

where A^* is defined as earlier and $q'=1-1/s$, $1\le s\le\infty$. Since above inequality holds for any polynomial $P_k(z)$, so we have

$$\|P_{k+1}(z)-P_k(z)\|/R'^k \le 2\ A^{*^{q'}}E^s_{k-1}(f),\quad 1\le s<\infty. \tag{19}$$

From (18), (19) and (15), we get

$$|\tilde{f}(z)|\le\sum_{k=0}^{\infty}\|P_{k+1}(z)-P_k(z)\|$$

or

$$\begin{aligned}\bar{M}(\Gamma_R,f)\ &\le\ |a_0|+2A^{*^{q'}}\sum_{k=0}^{\infty}E^s_{k-1}(f)(RR')^R,\\ &\le\ |a_0|+2A^{*q'}(RR')^{1/\zeta}\mu(RR^*,\tilde{H}),\qquad z\in B_R.\end{aligned} \tag{20}$$

The right hand side of series (18) converges for every R and therefore, the series on the right of (20) converges uniformly on every compact subset of complex plane and so $\tilde{f}(z)$ is entire and $\tilde{f}(z)=f(z)$. Since $\lim_{\zeta\to\infty}\left[E^s_\zeta(f)\right]^{1/\zeta}=0$ by (15) it follows that $\tilde{H}(t)$ is entire. By virtue of Lemma 1, Lemma 2 and (20) for all $p\ge2$ and $q\ge1$, we have

$$\lambda(p,q,f)/\zeta\le\lambda(p,q,\tilde{H})\quad\text{and}\quad t^*(p,q,f)\le t^*(p,q,\tilde{H}). \tag{21}$$

(17) and (21) together proves the Theorem 2. ■

Theorem 3. *Let* $f(z)=\sum_{k=1}^{\infty}q_k(z)[\gamma(z)]^{k-1}$ *belongs to* $L^s(B)$, $1\le s\le\infty$ *and* α *be fixed. Then* $f(z)$ *can be extended to be an entire function of lower* (p,q)*-order* $\lambda(p,q)$ $(b<\lambda(p,q)<\infty)$ *if and only if, for* $(p,q)\ne(2,2)$,

$$\frac{\lambda(p,q,f)}{\zeta}=\max_{\{m_k\}}[P_\chi(l(p,q,H_\alpha))]=\max_{\{m_k\}}[P_\chi(l(p,q,\tilde{H}))] \tag{22}$$

and

$$\frac{\lambda(p,q,f)}{\zeta}=\max_{\{m_k\}}[P_\chi(l^*(p,q))]=\max_{\{m_k\}}[P_\chi(l^*(p,q,\tilde{H}))] \tag{23}$$

where

$$l(p,q,H_\alpha)=\liminf_{k\to\infty}\frac{\log^{[p-1]}m_{k-1}}{\log^{[q-1]}(\|q_{m_k}(z)\|_{\Gamma_\alpha})^{-1/m_k}}$$

$$l(p,q,\tilde{H})=\liminf_{k\to\infty}\frac{\log^{[p-1]}m_{k-1}}{\log^{[q-1]}(E^s_{\zeta_k}(f))^{-1/\zeta_k}},\quad \zeta_k=nm_k,$$

$$l^*(p,q,H_\alpha)=\liminf_{k\to\infty}\frac{\log^{[p-1]}m_{k-1}}{\log^{[q-1]}(\frac{1}{m_k-m_{k-1}}\log(\frac{\|q_{m_{k-1}(z)}\|_{\Gamma_\alpha}}{\|q_{m_k(z)}\|_{\Gamma_\alpha}}))},$$

$$l^*(p,q,H_\alpha)=\liminf_{k\to\infty}\frac{\log^{[p-1]}m_{k-1}}{\log^{[q-1]}(\frac{1}{m_k-m_{k-1}}\log(\frac{E^s_{\zeta_{k-1}}(f)}{E^s_{\zeta_k}(f)}))},$$

such that

$$\chi\equiv\chi\{m_k\}=\liminf_{k\to\infty}\frac{\log m_{k-1}}{\log m_k}.$$

Further, (22) and (23) hold for $(p,q)=(2,2)$ *also provided* $\{m_k\}$ *be the sequence of principal indices such that* $\log m_{k-1}\simeq\log m_k$ *as* $k\to\infty$.

Proof. In view of Theorem 1 and Theorem 2 we have concluded that $f\in L^s(B)$ can be extended to an entire function, if and only if, $H_\alpha(w)$ and $\tilde{H}(t)$ are entire functions. Applying Theorem 2 by Juneja et al [5, p. 62] to the functions $H_\alpha(w)=\sum_{k=1}^{\infty}\|q_{m_k}(z)\|_{\Gamma_\alpha}w^k$ and $\tilde{H}(t)=\sum_{k=1}^{\infty}E^s_{\zeta_k}(f)t^{\zeta_k}$, $\zeta_k=nm_k$, with left equalities of (11) and (14), it completes the proof of Theorem 3. ■

Theorem 4. *Let* α *be a fixed and* $f(z)=\sum_{k=1}^{\infty}q_k(z)[\gamma(z)]^{k-1}$ *belongs to* $L^s(B)$, $1\le s\le\infty$. *Then* $f(z)$ *can be extended to be an entire function of* (p,q)*-order* $\rho(p,q)$ $(b<\rho(p,q)<\infty)$ *and generalized lower* (p,q)*-type* $t^*(p,q)$ $(0<t^*(p,q)<\infty)$ *if and only if*

$$\begin{aligned}(24)\ t^*(p,q)&=\max_{\{m_k\}}\left\{\liminf_{k\to\infty}\left(\frac{\phi(\log^{[p-2]}m_{k-1})}{\log^{[q-1]}(\|q_{m_k}(z)\|_{\Gamma_\alpha})^{-1/\zeta_k}}\right)^{\rho(p,q)/\zeta_k}\right\},\\&=\max_{\{m_k\}}\left\{\liminf_{k\to\infty}\left(\frac{\phi(\log^{[p-2]}m_{k-1})}{\log^{[q-1]}(E^s_{\zeta_k}(f))^{-1/\zeta_k}}\right)^{\rho(p,q)/\zeta_k}\right\},\quad p\ge 3\end{aligned}$$

and further, if the sequence of principal indices $\{m_k\}$ satisfies $m_{k-1} \simeq m_k$ as $k > \infty$, then for $p = 2$,

$$\frac{t^*(p,q)}{M^*(p,q)} = \max_{\{m_k\}} \left\{ \liminf_{k\to\infty} \left(\frac{\phi(m_{k-1})}{\log^{[A]}(\|q_{m_k}(z)\|_{\Gamma_\alpha})^{-1/m_k}} \right)^{\frac{\rho(2,q)}{\zeta_k}-A} \right\},$$

$$= \max_{\{m_k\}} \left\{ \liminf_{k\to\infty} \left(\frac{\phi(m_{k-1})}{\log^{[A]}(E^s_{\zeta_k}(f))^{-1/\zeta_k}} \right)^{\frac{\rho(2,q)}{\zeta_k}-A} \right\}, \quad p \geq 3$$

where maximum is taken over all increasing sequence of positive integer and

$$M^*(p,q) = \begin{cases} \frac{(\gamma^*-1)^{(\gamma^*-1)}}{\gamma^{*\gamma^*}}, \gamma^* = \rho(2,2)/\zeta_k & \textit{if} \quad (p,q) = (2,2), \\ \frac{\zeta_k}{e\ \rho(2,1)} & \textit{if} \quad (p,q) = (2,1), \\ 1 & \textit{otherwise.} \end{cases}$$

Proof. In order to prove the theorem we apply Theorem 2 by Kasana ea al [7] to the function $H_\alpha(w) = \sum_{k=1}^{\infty} \|q_{m_k}(z)\|_{\Gamma_\alpha} w^{m_k}$ and $\tilde{H}(t) = \sum_{k=1}^{\infty} E^s_{\zeta_k}(f) t^{\zeta_k}$ with the resulting characterization of $t^*(p,q,H_\alpha)$ and $t^*(p,q,\tilde{H})$ in terms of $\|q_{m_k}(z)\|_{\Gamma_\alpha}$ and $E^s_{\zeta_k}(f)$ respectively. Now in view of right hand equality of (11) and (14), the proof of theorem is complete. ■

Note. Taking $\rho_{p,q}(R)/\zeta = \rho(p,q)/\zeta$, $\forall\ R > R_0$ and $\phi(x) = \frac{1}{x(\rho(p,q)/\zeta - A)}$, we have the following corollary which gives a formula for lower (p,q)-type $t(p,q)$ in terms of polynomial coefficients and approximation errors of an entire function $f(z) = \sum_{k=1}^{\infty} q_k(z)[\gamma]^{k-1}$.

Corollary 1. *Let α be fixed and $f(z) = \sum_{k=1}^{\infty} q_k(z)[\gamma(z)]^{k-1}$ belongs to $L^s(B)$, $1 \leq s \leq \infty$. Then $f(z)$ is the restriction of an entire function having (p,q)- order $\rho(p,q)$ $(b < \rho(p,q) < \infty)$ and lower (p,q)-type $t(p,q)$ $(0 < t(p,q) < \infty)$ if and only if*

$$\frac{t(p,q)}{M^*(p,q)} = \max_{\{m_k\}} \left\{ \liminf_{k\to\infty} \frac{\log^{[p-2]} m_{k-1}}{\left[\log^{[q-1]}(\|q_{m_k}(z)\|_{\Gamma_\alpha})^{-1/m_k}\right]^{\frac{\rho(p,q)}{m_k}-A}} \right\}$$

$$= \max_{\{m_k\}} \left\{ \liminf_{k\to\infty} \frac{\log^{[p-2]} m_{k-1}}{\left[\log^{[q-1]}(E^s_\zeta(f))^{-1/\zeta_k}\right]^{\frac{\rho(p,q)}{\zeta_k}-A}} \right\}.$$

Example 1. Consider the function $f(z) = \sum\limits_{k=1}^{\infty}[2k-1]^{-(2k-1)}z[z^2]^{k-1}$. Here $q_k(z) = [2k-1]^{-(2k-1)}z$ and $\gamma(z) = z^2$. From the coefficient formulae, we get $\rho = \lambda = 1$ and $T = t = \frac{1}{e}$. Further, we have $\rho = 2$ and $\|q_k(z)\|_{\Gamma_\alpha} = [2k-1]^{-(2k-1)}$. It can be easily seen that $[\ \|q_k(z)\|\ ]^{1/k} = 0$ as $k \longrightarrow \infty$ therefore $H_\alpha(w)$ is an entire function and the equations (7) and (8) are easily verified

Example 2. To verify the Theorem 3 and Corollary 1, Consider the function $f(z) = e^{z^2} + e^z = \sum\limits_{k=1}^{\infty} \frac{z}{\left(\frac{2k-1}{2}\right)!}[z^2]^{k-1} + \sum\limits_{k=1}^{\infty} \frac{z}{(2k-1)!}[z^2]^{k-1}$. For $(p,q) = (2,1)$, $\rho(f) = \lambda(f) = 2$ and $T(f) = t(f) = 1$. But if we rewrite the expansion of $f(z)$ in the form

$$\begin{aligned} f(z) &= \left(1 + \frac{z^2}{2} + \frac{z^4}{2!} + \frac{z^6}{3!} + \dots\right) + \left(1 + z + \frac{z^2}{2!} + \dots\right) \\ &= 2 + z + z^2\left(1 + \frac{1}{2!}\right) + \frac{z^3}{3!} + z^4\left(\frac{1}{2!} + \frac{1}{4!}\right) + \dots + z^{2m}\left[\frac{1}{m!} + \frac{1}{(2m)!}\right] \\ &\quad + \frac{z^{2m+1}}{(2m+1)!} + \dots \end{aligned}$$

we see by coefficient formulae $\rho(f) = 2$, $\lambda(f) = 1$, $T(f) = 1$ and $t(f) = 0$. This implies that equality does not hold for lower order and lower type cases. Hence we have taken max on the right hand side of (22), (23) and in Corollary 1. Now the verification of Theorem 3 and Corollary 1 needs some mechanical work and hence left for the reader.

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Huzoor H. Khan
Department of Mathematics
Aligarh Muslim University
Aligarh - 202002, U.P., India
e-mail: huzoorkhan@yahoo.com

Rifaqat Ali
Department of Mathematics
Aligarh Muslim University
Aligarh - 202002, U.P., India
e-mail: rifaqat.ali1@gmail.com

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