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ADAPTED QUADRATIC APPROXIMATION FOR LOGARITHMIC KERNEL INTEGRALS

Abstract. In this work, we explain a new numerical schemes of collocation methods based on the adapted quadratic approximation of singular integral with logarithmic kernel. This approximation leads to obtain the numerical solution of singular integral equations with logarithmic kernel on an oriented smooth contour.

Key words: weakly singular integral, quadratic interpolation, Hőlder space, Hőlder condition.

AMS Mathematics Subject Classification: 45D05, 45E05, 45L05, 65R20.

1. Introduction

Singular integral equations with logarithmic kernels is taken to model many problems of mathematical physics, problems in the elasticity theory, aerodynamics and thermoplasticity [2], [3].

$$a(t_0)\varphi(t_0) + \frac{b(t_0)}{\pi i}\int_\Gamma \ln(t-t_0)\varphi(t)dt + \int_\Gamma k(t,t_0)\varphi(t)dt = f(t_0), \tag{1}$$

where Γ represents an oriented smooth contour, the points t and t_0 are on Γ. This equation plays an important role in modern numerical computations in the applied sciences, in particular in the applied mathematics.

In this work, we study a new numerical approximation of singular integrals with logarithmic kernel based on the quadratic approximation

$$F(t_0) = \frac{1}{\pi i}\int_\Gamma \ln(t-t_0)\varphi(t)dt, \quad t, t_0 \in \Gamma, \tag{2}$$

Noting that, the density $\varphi(t)$ has to satisfy the Holder condition $H(\mu)$ [2]. In other words, for any two points t_1 and t_2 of Γ , we get

$$\mid \varphi(t_2) - \varphi(t_1) \mid \leq A \mid t_2 - t_1 \mid^\mu, \quad 0 < \mu \leq 1,$$

where A is a positive constant, called the Holder constant and μ the Holder index.

2. Quadrature

We denote by t the parametric complex function $t(s)$ of the curve Γ defined by

$$t(s) \ = x(s) + iy(s), \quad a \leq s \leq b,$$

where $x(s)$ and $y(s)$ are continuous functions on the finite interval of definition $[a, b]$ and have continuous first derivatives $x'(s)$ and $y'(s)$ never simultaneously null. Let N be an arbitrary natural number, generally we take it large enough and divide the interval $[a, b]$ into N equal subintervals $I_1, I_2, \ldots, I_N$ by the points

$$s_\sigma = a + \sigma \frac{l}{N}, \quad l = b - a, \quad \sigma = 0, 1, 2, \ldots, N.$$

Further, we fix a natural number $M > 1$, and divide each of segments $[s_\sigma, s_{\sigma+1}]$ by the equidistant points

$$s_{\sigma k} = s_\sigma + k\frac{h}{2M}, \quad h = \frac{l}{N}, \quad k = 0, 1, \ldots, 2M.$$

In other words, we have for each subinterval $[s_\sigma, s_{\sigma+1}]$ the following subdivision

$$[s_\sigma, s_{\sigma+1}] = \{s_\sigma = s_{\sigma 0} < s_{\sigma 1} < \ldots < s_{\sigma 2M} = s_{\sigma+1}\}.$$

We introduce the notation

$$t_\sigma = t(s_\sigma), \ \ t_{\sigma k} = t(s_{\sigma k}); \quad \sigma = 0, 1, 2, ..., N; \quad k = 0, 1, \ldots, 2M.$$

Assuming that, for the indices $\sigma, \nu = 0, 1, 2, \ldots, N-1$, the points t and t_0 belong respectively to the arcs $\widehat{t_\sigma t_{\sigma+1}}$ and $\widehat{t_\nu t_{\nu+1}}$ where $\widehat{t_\alpha t_{\alpha+1}}$ designates the smallest arc with ends t_α and $t_{\alpha+1}$ [3], [5], [6] and [7].

For an arbitrary number $\sigma = 0, 1, 2, \ldots, N-1$, we define the piecewise quadratic Lagrange interpolation polynomial $S_2(\varphi; t, \sigma)$ dependent on φ, t and σ which represents the quadratic approximation of the function density $\varphi(t)$ on the subinterval $[t_\sigma, t_{\sigma+1}]$ of the curve Γ. As we know, the interval $[t_\sigma, t_{\sigma+1}]$ is divided into subintervals $[t_{\sigma k}, t_{\sigma(k+2)}]$ of length $(t_{\sigma(k+2)} - t_{\sigma k})$, $k = 2i$, $i = 0, 1, \ldots, M-1$. We interpolate the function density $\varphi(t)$ with respect to the values $\varphi(t_{\sigma k})$, $\varphi(t_{\sigma(k+1)})$ and $\varphi(t_{\sigma(k+2)})$ at the points $t_{\sigma k}$, $t_{\sigma(k+1)}$ and $t_{\sigma(k+2)}$ respectively with a quadratic polynomial, given by the following formula.

For $t_{\sigma k} \le t \le t_{\sigma(k+2)}$,

$$S_2(\varphi;t,\sigma) = \frac{(t-t_{\sigma(k+1)})(t-t_{\sigma(k+2)})}{(t_{\sigma(k+1)}-t_{\sigma k})(t_{\sigma(k+2)}-t_{\sigma k})}\varphi(t_{\sigma k}) - \frac{(t-t_{\sigma k})(t-t_{\sigma(k+2)})}{(t_{\sigma(k+1)}-t_{\sigma k})(t_{\sigma(k+2)}-t_{\sigma(k+1)})}\varphi(t_{\sigma(k+1)}) + \frac{(t-t_{\sigma k})(t-t_{\sigma(k+1)})}{(t_{\sigma(k+2)}-t_{\sigma k})(t_{\sigma(k+2)}-t_{\sigma(k+1)})}\varphi(t_{\sigma(k+2)}), \tag{3}$$

this piecewise quadratic interpolating polynomial exists and is unique.

We define for an arbitrary numbers σ and ν, such that $0 \le \sigma, \nu \le N-1$, the following continuous function $\beta_{\sigma\nu}(\varphi;t,t_0)$, dependents on φ, t and t_0

$$\beta_{\sigma\nu}(\varphi;t,t_0) = \begin{cases} U(\varphi;t,\sigma) - V(\varphi;t_0,\sigma,\nu) & \text{for } t \ne t_0, \\ 0 & \text{for } t = t_0. \end{cases} \tag{4}$$

The function $U(\varphi;t,\sigma)$ represents a modified quadratic interpolation of the function density $\varphi(t)$ on the subinterval $[t_\sigma, t_{\sigma+1}]$ of the curve Γ.

Indeed, for $t_{\sigma k} \le t \le t_{\sigma(k+2)}$ and $t - t_0 \ne 1$, we put

$$U(\varphi;t,\sigma) = \frac{(t-t_{\sigma(k+1)})(t-t_{\sigma(k+2)})}{(t_{\sigma(k+1)}-t_{\sigma k})(t_{\sigma(k+2)}-t_{\sigma k})}\varphi(t_{\sigma k})\frac{\ln(t_{\sigma k}-t_0)}{\ln(t-t_0)} - \frac{(t-t_{\sigma k})(t-t_{\sigma(k+2)})}{(t_{\sigma(k+1)}-t_{\sigma k})(t_{\sigma(k+2)}-t_{\sigma(k+1)})}\varphi(t_{\sigma(k+1)})\frac{\ln(t_{\sigma(k+1)}-t_0)}{\ln(t-t_0)} + \frac{(t-t_{\sigma k})(t-t_{\sigma(k+1)})}{(t_{\sigma(k+2)}-t_{\sigma k})(t_{\sigma(k+2)}-t_{\sigma(k+1)})}\varphi(t_{\sigma(k+2)})\frac{\ln(t_{\sigma(k+2)}-t_0)}{\ln(t-t_0)},$$

and the function $V(\varphi;t_0,\sigma,\nu)$ is given by

$$V(\varphi;t_0,\sigma,\nu) = S_2(\varphi;t_0,\nu)\frac{(t-t_{\sigma(k+1)})(t-t_{\sigma(k+2)})}{(t_{\sigma(k+1)}-t_{\sigma k})(t_{\sigma(k+2)}-t_{\sigma k})}\frac{\ln(t_{\sigma k}-t_0)}{\ln(t-t_0)} - S_2(\varphi;t_0,\nu)\frac{(t-t_{\sigma k})(t-t_{\sigma(k+2)})}{(t_{\sigma(k+1)}-t_{\sigma k})(t_{\sigma(k+2)}-t_{\sigma(k+1)})}\frac{\ln(t_{\sigma(k+1)}-t_0)}{\ln(t-t_0)} + S_2(\varphi;t_0,\nu)\frac{(t-t_{\sigma k})(t-t_{\sigma(k+1)})}{(t_{\sigma(k+2)}-t_{\sigma k})(t_{\sigma(k+2)}-t_{\sigma(k+1)})}\frac{\ln(t_{\sigma(k+2)}-t_0)}{\ln(t-t_0)},$$

where the function φ represents a given function on the curve Γ and of the class $H(\mu)$.

Denoting by $\psi_{\sigma\nu}(\varphi;t,t_0)$ the cubic approximation of the density $\varphi(t)$ at the point $t \in [t_\sigma, t_{\sigma+1}]$, $t_0 \in [t_\nu, t_{\nu+1}]$ and $0 \le \sigma, \nu \le N-1$ by

$$\psi_{\sigma\nu}(\varphi;t,t_0) = \varphi(t_0) + \beta_{\sigma\nu}(\varphi;t,t_0). \tag{5}$$

Our idea is to and replace the density $\varphi(t)$ by expansion (5) in the weakly singular integral (2)

$$F(t_0) = \frac{1}{\pi i}\int_\Gamma \varphi(t)\ln(t-t_0)dt,$$

and obtain the following approximation noting by $S(\varphi;t)$ given as

$$\begin{aligned} S(\varphi;t_0) &= \frac{1}{\pi i}\int_\Gamma \ln(t-t_0)\psi_{\sigma\nu}(\varphi;t,t_0)dt \\ &= \frac{1}{\pi i}\int_\Gamma \ln(t-t_0)[\varphi(t_0)+\beta_{\sigma\nu}(\varphi;t,t_0)]dt. \end{aligned} \tag{6}$$

3. Main Rresult

Theorem. *Let Γ be an oriented smooth contour and let φ be a density function defined on Γ and satisfying the Hölder condition $H(\mu)$ then, the following estimation*

$$|F(t_0)-S(\varphi;t_0)| \le \max\left(\frac{C_1\ln(2MN)}{(2MN)^\mu}, \frac{C_2}{(2MN)^{\mu+1}}\right), \quad N,M>1,$$

holds, where the constant C_1, C_2 depends only on the contour Γ.

Proof. Taking the points $t\in[t_\sigma,t_{\sigma+1}]$ and $t_0\in[t_\nu,t_{\nu+1}]$, we can write for $t_{\sigma k}\le t\le t_{\sigma(k+2)}$ and $t_{\nu k}\le t_0\le t_{\nu(k+2)}$

$$\begin{aligned} &\varphi(t)-\psi_{\sigma\nu}(\varphi;t,t_0) = \varphi(t)-\varphi(t_0)-\beta_{\sigma\nu}(\varphi;t,t_0) = \varphi(t)-\varphi(t_0) \\ &-\Big\{\frac{(t-t_{\sigma(k+1)})(t-t_{\sigma(k+2)})}{(t_{\sigma(k+1)}-t_{\sigma k})(t_{\sigma(k+2)}-t_{\sigma k})}\varphi(t_{\sigma k})\frac{\ln(t_{\sigma k}-t_0)}{\ln(t-t_0)}\frac{\sqrt{(t-t_0)}}{\sqrt{(t_{\sigma k}-t_0)}} \\ &-\frac{(t-t_{\sigma k})(t-t_{\sigma(k+2)})}{(t_{\sigma(k+1)}-t_{\sigma k})(t_{\sigma(k+2)}-t_{\sigma(k+1)})}\varphi(t_{\sigma(k+1)})\frac{\ln(t_{\sigma(k+1)}-t_0)}{\ln(t-t_0)}\frac{\sqrt{(t-t_0)}}{\sqrt{(t_{\sigma(k+1)}-t_0)}} \\ &+\frac{(t-t_{\sigma k})(t-t_{\sigma(k+1)})}{(t_{\sigma(k+2)}-t_{\sigma k})(t_{\sigma(k+2)}-t_{\sigma(k+1)})}\varphi(t_{\sigma(k+2)})\frac{\ln(t_{\sigma(k+2)}-t_0)}{\ln(t-t_0)}\frac{\sqrt{(t-t_0)}}{\sqrt{(t_{\sigma(k+2)}-t_0)}} \\ &-\frac{S_2(\varphi;t_0,\nu)(t-t_{\sigma(k+1)})(t-t_{\sigma(k+2)})}{(t_{\sigma(k+2)}-t_{\sigma k})(t_{\sigma(k+1)}-t_{\sigma k})}\frac{\ln(t_{\sigma k}-t_0)}{\ln(t-t_0)}\frac{\sqrt{(t-t_0)}}{\sqrt{(t_{\sigma k}-t_0)}} \\ &+\frac{S_2(\varphi;t_0,\nu)(t-t_{\sigma k)})(t-t_{\sigma(k+2)})}{(t_{\sigma(k+2)}-t_{\sigma(k+1)})(t_{\sigma(k+1)}-t_{\sigma k})}\frac{\ln(t_{\sigma(k+1)}-t_0)}{\ln(t-t_0)}\frac{\sqrt{(t-t_0)}}{\sqrt{(t_{\sigma(k+1)}-t_0)}} \\ &-\frac{S_2(\varphi;t_0,\nu)(t-t_{\sigma 2k)})(t-t_{\sigma(2k+1)})}{(t_{\sigma(2k+2)}-t_{\sigma(2k+1)})(t_{\sigma(2k+2)}-t_{\sigma 2k})}\frac{\ln(t_{\sigma(2k+2)}-t_0)}{\ln(t-t_0)}\frac{\sqrt{(t-t_0)}}{\sqrt{(t_{\sigma(k+2)}-t_0)}}\Big\}. \end{aligned} \tag{7}$$

Taking into account the expression (7) we get

$$\int_{\Gamma} \ln(t-t_0)[\varphi(t)-\psi_{\sigma\nu}(\varphi;t,t_0)]dt = \sum_{\sigma=0}^{N-1}\int_{t_\sigma t_{\sigma+1}} \ln(t-t_0)[\varphi(t)-\psi_{\sigma\nu}(\varphi;t,t_0)]dt, \tag{8}$$

hence

$$\begin{aligned}
F(t_0)-S(\varphi;t_0) &= \frac{1}{\pi i}\sum_{\sigma=0}^{N-1}\sum_{k=0}^{M-1}\int_{t_{\sigma 2k}t_{\sigma(2k+2)}} \ln(t-t_0)[\varphi(t)-\varphi(t_0)] \\
&-\{\frac{(t-t_{\sigma(2k+1)})(t-t_{\sigma(2k+2)})}{(t_{\sigma(2k+1)}-t_{\sigma 2k})(t_{\sigma(2k+2)}-t_{\sigma 2k})}\varphi(t_{\sigma 2k})\frac{\ln(t_{\sigma 2k}-t_0)}{\ln(t-t_0)}\frac{\sqrt{(t-t_0)}}{\sqrt{(t_{\sigma 2k}-t_0)}} \\
&-\frac{(t-t_{\sigma 2k})(t-t_{\sigma(2k+2)})}{(t_{\sigma(2k+1)}-t_{\sigma 2k})(t_{\sigma(2k+2)}-t_{\sigma(2k+1)})}\varphi(t_{\sigma(2k+1)}) \\
&\times\frac{\ln(t_{\sigma(2k+1)}-t_0)}{\ln(t-t_0)}\frac{\sqrt{(t-t_0)}}{\sqrt{(t_{\sigma(2k+1)}-t_0)}} \\
&+\frac{(t-t_{\sigma 2k})(t-t_{\sigma(2k+1)})}{(t_{\sigma(2k+2)}-t_{\sigma 2k})(t_{\sigma(2k+2)}-t_{\sigma(2k+1)})}\varphi(t_{\sigma(2k+2)}) \\
&\times\frac{\ln(t_{\sigma(2k+2)}-t_0)}{\ln(t-t_0)}\frac{\sqrt{(t-t_0)}}{\sqrt{(t_{\sigma(2k+2)}-t_0)}} \\
&-\frac{S_2(\varphi;t_0,\nu)(t-t_{\sigma(2k+1)})(t-t_{\sigma(2k+2)})}{(t_{\sigma(2k+2)}-t_{\sigma 2k})(t_{\sigma(2k+1)}-t_{\sigma 2k})}\frac{\ln(t_{\sigma 2k}-t_0)}{\ln(t-t_0)}\frac{\sqrt{(t-t_0)}}{\sqrt{(t_{\sigma 2k}-t_0)}} \\
&+\frac{S_2(\varphi;t_0,\nu)(t-t_{\sigma 2k)})(t-t_{\sigma(2k+2)})}{(t_{\sigma(2k+2)}-t_{\sigma(2k+1)})(t_{\sigma(2k+1)}-t_{\sigma 2k})}\frac{\ln(t_{\sigma(2k+1)}-t_0)}{\ln(t-t_0)}\frac{\sqrt{(t-t_0)}}{\sqrt{(t_{\sigma(2k+1)}-t_0)}} \\
&-\frac{S_2(\varphi;t_0,\nu)(t-t_{\sigma 2k)})(t-t_{\sigma(2k+1)})}{(t_{\sigma(2k+2)}-t_{\sigma(2k+1)})(t_{\sigma(2k+2)}-t_{\sigma 2k})}\frac{\ln(t_{\sigma(2k+2)}-t_0)}{\ln(t-t_0)} \\
&\times\frac{\sqrt{(t-t_0)}}{\sqrt{(t_{\sigma(2k+2)}-t_0)}}\}\ln(t-t_0)dt.
\end{aligned}$$

Seeing that, the equalities $t_{\sigma 2k}-t_0=0$, $t_{\sigma(2k+1)}-t_0=0$ and $t_{\sigma(2k+2)}-t_0=0$ are possible only when $\sigma=\nu-1,\nu+1$ and ν. For these cases, it is easy to see that the integral (8) exists when $t_{\sigma 2k}$ tends to t_0 or $t_{\sigma(2k+1)}$ tends to t_0 or or $t_{\sigma(2k+2)}$ tends to t_0; the other case, if $\sigma=\nu$ we can easily seeing that, the function $\beta_{\sigma\sigma}(\varphi;t,t_0)$ contains $(t_{\sigma 2k}-t_0)$, $(t_{\sigma(2k+1)}-t_0)$ and $(t_{\sigma(2k+2)}-t_0)$ as factors, so for the four cases $t=t_0$ or $t_{\sigma 2k}=t_0$ or $t_{\sigma(2k+1)}=t_0$ or $t_{\sigma(2k+2)}=t_0$ the function $\beta_{\sigma\nu}(\varphi;t,t_0)$ has a well sense.

Indeed, for the points $t,\ t_0\in[t_\sigma,t_{\sigma+1}]$ such that $t_{\sigma 2k}\le t,\ t_0\le t_{\sigma(2k+2)}$, we write

$$\beta_{\sigma\sigma}(\varphi;t,t_0)=U(\varphi;t,\sigma)-V(\varphi;t_0,\sigma,\sigma),$$

hence

$$
\begin{aligned}
(9)\qquad \beta_{\sigma\sigma}(\varphi;t,t_0) \;=\; & \frac{(t-t_{\sigma(2k+1)})(t-t_{\sigma(2k+2)})\ln(t_{\sigma2k}-t_0)}{(t_{\sigma(2k+1)}-t_{\sigma2k})(t_{\sigma(2k+2)}-t_{\sigma2k})\ln(t-t_0)} \\
& \times\frac{\sqrt{(t-t_0)}}{\sqrt{(t_{\sigma2k}-t_0)}}(\varphi(t_{\sigma2k})-S_2(\varphi;t_0,\sigma)) \\
& -\frac{(t-t_{\sigma2k})(t-t_{\sigma(2k+2)})\ln(t_{\sigma(2k+1)}-t_0)}{(t_{\sigma(2k+1)}-t_{\sigma2k})(t_{\sigma(2k+2)}-t_{\sigma(2k+1)})\ln(t-t_0)} \\
& \times\frac{\sqrt{(t-t_0)}}{\sqrt{(t_{\sigma(2k+1)}-t_0)}}(\varphi(t_{\sigma(2k+1)})-S_2(\varphi;t_0,\sigma)) \\
& +\frac{(t-t_{\sigma2k})(t-t_{\sigma(2k+1)})\ln(t_{\sigma(2k+2)}-t_0)}{(t_{\sigma(2k+2)}-t_{\sigma2k})(t_{\sigma(2k+2)}-t_{\sigma(2k+1)})\ln(t-t_0)} \\
& \times\frac{\sqrt{(t-t_0)}}{\sqrt{(t_{\sigma(2k+2)}-t_0)}}(\varphi(t_{\sigma(2k+2)})-S_2(\varphi;t_0,\sigma)).
\end{aligned}
$$

In other words, we write

$$\beta_{\sigma\sigma}(\varphi;t,t_0)=\frac{\sqrt{(t-t_0)}}{\ln(t-t_0)}Q(\varphi;t,t_0),$$

where the expression $Q(\varphi;t,t_0)$ is given by

$$
\begin{aligned}
Q(\varphi;t,t_0) \;=\; & \frac{(t-t_{\sigma(2k+1)})(t-t_{\sigma(2k+2)})\sqrt{(t_{\sigma2k}-t_0)}\ln(t_{\sigma2k}-t_0)}{(t_{\sigma(2k+1)}-t_{\sigma2k})(t_{\sigma(2k+2)}-t_{\sigma2k})} \\
& \times\{\frac{(t_{\sigma(2k+2)}-t_0)}{(t_{\sigma(2k+1)}-t_{\sigma2k})(t_{\sigma(2k+2)}-t_{\sigma(2k+1)})}(\varphi(t_{\sigma2k+1})-\varphi(t_{\sigma2k})) \\
& +\frac{(t_{0-}t_{\sigma(2k+1)})}{(t_{\sigma(2k+1)}-t_{\sigma2k})(t_{\sigma(2k+2)}-t_{\sigma(2k+1)})}(\varphi(t_{\sigma2k+2})-\varphi(t_{\sigma2k}))\} \\
& -\frac{(t-t_{\sigma2k})(t-t_{\sigma(2k+2)})\sqrt{(t_{\sigma(2k+1)}-t_0)}\ln(t_{\sigma(2k+1)}-t_0)}{(t_{\sigma(2k+1)}-t_{\sigma2k})(t_{\sigma(2k+2)}-t_{\sigma(2k+1)})} \\
& \times\{\frac{(t_{\sigma(2k+2)}-t_0)}{(t_{\sigma(2k+1)}-t_{\sigma2k})(t_{\sigma(2k+2)}-t_{\sigma(2k+1)})}(\varphi(t_{\sigma2k+1})-\varphi(t_{\sigma2k})) \\
& +\frac{(t_{0-}t_{\sigma2k})}{(t_{\sigma(2k+1)}-t_{\sigma2k})(t_{\sigma(2k+2)}-t_{\sigma(2k+1)})}(\varphi(t_{\sigma2k+2})-\varphi(t_{\sigma(2k+1)}))\} \\
& +\frac{(t-t_{\sigma2k})(t-t_{\sigma(2k+1)})\sqrt{(t_{\sigma(2k+2)}-t_0)}\ln(t_{\sigma(2k+2)}-t_0)}{(t_{\sigma(2k+2)}-t_{\sigma2k})(t_{\sigma(2k+2)}-t_{\sigma(2k+1)})} \\
& \times\{\frac{(t_{\sigma(2k+1)}-t_0)}{(t_{\sigma(2k+1)}-t_{\sigma2k})(t_{\sigma(2k+2)}-t_{\sigma(2k+1)})}(\varphi(t_{\sigma2k+2})-\varphi(t_{\sigma2k})) \\
& +\frac{(t_{0-}t_{\sigma2k})}{(t_{\sigma(2k+1)}-t_{\sigma2k})(t_{\sigma(2k+2)}-t_{\sigma(2k+1)})}(\varphi(t_{\sigma2k+2})-\varphi(t_{\sigma(2k+1)}))\}
\end{aligned}
$$

with

$$\lim_{t_0 \to t_{\sigma 2k}} Q(\varphi; t, t_0) = Q(\varphi; t, t_{\sigma 2k}),$$

and

$$\lim_{t_0 \to t_{\sigma(2k+1)}} Q(\varphi; t, t_0) = Q(\varphi; t, t_{\sigma(2k+1)}),$$

and

$$\lim_{t_0 \to t_{\sigma(2k+2)}} Q(\varphi; t, t_0) = Q(\varphi; t, t_{\sigma(2k+2)}).$$

Passing now to the estimation of the expression (8), for $t_0 \in \widehat{t_\nu t_{\nu+1}}$ and $\sigma \neq \nu-1,\ \nu+1$ and ν we have

$$\begin{aligned}
&\Bigg| \frac{1}{\pi i} \sum_{\sigma=0}^{N-1} \sum_{k=0}^{M-1} \int_{t_{\sigma 2k} t_{\sigma(2k+2)}} \ln(t-t_0)[\varphi(t)-\varphi(t_0)] \\
&-\{ \frac{(t-t_{\sigma(2k+1)})(t-t_{\sigma(2k+2)})}{(t_{\sigma(2k+1)}-t_{\sigma 2k})(t_{\sigma(2k+2)}-t_{\sigma 2k})} \varphi(t_{\sigma 2k}) \frac{\ln(t_{\sigma 2k}-t_0)}{\ln(t-t_0)} \frac{\sqrt{(t-t_0)}}{\sqrt{(t_{\sigma 2k}-t_0)}} \\
&- \frac{(t-t_{\sigma 2k})(t-t_{\sigma(2k+2)})}{(t_{\sigma(2k+1)}-t_{\sigma 2k})(t_{\sigma(2k+2)}-t_{\sigma(2k+1)})} \varphi(t_{\sigma(2k+1)}) \\
&\times \frac{\ln(t_{\sigma(2k+1)}-t_0)}{\ln(t-t_0)} \frac{\sqrt{(t-t_0)}}{\sqrt{(t_{\sigma(2k+1)}-t_0)}} \\
&+ \frac{(t-t_{\sigma 2k})(t-t_{\sigma(2k+1)})}{(t_{\sigma(2k+2)}-t_{\sigma 2k})(t_{\sigma(2k+2)}-t_{\sigma(2k+1)})} \varphi(t_{\sigma(2k+2)}) \\
&\times \frac{\ln(t_{\sigma(2k+2)}-t_0)}{\ln(t-t_0)} \frac{\sqrt{(t-t_0)}}{\sqrt{(t_{\sigma(2k+2)}-t_0)}} \\
&- \frac{S_2(\varphi; t_0, \nu)(t-t_{\sigma(2k+1)})(t-t_{\sigma(2k+2)})}{(t_{\sigma(2k+2)}-t_{\sigma 2k})(t_{\sigma(2k+1)}-t_{\sigma 2k})} \frac{\ln(t_{\sigma 2k}-t_0)}{\ln(t-t_0)} \frac{\sqrt{(t-t_0)}}{\sqrt{(t_{\sigma 2k}-t_0)}} \\
&+ \frac{S_2(\varphi; t_0, \nu)(t-t_{\sigma 2k}))(t-t_{\sigma(2k+2)})}{(t_{\sigma(2k+2)}-t_{\sigma(2k+1)})(t_{\sigma(2k+1)}-t_{\sigma 2k})} \frac{\ln(t_{\sigma(2k+1)}-t_0)}{\ln(t-t_0)} \frac{\sqrt{(t-t_0)}}{\sqrt{(t_{\sigma(2k+1)}-t_0)}} \\
&- \frac{S_2(\varphi; t_0, \nu)(t-t_{\sigma 2k}))(t-t_{\sigma(2k+1)})}{(t_{\sigma(2k+2)}-t_{\sigma(2k+1)})(t_{\sigma(2k+2)}-t_{\sigma 2k})} \frac{\ln(t_{\sigma(2k+2)}-t_0)}{\ln(t-t_0)} \\
&\times \frac{\sqrt{(t-t_0)}}{\sqrt{(t_{\sigma(2k+2)}-t_0)}} \} \ln(t-t_0) dt \Bigg| = O(\frac{\ln 2MN}{(2MN)^\mu}).
\end{aligned}$$

Indeed, it is clear that

$$\max_{t_0 \in \widehat{t_\nu t_{\nu+1}}} \mid \frac{1}{\pi i} \sum_{\sigma=0}^{N-1} \sum_{k=0}^{M-1} \int_{t_{\sigma 2k}}^{t_{\sigma(2k+2)}} (\varphi(t)-\varphi(t_0)) \ln(t-t_0) dt \mid = O(\frac{\ln 2MN}{(2MN)^\mu})$$

and also we estimate the expression

$$
\begin{aligned}
&\frac{1}{\pi i}\sum_{\sigma=0}^{N-1}\sum_{k=0}^{M-1}\int_{t_{\sigma 2k}}^{t_{\sigma(2k+2)}} -\{\frac{(t-t_{\sigma(2k+1)})(t-t_{\sigma(2k+2)})}{(t_{\sigma(2k+1)}-t_{\sigma 2k})(t_{\sigma(2k+2)}-t_{\sigma 2k})}\varphi(t_{\sigma 2k}) \\
&\times\frac{\ln(t_{\sigma 2k}-t_0)}{\ln(t-t_0)}\frac{\sqrt{(t-t_0)}}{\sqrt{(t_{\sigma 2k}-t_0)}} \\
&-\frac{(t-t_{\sigma 2k})(t-t_{\sigma(2k+2)})}{(t_{\sigma(2k+1)}-t_{\sigma 2k})(t_{\sigma(2k+2)}-t_{\sigma(2k+1)})}\varphi(t_{\sigma(2k+1)}) \\
&\times\frac{\ln(t_{\sigma(2k+1)}-t_0)}{\ln(t-t_0)}\frac{\sqrt{(t-t_0)}}{\sqrt{(t_{\sigma(2k+1)}-t_0)}} \\
&+\frac{(t-t_{\sigma 2k})(t-t_{\sigma(2k+1)})}{(t_{\sigma(2k+2)}-t_{\sigma 2k})(t_{\sigma(2k+2)}-t_{\sigma(2k+1)})}\varphi(t_{\sigma(2k+2)}) \\
&\times\frac{\ln(t_{\sigma(2k+2)}-t_0)}{\ln(t-t_0)}\frac{\sqrt{(t-t_0)}}{\sqrt{(t_{\sigma(2k+2)}-t_0)}} \\
&-\frac{S_2(\varphi;t_0,\nu)(t-t_{\sigma(2k+1)})(t-t_{\sigma(2k+2)})}{(t_{\sigma(2k+2)}-t_{\sigma 2k})(t_{\sigma(2k+1)}-t_{\sigma 2k})}\frac{\ln(t_{\sigma 2k}-t_0)}{\ln(t-t_0)}\frac{\sqrt{(t-t_0)}}{\sqrt{(t_{\sigma 2k}-t_0)}} \\
&+\frac{S_2(\varphi;t_0,\nu)(t-t_{\sigma 2k)})(t-t_{\sigma(2k+2)})}{(t_{\sigma(2k+2)}-t_{\sigma(2k+1)})(t_{\sigma(2k+1)}-t_{\sigma 2k})}\frac{\ln(t_{\sigma(2k+1)}-t_0)}{\ln(t-t_0)}\frac{\sqrt{(t-t_0)}}{\sqrt{(t_{\sigma(2k+1)}-t_0)}} \\
&-\frac{S_2(\varphi;t_0,\nu)(t-t_{\sigma 2k)})(t-t_{\sigma(2k+1)})}{(t_{\sigma(2k+2)}-t_{\sigma(2k+1)})(t_{\sigma(2k+2)}-t_{\sigma 2k})}\frac{\ln(t_{\sigma(2k+2)}-t_0)}{\ln(t-t_0)} \\
&\times\frac{\sqrt{(t-t_0)}}{\sqrt{(t_{\sigma(2k+2)}-t_0)}}\}\ln(t-t_0)dt
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{\pi i}\sum_{\sigma=0}^{N-1}\sum_{k=0}^{M-1}\int_{t_{\sigma 2k}t_{\sigma(2k+2)}} -\frac{(t-t_{\sigma 2k})(t-t_{\sigma(2k+2)})}{(t_{\sigma(2k+1)}-t_{\sigma 2k})(t_{\sigma(2k+2)}-t_{\sigma(2k+1)})}\varphi(t_{\sigma(2k+1)}) \\
&\times\frac{\ln(t_{\sigma(2k+1)}-t_0)}{\ln(t-t_0)}\frac{\sqrt{(t-t_0)}}{\sqrt{(t_{\sigma(2k+1)}-t_0)}} \\
&+\frac{(t-t_{\sigma 2k})(t-t_{\sigma(2k+1)})}{(t_{\sigma(2k+2)}-t_{\sigma 2k})(t_{\sigma(2k+2)}-t_{\sigma(2k+1)})}\varphi(t_{\sigma(2k+2)}) \\
&\times\frac{\ln(t_{\sigma(2k+2)}-t_0)}{\ln(t-t_0)}\frac{\sqrt{(t-t_0)}}{\sqrt{(t_{\sigma(2k+2)}-t_0)}} \\
&-\frac{S_2(\varphi;t_0,\nu)(t-t_{\sigma(2k+1)})(t-t_{\sigma(2k+2)})}{(t_{\sigma(2k+2)}-t_{\sigma 2k})(t_{\sigma(2k+1)}-t_{\sigma 2k})}\frac{\ln(t_{\sigma 2k}-t_0)}{\ln(t-t_0)}\frac{\sqrt{(t-t_0)}}{\sqrt{(t_{\sigma 2k}-t_0)}} \\
&+\frac{S_2(\varphi;t_0,\nu)(t-t_{\sigma 2k)})(t-t_{\sigma(2k+2)})}{(t_{\sigma(2k+2)}-t_{\sigma(2k+1)})(t_{\sigma(2k+1)}-t_{\sigma 2k})}\frac{\ln(t_{\sigma(2k+1)}-t_0)}{\ln(t-t_0)}\frac{\sqrt{(t-t_0)}}{\sqrt{(t_{\sigma(2k+1)}-t_0)}} \\
&-\frac{S_2(\varphi;t_0,\nu)(t-t_{\sigma 2k)})(t-t_{\sigma(2k+1)})}{(t_{\sigma(2k+2)}-t_{\sigma(2k+1)})(t_{\sigma(2k+2)}-t_{\sigma 2k})}
\end{aligned}
$$

$$\times \frac{\ln(t_{\sigma(2k+2)} - t_0)}{\ln(t - t_0)} \frac{\sqrt{(t - t_0)}}{\sqrt{(t_{\sigma(2k+2)} - t_0)}}\} \ln(t - t_0) dt$$

$$\simeq \left| \frac{1}{\pi i} \sum_{\sigma=0}^{N-1} \sum_{k=0}^{M-1} \int_{t_{\sigma k} t_{\sigma(k+1)}} \frac{\varphi(t_{\nu k}) - \varphi(t_{\sigma k})}{t_{\nu k} - t_{\sigma k}} + \frac{\varphi(t_{\nu(k+1)}) - \varphi(t_{\sigma(k+1)})}{t_{\nu(k+1)} - t_{\sigma(k+1)}} dt \right|$$

$$= O(\frac{\ln MN}{M^{\mu} N^{\mu}}).$$

Naturally, the estimation given above is obtained by using the density φ, as an element of the Holder space $H(\mu)$ [2], and the following natural estimation

$$\left| \frac{(t - t_{\sigma(2k+1)})(t - t_{\sigma(2k+2)})}{(t_{\sigma(2k+2)} - t_{\sigma 2k})(t_{\sigma(2k+1)} - t_{\sigma 2k})} \frac{\ln(t_{\sigma 2k} - t_0)}{\ln(t - t_0)} \frac{\sqrt{(t - t_0)}}{\sqrt{(t_{\sigma 2k} - t_0)}} \right| = O(1),$$

$$\left| \frac{(t - t_{\sigma 2k}))(t - t_{\sigma(2k+2)})}{(t_{\sigma(2k+2)} - t_{\sigma(2k+1)})(t_{\sigma(2k+1)} - t_{\sigma 2k})} \frac{\ln(t_{\sigma(2k+1)} - t_0)}{\ln(t - t_0)} \frac{\sqrt{(t - t_0)}}{\sqrt{(t_{\sigma(2k+1)} - t_0)}} \right| = O(1),$$

$$\left| \frac{(t - t_{\sigma 2k}))(t - t_{\sigma(2k+1)})}{(t_{\sigma(2k+2)} - t_{\sigma(2k+1)})(t_{\sigma(2k+2)} - t_{\sigma 2k})} \frac{\ln(t_{\sigma(2k+2)} - t_0)}{\ln(t - t_0)} \frac{\sqrt{(t - t_0)}}{\sqrt{(t_{\sigma(2k+2)} - t_0)}} \right| = O(1).$$

Further, for the cases where $\sigma = \nu - 1,\ \nu + 1$ and ν, using the relation (9) and the smoothness of Γ with the condition of the function φ in the space $H(\mu)$, we get

$$| \int_{t_\nu t_{\nu+1}} \ln(t - t_0)[\varphi(t) - \varphi(t_0)] dt \mid \leq A \int_{s_\nu}^{s_{\nu+1}} \ln(s - s_0) \mid s - s_0 \mid^{\mu} ds$$

$$= O(\frac{\ln(2MN)}{(2MN)^{\mu+1}}).$$

■

4. Numerical experiments

Using our approximation, we apply the algorithm to singular integrals with logarithmic kernel and we present results concerning the accuracy of the calculations. In each table I represents the exact value of the singular integral and $\widetilde{I}$ corresponds to the approximate calculation produced by our approximation (6).

Example 1. Consider the weakly singular integral,

$$I = F(t_0) = \int_{\Gamma} \ln(t - t_0) \varphi(t) dt,$$

where the curve Γ designate the unit circle and the function density φ is given by the following expression

$$\varphi(t) = -\frac{1}{t^2}.$$

N	M	$\| I - \widetilde{I} \|_1$	$\| I - \widetilde{I} \|_2$	$\| I - \widetilde{I} \|_\infty$
10	2	5.7174414E-02	3.5285469E-02	2.9367447E-02
10	3	5.5649281E-03	2.9125938E-03	1.8216968E-03
10	4	1.4109910E-03	7.3831965E-04	5.1295757E-04

Example 2. Consider the weakly singular integral,

$$I = F(t_0) = \int_\Gamma \varphi(t) \ln(t - t_0) dt,$$

where the curve Γ designate the unit circle and the function density φ is given by the following expression

$$\varphi(t) = \frac{2}{t^3}.$$

N	M	$\| I - \widetilde{I} \|_1$	$\| I - \widetilde{I} \|_2$	$\| I - \widetilde{I} \|_\infty$
10	2	1.5601690E-01	9.2777811E-02	6.5414310E-02
10	3	1.5781343E-02	9.7699165E-03	6.8665147E-03
10	4	2.1157265E-03	1.1530236E-03	7.7348948E-04

5. Conclusion

The proposed approximation can be used to remove the weakly singularity in the singular integrals with logarithmic kernel of the form (2). It was tested for the numerical calculus of many singular integrals, where it gave good results.

References

[1] ANTIDZE D.J., *On the Approximate Solution of Singular Integral Equations,* Seminar of Institute of Applied Mathematics, Tbilissi, 1975.

[2] MUSKHELISHVILI N.I., Singular integral equations, *Nauka,* Moscow, 1968, English transl, of 1sted Noordhoff, 1953; reprint, 1972.

[3] NADIR M., ANTIDZE J., On the numerical solution of singular integral equations using Sanikidze's approximation, *Comp. Meth. in Sc. Tech.,* 10(1)(2004), 83-89.

[4] NADIR M., LAKEHALI B., On the Approximation of Singular Integrals, *FEN DERGİSİ (E-DERGİ),* 2(2)(2007), 236-240.

[5] Nadir M., Adapted quadratic approximation for singular integrals, *Journal of Mathematical Inequalities,* 4(3)(2010), 423-430.
[6] Nadir M., *Adapted Quadratic Approximation for Singular Integrals Equations,* to appear in International Journal of Applied Mathematics & Statistics, (IJAMAS), 2012.
[7] Sanikidze J., *On Approximate Calculation of Singular Line Integral,* Seminar of Institute of Applied Mathematics, Tbilissi, 1970.
[8] Sanikidze J., *Approximate Solution of Singular Integral Equations in the Case of Closed Contours of Integration,* Seminar of Institute of Applied Mathematics, Tbilissi, 1971.

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