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SEQUENCE SPACES DEFINED BY A SEQUENCE OF MODULUS FUNCTION IN n-NORMED SPACES

Abstract. In the present paper we introduce the sequence spaces defined by a sequence of modulus function $F = (f_k)$ in n-normed spaces. We study some topological properties and prove some inclusion relations between these spaces.

Key words: paranorm space, difference sequence space, modulus function n-normed space.

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1. Introduction and preliminaries

The concept of 2-normed spaces was initially developed by Gähler[3] in the mid of 1960's, while that of n-normed spaces one can see in Misiak [13]. Since then, many others have studied this concept and obtained various results, see Gunawan ([4], [5]) and Gunawan and Mashadi [6]. For more details about the sequence spaces over n-normed spaces (see [15], [16]). Let $n \in \mathbb{N}$ and X be a linear space over the field $\mathbb{K}$, where $\mathbb{K}$ is the field of real or complex numbers of dimension d, where $d \geq n \geq 2$. A real valued function $||\cdot, \cdots, \cdot||$ on X^n satisfying the following four conditions:

(a) $||x_1, x_2, \cdots, x_n|| = 0$ if and only if $x_1, x_2, \cdots, x_n$ are linearly dependent in X;

(b) $||x_1, x_2, \cdots, x_n||$ is invariant under permutation;

(c) $||\alpha x_1, x_2, \cdots, x_n|| = |\alpha| \ ||x_1, x_2, \cdots, x_n||$ for any $\alpha \in \mathbb{K}$, and

(d) $||x + x', x_2, \cdots, x_n|| \leq ||x, x_2, \cdots, x_n|| + ||x', x_2, \cdots, x_n||$

is called an n-norm on X, and the pair $(X, ||\cdot, \cdots, \cdot||)$ is called a n-normed space over the field $\mathbb{K}$. For example, we may take $X = \mathbb{R}^n$ being equipped with the n-norm $||x_1, x_2, \cdots, x_n||_E =$ the volume of the n-dimensional parallelopiped spanned by the vectors $x_1, x_2, \cdots, x_n$ which may be given explicitly by the formula

$$||x_1, x_2, \cdots, x_n||_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots, n$. Let $(X, ||\cdot, \cdots, \cdot||)$ be an n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \cdots, a_n\}$ be linearly independent set in X. Then the following function $||\cdot,\cdots,\cdot||_\infty$ on X^{n-1} defined by

$$||x_1,x_2,\cdots,x_{n-1}||_\infty = \max\{||x_1,x_2,\cdots,x_{n-1},a_i|| : i=1,2,\cdots,n\}$$

defines an $(n-1)$-norm on X with respect to $\{a_1,a_2,\cdots,a_n\}$.

A sequence (x_k) in a n-normed space $(X,||\cdot,\cdots,\cdot||)$ is said to converge to some $L\in X$ if

$$\lim_{k\to\infty} ||x_k-L,z_1,\cdots,z_{n-1}|| = 0 \text{ for every } z_1,\cdots,z_{n-1}\in X.$$

A sequence (x_k) in a n-normed space $(X,||\cdot,\cdots,\cdot||)$ is said to be Cauchy if

$$\lim_{k,p\to\infty} ||x_k-x_p,z_1,\cdots,z_{n-1}|| = 0 \text{ for every } z_1,\cdots,z_{n-1}\in X.$$

If every Cauchy sequence in X converges to some $L\in X$, then X is said to be complete with respect to the n-norm. Any complete n-normed space is said to be n-Banach space. Let w be the set of all sequences of real or complex numbers and l_∞, c and c_o be the sequence spaces of bounded, convergent and null sequences $x=(x_k)$, respectively. A sequence $x\in l_\infty$ is said to be almost convergent if all Banach limits of x coincide. Lorentz [8] proved that

$$\hat{c} = \Big\{x=(x_k) : \lim_n \frac{1}{n}\sum_{k=1}^{n} x_{k+s} \text{ exists, uniformly in } s\Big\}.$$

Maddox ([8], [10]) has defined x to be strongly almost convergent to a number L if

$$\lim_n \frac{1}{n}\sum_{k=1}^{n} |x_{k+s}-L| = 0, \text{ uniformly in } s.$$

Let $p=(p_k)$ be a sequence of strictly positive real numbers. Nanda [14] has defined the following sequence spaces:

$$[\hat{c},p] = \Big\{x=(x_k) : \lim_n \frac{1}{n}\sum_{k=1}^{n} |x_{k+s}-L|^{p_k} = 0, \text{ uniformly in } s\Big\},$$

$$[\hat{c},p]_0 = \Big\{x=(x_k) : \lim_n \frac{1}{n}\sum_{k=1}^{n} |x_{k+s}|^{p_k} = 0, \text{ uniformly in } s\Big\}$$

and

$$[\hat{c},p]_\infty = \Big\{x=(x_k) : \sup_{s,n} \frac{1}{n}\sum_{k=1}^{n} |x_{k+s}|^{p_k} < \infty\Big\}.$$

The notion of difference sequence spaces was introduced by Kizmaz [7], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Colak [2] by introducing the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$. Let w be the space of all complex or real sequences $x = (x_k)$ and let m, r be non-negative integers, then for $Z = l_\infty,\ c,\ c_0$ we have sequence spaces

$$Z(\Delta_r^m) = \{x = (x_k) \in w : (\Delta_r^m x_k) \in Z\},$$

where $\Delta_r^m x = (\Delta_r^m x_k) = (\Delta_r^{m-1} x_k - \Delta_r^{m-1} x_{k+r})$ and $\Delta_r^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta_r^m x_k = \sum_{v=0}^{m} (-1)^v \binom{m}{v} x_{k+rv}.$$

Taking $r = 1$, we get the spaces which were studied by Et and Colak [2]. Taking $m = r = 1$, we get the spaces which were introduced and studied by Kizmaz [7].

Let X be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm, if

(a) $p(x) \geq 0$, for all $x \in X$,
(b) $p(-x) = p(x)$, for all $x \in X$,
(c) $p(x + y) \leq p(x) + p(y)$, for all $x, y \in X$,
(d) if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [19], Theorem 10.4.2, P-183).

A modulus function is a function $f : [0, \infty) \to [0, \infty)$ such that

(a) $f(x) = 0$ if and only if $x = 0$,
(b) $f(x + y) \leq f(x) + f(y)$ for all $x \geq 0$, $y \geq 0$,
(c) f is increasing
(d) f is continuous from right at 0.

It follows that f must be continuous everywhere on $[0, \infty)$. The modulus function may be bounded or unbounded. For example, if we take $f(x) = \frac{x}{x+1}$, then $f(x)$ is bounded. If $f(x) = x^p$, $0 < p < 1$, then the modulus $f(x)$ is unbounded. Subseqentially, modulus function has been discussed in ([1], [9], [11], [17], [18]) and many others.

Let $F = (f_k)$ be a sequence of modulus function and $(X, ||\cdot, \cdots, \cdot||)$ be a n-normed space. Let $p = (p_k)$ be bounded sequence of strictly positive real numbers and $u = (u_k)$ be any sequence of strictly positive real numbers. By

$S(n-X)$ we denote the space of all sequences defined over $(X, ||\cdot, \cdots, \cdot||)$. In the present paper we define the following sequence spaces:

$$\Big[\hat{c}, F, u, p, ||\cdot, \cdots, \cdot||\Big](\Delta_r^m) = \Big\{x = (x_k) \in S(n-X) : \lim_n \frac{1}{n} \sum_{k=1}^{n} \Big[f_k\Big(||\frac{u_k \Delta_r^m x_{k+s} - L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} = 0, \text{ uniformly in } s, \text{ for some } \rho > 0 \text{ and } L > 0 \Big\},$$

$$\Big[\hat{c}, F, u, p, ||\cdot, \cdots, \cdot||\Big]_0(\Delta_r^m) = \Big\{x = (x_k) \in S(n-X) : \lim_n \frac{1}{n} \sum_{k=1}^{n} \Big[f_k\Big(||\frac{u_k \Delta_r^m x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} = 0, \text{ uniformly in s, for some } \rho > 0\Big\},$$

and

$$\Big[\hat{c}, F, u, p, ||\cdot, \cdots, \cdot||\Big]_\infty(\Delta_r^m) = \Big\{x = (x_k) \in S(n-X) : \sup_{s,n} \frac{1}{n} \sum_{k=1}^{n} \Big[f_k\Big(||\frac{u_k \Delta_r^m x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} < \infty, \text{ for some } \rho > 0\Big\}.$$

If we take $F(x) = x$, we have

$$\Big[\hat{c}, u, p, ||\cdot, \cdots, \cdot||\Big](\Delta_r^m) = \Big\{x = (x_k) \in S(n-X) : \lim_n \frac{1}{n} \sum_{k=1}^{n} \Big[||\frac{u_k \Delta_r^m x_{k+s} - L}{\rho}, z_1, \cdots, z_{n-1}||\Big]^{p_k} = 0, \text{ uniformly in } s, \text{ for some } \rho > 0 \text{ and } L > 0 \Big\},$$

$$\Big[\hat{c}, u, p, ||\cdot, \cdots, \cdot||\Big]_0(\Delta_r^m) = \Big\{x = (x_k) \in S(n-X) : \lim_n \frac{1}{n} \sum_{k=1}^{n} \Big[||\frac{u_k \Delta_r^m x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}||\Big]^{p_k} = 0, \text{ uniformly in } s, \text{ for some } \rho > 0\Big\},$$

and

$$\Big[\hat{c}, u, p, ||\cdot, \cdots, \cdot||\Big]_\infty(\Delta_r^m) = \Big\{x = (x_k) \in S(n-X) : \sup_{s,n} \frac{1}{n} \sum_{k=1}^{n} \Big[||\frac{u_k \Delta_r^m x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}||\Big]^{p_k} < \infty, \text{ for some } \rho > 0\Big\}.$$

If we take $p = (p_k) = 1, \ \forall k \in \mathbb{N}$, we have

$$\Big[\hat{c}, F, u, ||\cdot, \cdots, \cdot||\Big](\Delta_r^m) = \Big\{x = (x_k) \in S(n-X) : \lim_n \frac{1}{n} \sum_{k=1}^{n} f_k\Big(||\frac{u_k \Delta_r^m x_{k+s} - L}{\rho}, z_1, \cdots, z_{n-1}||\Big) = 0, \text{ uniformly in } s, \text{ for some } \rho > 0 \text{ and } L > 0 \ \Big\},$$

$$\Big[\hat{c}, F, u, ||\cdot, \cdots, \cdot||\Big]_0(\Delta_r^m) = \Big\{x = (x_k) \in S(n-X) : \lim_n \frac{1}{n} \sum_{k=1}^{n} f_k\Big(||\frac{u_k \Delta_r^m x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}||\Big) = 0, \text{ uniformly in } s, \text{ for some } \rho > 0\Big\},$$

and

$$\Big[\hat{c}, F, u, ||., \cdots, .||\Big]_\infty(\Delta_r^m) = \Big\{x = (x_k) \in S(n-X) : \sup_{s,n} \frac{1}{n} \sum_{k=1}^{n} f_k\Big(||\frac{u_k \Delta_r^m x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}||\Big) < \infty, \ \text{for some} \ \ \rho > 0\Big\}.$$

The following inequality will be used throughout the paper. If $0 \leq p_k \leq \sup p_k = G, \ K = \max(1, 2^{G-1})$ then

$$|a_k + b_k|^{p_k} \leq K\{|a_k|^{p_k} + |b_k|^{p_k}\} \tag{1}$$

for all k and $a_k, b_k \in \mathbb{C}$. Also $|a|^{p_k} \leq \max(1, |a|^G)$ for all $a \in \mathbb{C}$.

In this paper we study some topological properties and prove some inclusion relation between the sequence spaces $[\hat{c}, F, u, p, ||\cdot, \cdots, \cdot||](\Delta_r^m)$, $[\hat{c}, F, u, p, ||\cdot, \cdots, \cdot||]_0(\Delta_r^m)$ and $[\hat{c}, F, u, p, ||\cdot, \cdots, \cdot||]_\infty(\Delta_r^m)$.

2. Main results

Theorem 1. *Let $F = (f_k)$ be a sequence of modulus function and $p = (p_k)$ be a bounded sequence of strictly positive real numbers, then the classes of sequence $[\hat{c}, F, u, p, || \cdot, \cdots, \cdot ||](\Delta_r^m)$, $[\hat{c}, F, u, p, || \cdot, \cdots, \cdot ||]_0(\Delta_r^m)$ and $[\hat{c}, F, u, p, || \cdot, \cdots, \cdot||]_\infty(\Delta_r^m)$ are linear spaces over the set of complex number $\mathbb{C}$.*

Proof. Let $x = (x_k), \ y = (y_k) \in [\hat{c}, F, u, p, || \cdot, \cdots, \cdot||]_0(\Delta_r^m)$ and $\alpha, \ \beta$ be any scalars. Then there exist positive numbers ρ_1 and ρ_2 such that

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} \Big[f_k\Big(||\frac{u_k \Delta_r^m x_{k+s}}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} = 0$$

and

$$\lim_n \frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(||\frac{u_k\Delta_r^m y_{k+s}}{\rho_2}, z_1, \cdots z_{n-1}||\Big)\Big]^{p_k} = 0.$$

Let $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $F = (f_k)$ is non-decreasing function, by using inequality (1), we have

$$\begin{aligned}
&\lim_n \frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(||\frac{u_k\Delta_r^m(\alpha x_{k+s} + \beta y_{k+s})}{\rho_3}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \\
&\quad\le \lim_n \frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(||\frac{u_k\Delta_r^m(\alpha x_{k+s})}{\rho_3}, z_1, \cdots, z_{n-1}|| \\
&\qquad + ||\frac{u_k\Delta_r^m(\beta y_{k+s})}{\rho_3}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \\
&\quad\le \lim_n \frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(||\frac{u_k\Delta_r^m x_{k+s}}{\rho_1}, z_1, \cdots, z_{n-1}|| \\
&\qquad + ||\frac{u_k\Delta_r^m y_{k+s}}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \\
&\quad\le K\lim_n \frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(||\frac{u_k\Delta_r^m x_{k+s}}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \\
&\qquad + K\lim_n \frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(||\frac{u_k\Delta_r^m y_{k+s}}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \\
&\qquad \to 0 \quad \text{as } n \to \infty, \ \text{ uniformly in } \ s.
\end{aligned}$$

So that $\alpha x + \beta y \in [\hat{c}, F, u, p, ||\cdot, \cdots, \cdot||]_0(\Delta_r^m)$. This proves that $[\hat{c}, F, u, p, ||\cdot, \cdots, \cdot||]_0(\Delta_r^m)$ is a linear space. Similarly, we can prove that $[\hat{c}, F, u, p, ||\cdot, \cdots, \cdot||]_\infty(\Delta_r^m)$ and $[\hat{c}, F, u, p, ||\cdot, \cdots, \cdot||](\Delta_r^m)$ are linear spaces. ■

Theorem 2. *Let $F = (f_k)$ be a sequence of modulus function, $p = (p_k)$ be a bounded sequence of positive real numbers. Then $[\hat{c}, F, u, p, ||\cdot, \cdots, \cdot||]_0(\Delta_r^m)$ is a paranormed space with respect to the paranorm defined by*

$$g(x) = \inf\Big\{\rho^{\frac{p_n}{H}} : \Big(\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(||\frac{u_k\Delta_r^m x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \le 1\Big\},$$

where $H = \max(1, \sup_k p_k < \infty)$.

Proof. Clearly $g(x) \ge 0$ for $x = (x_k) \in \Big[\hat{c}, F, u, p, ||\cdot, \cdots, \cdot||\Big]_0(\Delta_r^m)$. Since $F(0) = 0$, we get $g(0) = 0$.

Conversely, suppose that $g(x) = 0$, then

$$\inf\Big\{\rho^{\frac{p_n}{H}} : \Big(\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(||\frac{u_k\Delta_r^m x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1\Big\} = 0.$$

This implies that for a given $\epsilon > 0$, there exists some $\rho_\epsilon (0 < \rho_\epsilon < \epsilon)$ such that

$$\Big(\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(||\frac{u_k\Delta_r^m x_{k+s}}{\rho_\epsilon}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1.$$

Thus

$$\Big(\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(||\frac{u_k\Delta_r^m x_{k+s}}{\epsilon}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}}$$

$$\leq \Big(\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(||\frac{u_k\Delta_r^m x_{k+s}}{\rho_\epsilon}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1,$$

for each n. Suppose that $x_k \neq 0$ for each $k \in \mathbb{N}$. This implies that $\Delta_r^m x_{k+s} \neq 0$, for each $k, s \in \mathbb{N}$. Let $\epsilon \to 0$, then $||\frac{u_k\Delta_r^m x_{k+s}}{\epsilon}, z_1, \cdots, z_{n-1}|| \to \infty$. It follows that

$$\Big(\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(||\frac{u_k\Delta_r^m x_{k+s}}{\epsilon}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \to \infty$$

which is a contradiction. Therefore, $\Delta_r^m x_{k+s} = 0$ for each k and thus $x_k = 0$ for each $k \in \mathbb{N}$. Let $\rho_1 > 0$ and $\rho_2 > 0$ be such that

$$\Big(\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(||\frac{u_k\Delta_r^m x_{k+s}}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1$$

and

$$\Big(\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(||\frac{u_k\Delta_r^m x_{k+s}}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1$$

for each n. Let $\rho = \rho_1 + \rho_2$. Then by using Minkowski's inequality, we have

$$\Big(\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(||\frac{u_k\Delta_r^m (x_{k+s} + y_{k+s})}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}}$$

$$\leq \Big(\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(||\frac{u_k\Delta_r^m x_{k+s} + \Delta_r^m y_{k+s}}{\rho_1 + \rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}}$$

$$\leq \Big(\frac{1}{n}\sum_{k=1}^{n}\Big[\frac{\rho_1}{\rho_1+\rho_2}f_k\Big(||\frac{\Delta_r^m x_{k+s}}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)$$

$$+\frac{\rho_2}{\rho_1+\rho_2}f_k\Big(||\frac{u_k\Delta_r^m x_{k+s}}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}}$$

$$\leq \Big(\frac{\rho_1}{\rho_1+\rho_2}\Big)\Big(\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(||\frac{\Delta_r^m x_{k+s}}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}}$$

$$+\Big(\frac{\rho_2}{\rho_1+\rho_2}\Big)\Big(\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(||\frac{u_k\Delta_r^m y_{k+s}}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1.$$

Since ρ's are non-negative, so we have

$$g(x+y) = \inf\Big\{\rho^{\frac{p_n}{H}} : \Big(\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(||\frac{u_k\Delta_r^m(x_{k+s}+y_{k+s})}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1\Big\},$$

$$\leq \inf\Big\{\rho_1^{\frac{p_n}{H}} : \Big(\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(||\frac{u_k\Delta_r^m x_{k+s}}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1\Big\}$$

$$+\inf\Big\{\rho_2^{\frac{p_n}{H}} : \Big(\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(||\frac{u_k\Delta_r^m y_{k+s}}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1\Big\}.$$

Therefore,

$$g(x+y) \leq g(x)+g(y).$$

Finally, we prove that the scalar multiplication is continuous. Let λ be any complex number. By definition,

$$g(\lambda x) = \inf\Big\{\rho^{\frac{p_n}{H}} : \Big(\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(||\frac{u_k\Delta_r^m \lambda x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1\Big\}.$$

Then

$$g(\lambda x) = \inf\Big\{(|\lambda|t)^{\frac{p_n}{H}} : \Big(\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(||\frac{u_k\Delta_r^m x_{k+s}}{t}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1\Big\}.$$

where $t = \frac{\rho}{|\lambda|}$. Since $|\lambda|^{p_n} \leq \max(1, |\lambda|^{\sup p_n})$, we have

$$g(\lambda x) \leq \max(1, |\lambda|^{\sup p_n})$$
$$\inf\Big\{t^{\frac{p_n}{H}} : \Big(\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(||\frac{u_k\Delta_r^m x_{k+s}}{t}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1\Big\}.$$

So, the fact that scalar multiplication is continuous follows from the above inequality. This completes the proof of the theorem. ■

Theorem 3. *Let $F = (f_k)$ be a sequence of modulus function. Then the following statements are equivalent:*

(i) $[\hat{c}, u, p, ||\cdot, \cdots, \cdot||]_\infty(\Delta_r^m) \subseteq [\hat{c}, F, u, p, ||\cdot, \cdots, \cdot||]_\infty(\Delta_r^m)$,

(ii) $[\hat{c}, u, p, ||\cdot, \cdots, \cdot||]_0(\Delta_r^m) \subseteq [\hat{c}, F, u, p, ||\cdot, \cdots, \cdot||]_\infty(\Delta_r^m)$,

(iii) $\sup_n \frac{1}{n} \sum_{k=1}^{n} [f_k(t)]^{p_k} < \infty$, *where* $t = ||\frac{u_k \Delta_r^m x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}|| > 0$.

Proof. $(i) \Rightarrow (ii)$ is obvious.

$(ii) \Rightarrow (iii)$ Suppose $[\hat{c}, u, p, ||\cdot, \cdots, \cdot||]_0(\Delta_r^m) \subseteq [\hat{c}, F, u, p, ||\cdot, \cdots, \cdot||]_\infty(\Delta_r^m)$ and let (iii) does not hold. Then for some $t > 0$

$$\sup_n \frac{1}{n} \sum_{k=1}^{n} [f_k(t)]^{p_k} = \infty,$$

and therefore there is a sequence (n_i) of positive integers such that

$$\frac{1}{n_i} \sum_{k=1}^{n_i} [f_k(i^{-1})]^{p_k} > i, \quad i = 1, 2, \dots . \tag{2}$$

Define $x = (x_k)$ by

$$x_k = \begin{cases} i^{-1}, & 1 \le k \le n_i, \quad i = 1, 2, ... \\ 0, & \qquad k \ge n_i. \end{cases}$$

Then $x = (x_k) \in [\hat{c}, u, p, ||\cdot, \cdots, \cdot||]_0(\Delta_r^m)$ but $x = (x_k) \notin [\hat{c}, F, u, p, ||\cdot, \cdots, \cdot||]_\infty(\Delta_r^m)$ which contradicts (ii). Hence (iii) must hold.

$(iii) \Rightarrow (i)$ Suppose $x = (x_k) \in [\hat{c}, u, p, ||\cdot, \cdots, \cdot||]_\infty(\Delta_r^m)$ and $x = (x_k) \notin [\hat{c}, F, u, p, ||\cdot, \cdots, \cdot||]_\infty(\Delta_r^m)$. Then

$$\sup_{s,n} \frac{1}{n} \sum_{k=1}^{n} \left[f_k \Big(||\frac{u_k \Delta_r^m x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}|| \Big) \right]^{p_k} = \infty. \tag{3}$$

Let $t = ||\frac{u_k \Delta_r^m x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}||$ for each k and fixed s, then by Eq.(3)

$$\sup_n \frac{1}{n} \sum_{k=1}^{n} [f_k(t)]^{p_k} = \infty,$$

which contradicts (iii). Hence (i) must hold. ■

Theorem 4. *Let* $1 \leq p_k \leq \sup_k p_k < \infty$. *Then the following statements are equivalent :*

(i) $[\hat{c}, F, u, p, ||\cdot, \cdots, \cdot||]_0(\Delta_r^m) \subseteq [\hat{c}, u, p, ||\cdot, \cdots, \cdot||]_0(\Delta_r^m)$,

(ii) $[\hat{c}, F, u, p, ||\cdot, \cdots, \cdot||]_0(\Delta_r^m) \subseteq [\hat{c}, u, p, ||\cdot, \cdots, \cdot||]_\infty(\Delta_r^m)$,

(iii) $\inf_n \frac{1}{n} \sum\limits_{k=1}^{n} [f_k(t)]^{p_k} > 0$, $t > 0$.

Proof. *(i)* $\Rightarrow$ *(ii)* is obvious.

(ii) $\Rightarrow$ *(iii)* Suppose $[\hat{c}, F, u, p, ||\cdot, \cdots, \cdot||]_0(\Delta_r^m) \subseteq [\hat{c}, u, p, ||\cdot, \cdots, \cdot||]_\infty(\Delta_r^m)$ and let *(iii)* does not hold. Then

$$\inf_n \frac{1}{n} \sum_{k=1}^{n} [f_k(t)]^{p_k} = 0, \quad t > 0. \tag{4}$$

We can choose an index sequence (n_i) such that

$$\frac{1}{n_i} \sum_{k=1}^{n_i} [f_k(i)]^{p_k} < i^{-1}, \quad i = 1, 2, \ldots, \ .$$

Define the sequence $x = (x_k)$ by

$$x_k = \begin{cases} i, & 1 \leq k \leq n_i, \quad i = 1, 2, ... \\ 0, & \qquad\quad k \geq n_i. \end{cases}$$

Thus by Eq.(4), $x = (x_k) \in [\hat{c}, F, u, p, ||\cdot, \cdots, \cdot||]_0(\Delta_r^m)$ but $x = (x_k) \notin [\hat{c}, u, p, ||\cdot, \cdots, \cdot||]_\infty(\Delta_r^m)$ which contradicts *(ii)*. Hence *(iii)* must hold.

(iii) $\Rightarrow$ *(i)* Let $x = (x_k) \in [\hat{c}, F, u, p, ||\cdot, \cdots, \cdot||]_0(\Delta_r^m)$. That is,

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} \left[f_k\left(|| \frac{u_k \Delta_r^m x_{k+s}}{\rho}, z_1, \cdots, z_{n-1} || \right) \right]^{p_k} = 0, \ \text{ uniformly in } \ s. \tag{5}$$

Suppose *(iii)* hold and $x = (x_k) \notin [\hat{c}, u, p, ||\cdot, \cdots, \cdot||]_0(\Delta_r^m)$. Then for some number $\epsilon_0 > 0$ and index n_0, we have $|| \frac{u_k \Delta_r^m x_{k+s}}{\rho}, z_1, \cdots, z_{n-1} || \geq \epsilon_0$, for some $s > s'$ and $1 \leq k \leq n_0$. Therefore

$$[f_k(\epsilon_0)]^{p_k} \leq \left[f_k\left(|| \frac{u_k \Delta_r^m x_{k+s}}{\rho}, z_1, \cdots, z_{n-1} || \right) \right]^{p_k}$$

and consequently by Eq. (5)

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} [f_k(\epsilon_0)]^{p_k} = 0,$$

which contradicts *(iii)*. Hence $[\hat{c}, F, u, p, ||\cdot, \cdots, \cdot||]_0(\Delta_r^m) \subseteq [\hat{c}, u, p, ||\cdot, \cdots, \cdot||]_0(\Delta_r^m)$. ■

Theorem 5. *Let* $F = (f_k)$ *be a sequence of modulus function. Let* $1 \leq p_k \leq \sup_k p_k < \infty$. *Then*

$$\Big[\hat{c}, F, u, p, ||\cdot, \cdots, \cdot||\Big]_\infty(\Delta_r^m) \subseteq \Big[\hat{c}, u, p, ||\cdot, \cdots, \cdot||\Big]_0(\Delta_r^m)$$

hold if

$$\lim_n \frac{1}{n}\sum_{k=1}^{n}[f_k(t)]^{p_k} = \infty, \quad t > 0. \tag{6}$$

Proof. Suppose $[\hat{c}, F, u, p, ||\cdot, \cdots, \cdot||]_\infty(\Delta_r^m) \subseteq \Big[\hat{c}, u, p, ||\cdot, \cdots, \cdot||]_0(\Delta_r^m)$ and let Eq.(6) does not hold. Therefore there is a number $t_0 > 0$ and an index sequence (n_i) such that

$$\frac{1}{n_i}\sum_{k=1}^{n_i}[f_k(t_0)]^{p_k} \leq N < \infty, \quad i = 1, 2, \ldots . \tag{7}$$

Define the sequence $x = (x_k)$ by

$$x_k = \begin{cases} t_0, & 1 \leq k \leq n_i, \\ 0, & \quad\quad k \geq n_i. \end{cases} \quad i = 1, 2, \ldots .$$

Clearly, $x = (x_k) \in [\hat{c}, F, u, p, ||\cdot, \cdots, \cdot||]_\infty(\Delta_r^m)$ but $x = (x_k) \notin [\hat{c}, u, p, ||\cdot, \cdots, \cdot||]_0(\Delta_r^m)$. Hence Eq. (6) must hold.

Conversely, if $x = (x_k) \in \Big[\hat{c}, F, u, p, ||\cdot, \cdots, \cdot||\Big]_\infty(\Delta_r^m)$, then for each s and n

$$\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(||\frac{u_k\Delta_r^m x_{k+s}}{\rho}, z||\Big)\Big]^{p_k} \leq N < \infty. \tag{8}$$

Suppose that $x = (x_k) \notin \Big[\hat{c}, u, p, ||\cdot, \cdots, \cdot||\Big]_0(\Delta_r^m)$. Then for some number $\epsilon_0 > 0$ there is a number s_0

$$||\frac{u_k\Delta_r^m x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}|| \geq \epsilon_0, \;\text{ for }\; s \geq s_o.$$

Therefore

$$[f_k(\epsilon_o)]^{p_k} \leq \Big[f_k\Big(||\frac{u_k\Delta_r^m x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k},$$

and hence for each k and s we get

$$\frac{1}{n}\sum_{k=1}^{n}[f_k(\epsilon_o)]^{p_k} \leq N < \infty,$$

for some $N > 0$, which contradicts Eq. (6). Hence

$$\Big[\hat{c}, F, u, p, ||\cdot, \cdots, \cdot||\Big]_{\infty}(\Delta_r^m) \subseteq \Big[\hat{c}, u, p, ||\cdot, \cdots, \cdot||\Big]_0(\Delta_r^m).$$

This completes the proof. ■

Theorem 6. *Let $F = (f_k)$ be a sequence of modulus function and let $1 \leq p_k \leq \sup_k p_k < \infty$. Then*

$$\Big[\hat{c}, u, p, ||\cdot, \cdots, \cdot||\Big]_{\infty}(\Delta_r^m) \subseteq \Big[\hat{c}, F, u, p, ||\cdot, \cdots, \cdot||\Big]_0(\Delta_r^m)$$

hold if

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} [f_k(t)]^{p_k} = 0, \quad t > 0. \tag{9}$$

Proof. Let $[\hat{c}, u, p, ||\cdot, \cdots, \cdot||]_{\infty}(\Delta_r^m) \subseteq [\hat{c}, F, u, p, ||\cdot, \cdots, \cdot||]_0(\Delta_r^m)$. Suppose that Eq. (9) does not hold. Then for some $t_0 > 0$,

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} [f_k(t)]^{p_k} = L \neq 0. \tag{10}$$

Define $x = (x_k)$ by

$$x_k = t \sum_{v=0}^{k-m} (-1)^m \begin{pmatrix} m + k - v - 1 \\ k - v \end{pmatrix}$$

for $k = 1, 2, \ldots$. Then $x = (x_k) \notin [\hat{c}, F, u, p, ||\cdot, \cdots, \cdot||]_0(\Delta_r^m)$ but $x = (x_k) \in [\hat{c}, u, p, ||\cdot, \cdots, \cdot||]_{\infty}(\Delta_r^m)$. Hence Eq. (9) must hold.

Conversely, let $x = (x_k) \in [\hat{c}, u, p, ||\cdot, \cdots, \cdot||]_{\infty}(\Delta_r^m)$. Then for every k and s, we have

$$||\frac{u_k \Delta_r^m x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}|| \leq N < \infty.$$

Therefore

$$\Big[f_k||\frac{u_k \Delta_r^m x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}||\Big]^{p_k} \leq [f_k(N)]^{p_k}$$

and

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} \Big[f_k\Big(||\frac{u_k \Delta_r^m x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \leq \lim_n \frac{1}{n} \sum_{k=1}^{n} [f_k(N)]^{p_k} = 0.$$

Hence $x = (x_k) \in \Big[\hat{c}, F, u, p, ||\cdot, \cdots, \cdot||\Big]_0(\Delta_r^m)$. This completes the proof. ■

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References

[1] Altinok H., Altin Y., Isik M., The sequence space $Bv_\sigma(M, P, Q, S)$ on seminormed spaces, *Indian J. Pure and Appl. Math.,* 39(1)(2008), 49-58.

[2] Et M., Colak R., On some generalized difference sequence spaces, *Soochow J. Math.,* 21(4(1995), 377-386.

[3] Gahler S., Linear 2-normietre rume, *Math. Nachr.,* 28(1965), 1-43.

[4] Gunawan H., On n-inner product, n-norms, and the Cauchy-Schwartz inequality, *Sci. Math. Jpn.,* 5(2001), 47-54.

[5] Gunawan H., The space of p-summable sequence and its natural n-norm, *Bull. Aust. Math. Soc.,* 64(2001), 137-147.

[6] Gunawan H., Mashadi M., On n-normed spaces, *Int. J. Math. Math. Sci.,* 27(2001), 631-639.

[7] Kizmaz H., On certain sequence spaces, *Cand. Math. Bull.,* 24(2)(1981), 169-176.

[8] Lorentz G.G., A contribution to the theory of divergent series, *Act. Math.,* 80(1948), 167-190.

[9] Maddox I.J., Spaces of strongly summable sequences, *Quart. J. Math.,* 18(1967), 345-355.

[10] Maddox I.J, Elements of Functional Analysis, *Cambridge Univ. Press,* 1970.

[11] Maddox I.J., A new type of convergence, *Math. Proc. Camb. Phil. Soc.,* 83(1978), 61-64.

[12] Malkowsky E., Savas E., Some λ-sequence spaces defined by a modulus, *Archivum Mathematicum,* 36(2000), 219-228.

[13] Misiak A., n-inner product spaces, *Math. Nachr.,* 140(1989), 299-319.

[14] Nanda S., Strongly almost convergent sequences, *Bull. Call. Math. Soc.,* 76(1984), 236-240.

[15] Raj K., Sharma S.K., Sharma A.K., Some new sequence spaces defined by a sequence of modulus function in n-normed spaces, *Int. J. Math. Engg. Appls.,* 5(2)(2011), 395-403.

[16] Raj K., Sharma S.K., Sharma A.K., Difference sequence spaces in n-normed spaces defined by Musielak-Orlicz functions, *Armenian J. Math.,* 3(2010), 127-141.

[17] Raj K., Sharma S.K., Difference sequence spaces defined by sequence of modulus function, *Proyecciones J. Math.,* 30(2011), 189-199.

[18] Raj K., Sharma S.K., Some difference sequence spaces defined by sequence of modulus function, *Int. J. Math. Archive,* 2(2011), 236-240.

[19] Wilansky A., Summability through Functional Analysis, *North-Holland Math. Stud.,* 1984.

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