FASCICULI MATHEMATICI

Nr 50 2013

W. Aziz, J. A. Guerrero and N. Merentes

ON NEMYTSKII OPERATOR IN THE SPACE OF SET-VALUED FUNCTIONS OF BOUNDED p-VARIATION IN THE SENSE OF RIESZ WITH RESPECT TO THE WEIGHT FUNCTION

In memory of Professor Diómedes Bárcenas

Abstract. In this paper we consider the Nemytskii operator $(Hf)(t) = h(t, f(t))$, generated by a given set–valued function h is considered. It is shown that if H is globally Lipschitzian and maps the space of functions of bounded p-variation (with respect to a weight function α) into the space of set-valued functions of bounded q-variation (with respect to α) $1 < q < p$, then H is of the form $(H\varphi)(t) = A(t)\varphi(t) + B(t)$. On the other hand, if $1 < p < q$, then H is constant. It generalizes many earlier results of this type due to Chistyakov, Matkowski, Merentes-Nikodem, Merentes-Rivas, Smajdor-Smajdor and Zawadzka.

Key words: variation in the sense of Riesz, set-valued functions, weight function, composition operator, Jensen equation.

AMS Mathematics Subject Classification: 47B33, 26B30, 26E25, 26B40.

1. Introduction

Let $I, J \subset \mathbb{R}$ be intervals. By J^I denote the set of all functions $f : I \to J$. For a given function $h : I \times J \to \mathbb{R}$, the mapping $H : J^I \to \mathbb{R}^I$ defined by

$$(1) \qquad (Hf)(x) := h(x, f(x)), \qquad f \in J^I, \ x \in I,$$

is called a superposition operator (sometimes also composition operator, substitution operator, or Nemytskii operator) generated by h. The superposition operators play important role in the theory of differential equations, integral equations and functional equations. In 1982 J. Matkowski showed (cf. [5]) that a composition operator mapping the function space $\mathrm{Lip}(I, \mathbb{R})$,

$(I = [0,1])$ into itself is Lipschitzian with respect to the Lipschitzian norm if and only if its generator h has the form

$$h(x,y) = a(x)y + b(x), \qquad x \in I, \;\; y \in \mathbb{R}, \tag{2}$$

for some $a, b \in \mathrm{Lip}(I, \mathbb{R})$. This result was extended to a lot of spaces by J. Matkowski and others.

In [7] N. Merentes and K. Nikodem showed that Nemystkii operator H, generated by a set-valued function h, mapping the space of functions of bounded p-variation $(1 < p < \infty)$ into the space of set-valued functions of bounded p-variation and globally Lipschitzian has to be of the form (2), where $a(t)$ are linear continuous set-valued functions and b is a set-valued function of bounded p-variation. In 2000, V. V. Chistyakov in [3] proved that Lipschtzian Nemystkii operators H, which map between spaces of real valued functions of bounded generalized variation of Riesz-Orlicz type including weight is the form (2), where $a(t)$ and b are functions of bounded generalized variation of Riesz-Orlicz type including weight.

The aim of this paper is to prove an analogous result in the case when the Nemytskii operator H maps the space of set-valued functions of bounded p-variation in sense of Riesz with respect to the weight α into the space of set-valued functions of bounded q-variation in the sense of Riesz with respect to the weight α, where $1 < q \leq p < \infty$ and H is globally Lipschitzian. The particular case $p = q$ has been already considered by authors in [6, 7, 8, 13, 14], but the present case of possibly different spaces requires a different proof technique and this extension may turn out to be useful in some applications.

2. Preliminary results

The section is devoted to present some auxiliary facts which will be used later on.

Let $(X, \|\cdot\|)$ be a normed space and $p \geq 1$ be a fixed number. Given $\alpha : [a,b] \to \mathbb{R}$ a fixed continuous strictly increasing function called a weight, $f : [a,b] \to X$ and a partition $\pi : a = t_0 < t_1 < \cdots < t_n = b$ of the interval $[a,b]$, we define:

$$\sigma_{p,\alpha}(f;\pi) := \sum_{i=1}^{n} \frac{\|f(t_i) - f(t_{i-1})\|^p}{|\alpha(t_i) - \alpha(t_{i-1})|^{p-1}}.$$

The number:

$$V_{p,\alpha}(f,[a,b]) := \sup_{\pi} \sigma_{p,\alpha}(f,\pi),$$

where the supremum is taken over all partitions π of $[a,b]$, is called the p-variation in the sense Riesz of the function f with respect to the weight function α (cf. [3]). A function f is said to be of bounded p-variation if $V_{p,\alpha}(f,[a,b]) < \infty$. Denote by $RV_{p,\alpha}([a,b];X)$ the space of all functions $f:[a,b]\to X$ of bounded p-variation in the sense Riesz with respect to the weight function α equipped with the norm

$$\|f\|_p := \|f(a)\| + \big(V_{p,\alpha}(f,[a,b])\big)^{1/p}.$$

Clearly, for $p=1$ the space $RV_{1,\alpha}([a,b];X)$ coincides with classical space $BV([a,b];X)$ of functions of bounded variation. In the case when $X=\mathbb{R}$ and $1<p<\infty$, we have the space $RV_{p,\alpha}([a,b])$ of functions of bounded Riesz p-variation.

Let measure space $\big([a,b],\sum,\mu_\alpha\big)$ with μ_α the Lebesgue-Stieltjes measure defined in the σ-algebra $\sum$ and

$$L_{p,\alpha}[a,b] := \Big\{f:[a,b]\to\mathbb{R} \;\Big|\; f \text{ is } \mu_\alpha \text{ integrable and } \int_a^b |f|^p d\alpha < +\infty\Big\}.$$

Moreover, let α be a function strictly increasing and continuous in $[a,b]$. A set $E\subset[a,b]$ of α-measure (μ_α) zero is a set of values $x\in[a,b]$ which can be covered by a finite number or by a denumerable sequence of intervals whose total length (i.e. the sum of the individual lengths respect to α) is arbitrarily small (cf. [10], §25).

Definition 1 ([1, 2]). *A function $f:[a,b]\longrightarrow\mathbb{R}$ is said to be absolutely continuous with respect α, if for every $\epsilon>0$, there exists $\delta>0$ such that*

$$\sum_{j=1}^{n} |f(b_j)-f(a_j)| \le \epsilon,$$

for every finite number of nonoverlapping intervals (a_j,b_j), $j=1\cdots n$ with $[a_j,b_j]\subset[a,b]$ and

$$\sum_{j=1}^{n} |\alpha(b_j)-\alpha(a_j)| \le \delta.$$

The space of all absolutely continuous functions $f:[a,b]\longrightarrow\mathbb{R}$, with respect a function α strictly increasing, is denoted by $AC-\alpha$.

Also the following characterizations (cf. [1, 2, 4]) are well-known.

Lemma 1 (cf. M.C. Chakrabarty [2], Theorem 3.2). *If $f\in AC-\alpha$, then $f'_\alpha(x)$ exists and is finite except on a set of μ_α-measure zero.*

Lemma 2 (cf. M.C. Chakrabarty [2],Theorem 3.1). *If f is AC-α on $[a,b]$, then f'_α is Lebesgue-Stieltjes integrable and*

$$f(x) = f(a) + (LS)\int_a^x f'_\alpha(t)d\alpha, \quad x\in[a,b],$$

where $(LS)\int_{\ell_1}^{\ell_2}\varphi(t)d\alpha$ *denotes the Lebesgue-Stieltjes integral of* φ *over the closed interval* $[\ell_1,\ell_2]$.

Lemma 3. *If* $f \in RV_{p,\alpha}([a,b])$ *then* f *is AC-*α *on* $[a,b]$.

Proof. Let $f \in RV_{p,\alpha}[a,b]$ and (a_i,b_i), $i = 1,2,\dots,n$ be disjoint open intervals in $[a,b]$.

$$\begin{aligned}
\sum_{i=1}^{n}|f(b_i)-f(a_i)| &= \sum_{i=1}^{n}\frac{\big|f(b_i)-f(a_i)\big|}{\big|\alpha(b_i)-\alpha(a_i)\big|^{\frac{p-1}{p}}}\big|\alpha(b_i)-\alpha(a_i)\big|^{\frac{p-1}{p}}\\
&\le \left[\sum_{i=1}^{n}\frac{\big|f(b_i)-f(a_i)\big|^p}{\big|\alpha(b_i)-\alpha(a_i)\big|^{p-1}}\right]^{\frac{1}{p}}\left[\sum_{i=1}^{n}\big|\alpha(b_i)-\alpha(a_i)\big|\right]^{\frac{p-1}{p}}\\
&\le V_{p,\alpha}(f)\cdot\left[\sum_{i=1}^{n}\big|\alpha(b_i)-\alpha(a_i)\big|\right]^{\frac{p-1}{p}},
\end{aligned}$$

if we make $\sum_{i=1}^{n}\big|\alpha(b_i)-\alpha(a_i)\big|$ sufficiently small, for $p>1$, then we get $\sum_{i=1}^{n}|f(b_i)-f(a_i)|$ is as small as desired, i.e., f is $AC-\alpha$. ∎

The following statement is a generalization of Riesz Lemma [11].

Lemma 4 (Generalization Riesz Lemma)**.** *Let* $1<p<\infty$ *and* α *be a weight function. Then* $f \in RV_{p,\alpha}([a,b];X)$ *if and only if* f *is absolutely continuous on* $[a,b]$ *and its derivative* $f' \in L_{p,\alpha}[a,b]$. *Moreover*

$$V_{p,\alpha}(f,[a,b]) = \|f'\|^p_{L_{p,\alpha}[a,b]}.$$

Proof. Let f absolutely continuous on $[a,b]$ and its derivative $f' \in L_{p,\alpha}[a,b]$, let $\pi : a=t_0<\cdots<t_n=b$ be a partition of interval $[a,b]$. Since f is absolutely continuous on $[a,b]$ then f is α-absolutely continuous a.e. on $[a,b]$, and

$$\begin{aligned}
|f(t_i)-f(t_{i-1})|^p &= \Big|\int_{t_{i-1}}^{t_i}f'_\alpha(x)d\alpha(x)\Big|^p \le \Big[\int_{t_{i-1}}^{t_i}\big|f'_\alpha(x)\big|d\alpha(x)\Big]^p\\
&\le \left[\int_{t_{i-1}}^{t_i}\big|f'_\alpha(x)\big|^p d\alpha(x)\right]\left[\left(\int_{t_{i-1}}^{t_i}d\alpha(x)\right)^{\frac{p-1}{p}}\right]^p\\
&= \big[\alpha(t_i)-\alpha(t_{i-1})\big]^{p-1}\int_{t_{i-1}}^{t_i}\big|f'_\alpha(x)\big|^p d\alpha(x).
\end{aligned}$$

So
$$\sum_{i=1}^{n} \frac{|f(t_i)-f(t_{i-1})|^p}{|\alpha(t_i)-\alpha(t_{i-1})|^{p-1}} \le \int_a^b \left|f'_\alpha(x)\right|^p d\alpha(x) = \|f'_\alpha\|^p_{L_{p,\alpha}[a,b]}.$$
Thus
$$V_{p,\alpha}(f,[a,b];X) \le \|f'_\alpha\|^p_{L_{p,\alpha}[a,b]} < +\infty, \tag{3}$$
i.e. $f \in RV_{p,\alpha}([a,b];X)$.

For the converse, if $f \in RV_{p,\alpha}([a,b];X)$, then by Lemma 3 f is α-absolutely continuous on $[a,b]$ and, also by Lemma 1 f'_α exists a.e. on $[a,b]$. For every $n \in \mathbb{N}$ we consider $\pi_n : a = t_{0,n} < t_{1,n} < \cdots < t_{n,n} = b$ a partition of the interval $[a,b]$ defined by $t_{i,n} = a + \dfrac{b-a}{n} i, i = 0,1,\ldots,n$.

Let $\{f_n\}_n$ be a sequence of step functions, $f_n : [a,b] \to \mathbb{R}$, defined by
$$f_n(t) = \begin{cases} \dfrac{f(t_{i+1,n})-f(t_{i,n})}{\alpha(t_{i+1,n})-\alpha(t_{i,n})} & \text{for} \quad t_{i,n} \le t < t_{i+1,n}, \\ 0 & \text{for} \quad t = b. \end{cases}$$
Next, we show that $f_n \to f'_\alpha$ a.e. on $[a,b]$.

Indeed, for
$$\mathcal{A} = \Big\{t \in [a,b] \ \Big|\ f'_\alpha(t) \text{ exist}\Big\} - \Big\{t_{i,n} \ \Big|\ n \in \mathbb{N},\ i = 0,1,\ldots,n\Big\},$$
let $t \in \mathcal{A}$, then for every $n \in \mathbb{N}$ exists $k \in \{0,1,\ldots,n\}$ such that $t_{k,n} \le t < t_{k+1,n}$, thus
$$\begin{aligned} f_n(t) &= \frac{f(t_{k+1,n})-f(t_{k,n})}{\alpha(t_{k+1,n})-\alpha(t_{k,n})} \\ &= \frac{\alpha(t_{k+1,n})-\alpha(t)}{\alpha(t_{k+1,n})-\alpha(t_{k,n})} \frac{f(t_{k+1,n})-f(t)}{\alpha(t_{k+1,n})-\alpha(t)} \\ &\quad + \frac{\alpha(t)-\alpha(t_{k,n})}{\alpha(t_{k+1,n})-\alpha(t_{k,n})} \frac{f(t)-f(t_{k,n})}{\alpha(t)-\alpha(t_{k,n})}. \end{aligned}$$
Since
$$\frac{\alpha(t_{k+1,n})-\alpha(t)}{\alpha(t_{k+1,n})-\alpha(t_{k,n})} + \frac{\alpha(t)-\alpha(t_{k,n})}{\alpha(t_{k+1,n})-\alpha(t_{k,n})} = 1$$
it follows that $f_n(t)$ is a convex combination of the points $\dfrac{f(t_{k+1,n})-f(t)}{\alpha(t_{k+1,n})-\alpha(t)}$ and $\dfrac{f(t)-f(t_{k,n})}{\alpha(t)-\alpha(t_{k,n})}$. Now letting $n \to \infty$, we obtain that $t_{k,n} \to t$ and $t_{k+1,n} \to t$. Therefore
$$\lim_{n\to\infty} \frac{f(t_{k+1,n})-f(t)}{\alpha(t_{k+1,n})-\alpha(t)} = f'_\alpha(t) \quad \text{and} \quad \lim_{n\to\infty} \frac{f(t)-f(t_{k,n})}{\alpha(t)-\alpha(t_{k,n})} = f'_\alpha(t),$$

thus

$$\lim_{n\to\infty} f_n(t) = f'_\alpha(t), \qquad t \in \mathcal{A} \text{ a.e. on } [a,b].$$

By Fatou's Lemma

$$\begin{aligned}
(4) \int_a^b \left|f'_\alpha(t)\right|^p d\alpha(t) &= \int_a^b \lim_{n\to\infty} |f_n(t)|^p d\alpha(t) \\
&\leq \liminf_{n\to\infty} \int_a^b |f_n(t)|^p d\alpha(t) \\
&= \liminf_{n\to\infty} \sum_{i=0}^{n-1} \int_{t_{i,n}}^{t_{i+1,n}} \left|\frac{f(t_{i+1,n}) - f(t_{i,n})}{\alpha(t_{i+1,n}) - \alpha(t_{i,n})}\right|^p d\alpha(t) \\
&= \liminf_{n\to\infty} \sum_{i=0}^{n-1} \frac{|f(t_{i+1,n}) - f(t_{i,n})|^p}{|\alpha(t_{i+1,n}) - \alpha(t_{i,n})|^p} |\alpha(t_{i+1,n}) - \alpha(t_{i,n})| \\
&= \liminf_{n\to\infty} \sum_{i=0}^{n-1} \frac{|f(t_{i+1,n}) - f(t_{i,n})|^p}{|\alpha(t_{i+1,n}) - \alpha(t_{i,n})|^{p-1}} \\
&\leq V_{p,\alpha}(f,[a,b]) < +\infty.
\end{aligned}$$

Hence $f'_\alpha \in L_{p,\alpha}[a,b]$. From (3) and (4) we have

$$V_{p,\alpha}(f) = \|f'_\alpha\|^p_{L_{p,\alpha}[a,b]}.$$

■

Let $cc(X)$ be the family of all non-empty convex compact subsets of X and D be the *Hausdorff* metric in $cc(X)$, i.e.

$$D(A,B) := \inf\Big\{t > 0 : \; A \subseteq B + tS, \; B \subseteq A + tS\Big\},$$

where $S = \big\{y \in X : \; \|y\| \leq 1\big\}$.

We say that a set-valued function $F : [a,b] \to cc(X)$ has bounded p-variation in the sense Riesz with weight α $(1 < p < \infty)$ if

$$W_{p,\alpha}(F,[a,b]) := \sup_\pi \sum_{i=1}^n \frac{\big(D(F(t_i), F(t_{i-1}))\big)^p}{|\alpha(t_i) - \alpha(t_{i-1})|^{p-1}} < \infty,$$

where the supremum is taken over all partitions $\pi : a = t_0 < t_1 < \cdots < t_n = b$ of $[a,b]$. Denote by $RW_{p,\alpha}([a,b])$ the space of all set-valued functions $F : [a,b] \to cc(X)$ of bounded p-variation in the sense Riesz with respect to the weight function α equipped with the metric

$$\begin{aligned}
D_p(F_1,F_2) &:= D(F_1(a), F_2(a)) + \\
&\left[\sup_\pi \sum_{i=1}^n \frac{\Big(D(F_1(t_i) + F_2(t_{i-1}, F_1(t_{i-1}) + F_2(t_i))\Big)^p}{|\alpha(t_i) - \alpha(t_{i-1})|^{p-1}}\right]^{1/p}.
\end{aligned}$$

Clearly, for $p = 1$ the space $RW_{1,\alpha}([a,b]; cc(X))$ coincides with the space $BV([a,b]; cc(X))$ of set-valued functions of bounded variation.

Now, let $(X, \|\cdot\|)$, $(Y, \|\cdot\|)$ be two normed spaces and K be a convex cone in X. Given a set-valued function $h : [a,b] \times K \to cc(Y)$ we consider the Nemytskii operator H generated by h, that is the composition operator defined by

$$(Hf)(t) := h(t, f(t)), \quad t \in [a,b], \;\; f : [a,b] \to K.$$

We denote by $L(K; cc(Y))$ the space of all set-valued functions $A : K \to cc(Y)$ additive and positively homogeneous we say that A is linear if $A \in L(K; cc(Y))$.

In the proof of the main results of this paper we will use some facts which we list here as lemmas.

Lemma 5 (cf. H. Rådstrom [12], Lemma 3). *Let $(X, \|\cdot\|)$ be a normed space and let A, B, C be subsets of X. If A, B are convex compact and C is non-empty and bounded, then*

$$D(A + C, B + C) = D(A, B).$$

Lemma 6 (cf. K. Nikodem [9], Theorem 5.6). *Let $(X, \|\cdot\|)$, $(Y, \|\cdot\|)$ be normed spaces and K be a convex cone in X. A set-valued function $F : K \to cc(Y)$ satisfies the Jensen equation*

$$F\Big(\frac{x+y}{2}\Big) = \frac{1}{2}\Big(F(x) + F(y)\Big), \quad x, y \in K,$$

if and only if there exists an additive set-valued function $A : K \to cc(Y)$ and a set $B \in cc(Y)$ such that $F(x) = A(t) + B$, $x \in K$.

Lemma 7 (cf. Merentes and Rivas [8]). *If $F \in RW_{p,\alpha}([a,b]; cc(Y))$ with $p > 1$, then F is continuous. In the case $p = 1$, we have $F^-(\cdot, x) \in BW([a,b]; cc(Y))$ for all $x \in K$, where*

$$F^-(t,x) := \begin{cases} \lim\limits_{s \uparrow t} F(s,x), & t \in (a,b], \; x \in K, \\ F(a,x), & t = a, \; x \in K. \end{cases}$$

3. Main results

In this section we shall present a characterization of function $h : [a,b] \times K \to cc(Y)$ for which the Nemytskii operator $H = H_h$ generated by h maps the space $RV_{p,\alpha}([a,b]; K)$ into $RW_{q,\alpha}([a,b]; cc(Y))$, where $1 < q < p$, and it is globally Lipschitzian. On the other hand if $1 < p < q$, then the Nemytskii operator H is constant.

Theorem 1. *Let $\big(X, \|\cdot\|\big)$, $\big(Y, \|\cdot\|\big)$ be normed spaces, K be a convex cone in X and $1 < q < p$. If the Nemytskii operator H generated by a set-valued function $h : [a,b] \times K \to cc(Y)$ maps the space $RV_{p,\alpha}([a,b];K)$ into space $RW_{q,\alpha}([a,b];cc(Y))$ and if it is globally Lipschitzian, then the set-valued function H satisfies the following conditions*

(a) For all $t \in [a,b]$ there exists $M(t)$, such that

$$D_q(h(t,x),h(t,y)) \leq M(t)\|x-y\|, \quad x,y \in X. \tag{5}$$

(b) $h(t,x) = A(t)x + B(t), \quad t \in [a,b],\ x \in K$, where $A : [a,b] \to L(K, cc(Y))$ and $B \in RW_{q,\alpha}([a,b];cc(Y))$.

Proof. (*a*) Since $H : RV_{p,\alpha}(f,[a,b];K) \longrightarrow RW_{q,\alpha}([a,b];cc(Y))$ is globally Lipschitzian, there exists a constant M, such that

$$D_q(Hf_1, Hf_2) \leq M\|f_1 - f_2\|_p\,, \quad f_1, f_2 \in RV_{p,\alpha}([a,b];K).$$

Let $t \in (a,b]$. Using the definition of the operator H and of metric D_q, for $f_1, f_2 \in RV_{p,\alpha}([a,b];K)$, we have

$$\begin{aligned} &D_q\Big(h(t,f_1(t)) + h(a,f_2(a)), h(a,f_1(a)) + h(t,f_2(t))\Big) \\ &\qquad\qquad \leq M|\alpha(t)-\alpha(a)|^{1-\frac{1}{q}}\|f_1 - f_2\|_p\,. \end{aligned} \tag{6}$$

Define the auxiliary function $\eta : [a,b] \to [0,1]$ by

$$\eta(\tau) := \begin{cases} \dfrac{\alpha(\tau)-\alpha(a)}{\alpha(t)-\alpha(a)} & \text{for} \quad a \leq \tau \leq t \\ 1 & \text{for} \quad t \leq \tau \leq b. \end{cases}$$

The function $\eta \in RV_{p,\alpha}([a,b])$ and

$$V_{p,\alpha}(\eta,[a,b]) = \frac{1}{|\alpha(t)-\alpha(a)|^{p-1}}.$$

Let us fix $x, y \in K$ and define the functions $f_i : [a,b] \to K$ $(i = 1,2)$ by

$$f_1(\tau) := x, \quad f_2(\tau) := \eta(\tau)(y-x) + x, \quad \tau \in [a,b]. \tag{7}$$

The functions $f_i \in RV_{p,\alpha}([a,b];K)$ $(i = 1,2)$ and

$$\|f_1 - f_2\|_p = \Big(V_{p,\alpha}\big(\eta,[a,b]\big)\Big)^{\frac{1}{p}}\|x-y\| = \frac{\|x-y\|}{|\alpha(t)-\alpha(a)|^{1-\frac{1}{p}}}.$$

Hence, substituting in inequality (6) the functions f_i $(i = 1,2)$, we obtain

$$D_q\Big(h(t,x) + h(a,x), h(a,x) + h(t,y)\Big) \leq M\frac{|\alpha(t)-\alpha(a)|^{1-\frac{1}{q}}}{|\alpha(t)-\alpha(a)|^{1-\frac{1}{p}}}\|x-y\|, \tag{8}$$

for all $t \in [a,b]$, $x, y \in K$.

By Lemma 5 and the inequality (8) we have

$$D_q\Big(h(t,x), h(t,y)\Big) \le M \frac{|\alpha(t)-\alpha(a)|^{1-\frac{1}{q}}}{|\alpha(t)-\alpha(a)|^{1-\frac{1}{p}}} \|x-y\|,$$

for all $t \in [a,b]$, $x, y \in K$.

Now, let $t = a$. Define the function $\eta_1 : [a,b] \to [0,1]$ by

$$\begin{cases} \dfrac{\alpha(\tau)-\alpha(a)}{\alpha(t)-\alpha(a)}, & \tau \in (a,b]; \\ 0, & t = a \end{cases}$$

The function $\eta_1 \in RV_{p,\alpha}[a,b]$ and

$$V_{p,\alpha}(\eta_1) = \frac{1}{|\alpha(b)-\alpha(a)|^{p-1}}.$$

Let us fix $x, y \in K$ and define the functions $\widetilde{f}_i : [a,b] \to K$ $(i = 1,2)$ by

$$\widetilde{f}_1(\tau) := x, \quad \widetilde{f}_2(\tau) := \eta_1(\tau)(x-y) + y; \quad \tau \in [a,b]. \tag{9}$$

The functions $\widetilde{f}_i \in RV_{p,\alpha}([a,b];K)$ $(i = 1,2)$ and

$$\begin{aligned} \|\widetilde{f}_1 - \widetilde{f}_2\|_p &= \left(1 + \Big(V_{p,\alpha}\big(\eta_1, [a,b]\big)\Big)^{\frac{1}{p}}\right)\|x-y\| \\ &= \left(1 + \frac{1}{|\alpha(b)-\alpha(a)|^{1-\frac{1}{p}}}\right)\|x-y\|. \end{aligned}$$

Hence, substituting in the inequality (6), the functions $\widetilde{f}_i$ $(i = 1,2)$, we obtain

$$\begin{aligned} &D_q\Big(h(b,x) + h(a,y), h(a,x) + h(b,x)\Big) \\ &\qquad \le M|\alpha(b)-\alpha(a)|^{1-\frac{1}{q}} \left(1 + \frac{1}{|\alpha(b)-\alpha(a)|^{1-\frac{1}{p}}}\right)\|x-y\|. \end{aligned}$$

By Lemma 5 and the above inequality, we have

$$D_q\big(h(a,y), h(a,x)\big) \le M|\alpha(b)-\alpha(a)|^{1-\frac{1}{q}} \left(1 + \frac{1}{|\alpha(b)-\alpha(a)|^{1-\frac{1}{p}}}\right)\|x-y\|.$$

Define the function $M : [a,b] \to \mathbb{R}$ by

$$M(t) := \begin{cases} M\dfrac{|\alpha(t)-\alpha(a)|^{1-\frac{1}{q}}}{|\alpha(t)-\alpha(a)|^{1-\frac{1}{p}}} & \text{for} \quad a < t \le b, \\ M|\alpha(b)-\alpha(a)|^{1-\frac{1}{q}}\left(1 + \dfrac{1}{|\alpha(b)-\alpha(a)|^{1-\frac{1}{p}}}\right) & \text{for} \quad t = a. \end{cases}$$

Hence

$$D_q\big(h(t,x),h(t,y)\big) \le M(t)\|x-y\|, \quad x,y\in X,\ t\in[a,b],$$

and, consequently, for ever $t\in[a,b]$ the function $h:[a,b]\times K\to cc(Y)$ is continuous.

(b) Let us fix $t,t_0\in[a,b]$ such that $t_0<t$. Since the Nemytskii operator H is globally Lipschitzian, there exists a constant M, such that

$$D_q\Big(h(t,f_1(t))+h(t_0,f_2(t_0)),h(t_0,f_1(t_0))+h(t,f_2(t))\Big) \le M\|f_1-f_2\|_p|\alpha(t)-\alpha(t_0)|^{1-\frac{1}{q}}. \tag{10}$$

Define the function $\eta_2:[a,b]\to[0,1]$ by

$$\eta_2(\tau):=\begin{cases} \dfrac{\alpha(\tau)-\alpha(a)}{\alpha(t_0)-\alpha(a)} & \text{for} \quad a\le\tau\le t_0,\\ -\dfrac{\alpha(\tau)-\alpha(t)}{\alpha(t)-\alpha(t_0)} & \text{for} \quad t_0\le\tau\le t,\\ 0 & \text{for} \quad t\le\tau\le b.\end{cases}$$

The function $\eta_2\in RV_{p,\alpha}[a,b]$. Let us fix $x,y\in K$ and define the functions $f_i:[a,b]\to K$ by

$$\begin{cases} f_1(\tau):=\frac{1}{2}\eta_2(\tau)x+\Big(1-\frac{1}{2}\eta_2(\tau)\Big)y & \text{for} \quad \tau\in[a,b];\\ f_2(\tau):=\frac{1}{2}\Big(1+\eta_2(\tau)\Big)x+\frac{1}{2}\Big(1-\eta_2(\tau)\Big)y & \text{for} \quad \tau\in[a,b].\end{cases} \tag{11}$$

The functions $f_i\in RV_{p,\alpha}([a,b];K)$, $i=1,2$ and

$$\|f_1-f_2\|_p=\frac{\|x-y\|}{2}.$$

Substituting in the inequality (10) the functions f_i ($i=1,2$) defined by (11), we obtain

$$D_q\Big(h(t_0,x)+h(t,y),h\Big(t_0,\frac{x+y}{2}\Big)+h\Big(t,\frac{x+y}{2}\Big)\Big) \le \frac{1}{2}M|\alpha(t)-\alpha(t_0)|^{1-\frac{1}{q}}\|x-y\|. \tag{12}$$

Since H maps $RV_{p,\alpha}([a,b];K)$ into $RW_{q,\alpha}([a,b];cc(Y))$ $(1<q<p)$, then $h(\cdot,z)$ is continuous for all $z\in K$. Hence letting $t_0\uparrow t$ in the inequality (12), we get

$$D_q\Big(h(t,x)+h(t,y),h\Big(t,\frac{x+y}{2}\Big)+h\Big(t,\frac{x+y}{2}\Big)\Big)=0,$$

for all $t \in [a,b]$ and $x, y \in K$.

Thus for all $t \in [a,b]$, $x, y \in K$, we have

$$h\Big(t, \frac{x+y}{2}\Big) + h\Big(t, \frac{x+y}{2}\Big) = h(t,x) + h(t,y).$$

Since that values of h are convex, we obtain

$$h\Big(t, \frac{x+y}{2}\Big) = \frac{1}{2}\Big(h(t,x) + h(t,y)\Big), \tag{13}$$

for all $t \in [a,b]$, $x, y \in K$. Thus for all $t \in [a,b]$, the set-valued function $h(t,\cdot) : K \to cc(Y)$ satisfies the Jensen equation (13). Now by the Lemma 6, there exists an additive set-valued function $A(t) : K \to cc(Y)$ and a set $B(t) \in cc(Y)$, such that

$$h(t,x) = A(t)x + B(t), \quad t \in [a,b], \quad x \in K.$$

Substituting $h(t,x) = A(t)x + B(t)$ into inequality (5), we obtain for all $t \in [a,b]$ that there exists $M(t)$, such that

$$D_q(A(t)x, A(t)y) \leq M(t)\|x - y\|, \quad x, y \in K,$$

consequently, the set-valued function $A(t) : K \to cc(Y)$ is continuous, and $A(t)(\cdot) \in L(K, cc(Y))$.

Since $A(t)(\cdot)$ is additive and $0 \in K$, then $A(t)0 = \{0\}$, thus $h(t,0) = B(t)$, $t \in [a,b]$ and H maps $RV_{p,\alpha}([a,b];K)$ into $RW_{q,\alpha}([a,b];cc(Y))$, then $H(t,0) = B(t) \in RW_{q,\alpha}([a,b];K)$. ■

Theorem 2. *Let $\big(X, \|\cdot\|\big)$, $\big(Y, \|\cdot\|\big)$ be normed spaces, K a convex cone in X and $1 < p < q$. If the Nemytskii operator H generated by a set-valued function $h : [a,b] \times K \to cc(Y)$ maps the space $RV_{p,\alpha}([a,b];K)$ into the space $RW_{q,\alpha}([a,b];cc(Y))$ and it is globally Lipschizian, then the set-valued function h satisfies the condition*

$$h(t,x) = h(t,0), \quad t \in [a,b], \quad x \in K;$$

i.e. the Nemytskii operator is constant.

Proof. Since the Nemytskii operator H is globally Lipschizian between $RV_{p,\alpha}([a,b];K)$ and the space $RW_{q,\alpha}([a,b];cc(Y))$, $1 < p < q$, then there exists a constant M, such that

$$D_q(Hf_1, Hf_2) \leq M\|f_1 - f_2\|_p, \quad f_1, f_2 \in RV_{p,\alpha}([a,b];K).$$

Let us fix $t, t_0 \in [a,b]$ such that $t_0 < t$. Using the definitions of the operator H and of the metric D_q, we have

$$D_q\Big(h(t,f_1(t)) + h(t_0,f_2(t_0)), h(t_0,f_1(t_0)) + h(t,f_2(t))\Big) \le M|\alpha(t)-\alpha(t_0)|^{1-\frac{1}{q}}\|f_1-f_2\|_p, \quad f_1,f_2 \in RV_{p,\alpha}([a,b];K). \tag{14}$$

Define the auxiliary function $\eta_3 : [a,b] \to [0,1]$ by

$$\eta_3(\tau) := \begin{cases} 1 & \text{for} \quad a \le \tau \le t_0, \\ -\dfrac{\alpha(\tau)-\alpha(t)}{\alpha(t)-\alpha(t_0)} & \text{for} \quad t_0 \le \tau \le t, \\ 0 & \text{for} \quad t \le \tau \le b. \end{cases}$$

The function $\eta_3 \in RV_{p,\alpha}[a,b]$ and $V_{p,\alpha}(\eta_3;[a,b]) = \dfrac{1}{|\alpha(t)-\alpha(t_0)|^{p-1}}$.

Let us fix $x \in K$ and define the functions $f_i : [a,b] \to K$ $(i=1,2)$ by

$$f_1(\tau) := x, \quad f_2(\tau) := \eta_3(\tau)x, \quad \tau \in [a,b]. \tag{15}$$

We obtain that the functions $f_i \in RV_{p,\alpha}([a,b];K)$ $(i=1,2)$ and

$$\|f_1-f_2\|_p = \frac{\|x\|}{|\alpha(t)-\alpha(t_0)|^{1-\frac{1}{p}}}.$$

Hence, substituting in the inequality (14) the auxiliary functions f_i $(i = 1,2)$ defined by (15), we obtain

$$D_q\Big(h(t,x)+h(t_0,x), h(t_0,x)+h(t,0)\Big) \le M\frac{|\alpha(t)-\alpha(t_0)|^{1-\frac{1}{q}}}{|\alpha(t)-\alpha(t_0)|^{1-\frac{1}{p}}}\|x\|.$$

By Lemma 5 and the above inequality, we get

$$D_q\Big(h(t,x), h(t,0)\Big) \le M\frac{|\alpha(t)-\alpha(t_0)|^{1-\frac{1}{q}}}{|\alpha(t)-\alpha(t_0)|^{1-\frac{1}{p}}}\|x\|.$$

Since $q > p$. Letting $t \uparrow t_0$ in the above inequality, we have $D_q(h(t,x), h(t,0)) = 0$, thus for all $t \in [a,b]$ and for all $x \in K$, we get $h(t,x) = h(t,0)$. ∎

Theorem 3. *Let $(X, \|\cdot\|)$, $(Y, \|\cdot\|)$ be normed spaces, K a convex cone in X and $1 < p < \infty$. If the Nemytskii operator H generated by a set-valued function $h : [a,b] \times K \to cc(Y)$ maps the space $RV_{p,\alpha}([a,b];K)$ into the space*

$BW([a,b];cc(Y))$ and it is globally Lipschizian, then the left regularization $h^*:[a,b]\times K\to cc(Y)$ of the function h defined by

$$h^*(t,x):=\begin{cases} h^-(t,x), & t\in(a,b],\ x\in K;\\ \lim\limits_{s\downarrow a}(s,x), & t=a,\ x\in K,\end{cases}$$

satisfies the following conditions

(a) for all $t\in[a,b]$ there exists $M(t)$, such that

$$D_1(h^*(t,x),h^*(t,y))\le M(t)\|x-y\|,\quad x,y\in X.$$

(b) $h^*(t,x)=A(t)x+B(t)$, $t\in[a,b]$, $x\in K$, where $A(t)$ is linear continuous set–valued function, and $B\in BW([a,b];cc(Y))$.

Proof. (a) We take $t\in[a,b]$, and define the auxiliary function $\eta:[a,b]\to[0,1]$ by

$$\eta_4(\tau):=\begin{cases} 1 & \text{for}\quad a\le\tau\le t,\\ \dfrac{\alpha(\tau)-\alpha(b)}{\alpha(t)-\alpha(b)} & \text{for}\quad t\le\tau\le b.\end{cases}$$

The function $\eta_4\in RV_{p,\alpha}([a,b])$ and $V_{p,\alpha}(\eta_4,[a,b])=\dfrac{1}{|\alpha(b)-\alpha(t)|^{p-1}}$.

Let us fix $x,y\in K$ and define the functions $f_i:[a,b]\to K$ $(i=1,2)$ by

$$f_1(\tau):=x,\quad f_2(\tau):=\eta_4(\tau)(y-x)+x,\quad \tau\in[a,b]. \tag{16}$$

The functions $f_i\in RV_{p,\alpha}([a,b];K)$ $(i=1,2)$ and

$$\begin{aligned}\|f_1-f_2\|_p &= \Big(V_{p,\alpha}(\eta;[a,b])\Big)^{\frac{1}{p}}\|x-y\|\\ &= \left(1+\frac{1}{|\alpha(b)-\alpha(t)|^{1-\frac{1}{p}}}\right)\|x-y\|.\end{aligned}$$

Since the Nemytskii operator H is globally Lipschitzian between $RV_{p,\alpha}([a,b];K)$ and $BW([a,b];cc(Y))$, then there exists a constant M, such that

$$D\Big(h(b,f_1(b))+h(t,f_2(t)),h(t,f_1(t))+h(b,f_2(b))\Big)\le M\|f_1-f_2\|_p.$$

By Lemma 5, substituting the particular functions f_i $(i=1,2)$ defined by (16) in the above inequality, we obtain

$$D\Big(h(t,x),h(t,y)\Big)\le M(t)\|x-y\|,\quad x,y\in K,\ t\in[a,b], \tag{17}$$

where $M(t) := M\left[1 + \dfrac{1}{|\alpha(b)-\alpha(t)|^{1-\frac{1}{p}}}\right]$.

In the case where $t = b$, by a similar reasoning as above, we obtain that there exists a constant $M(b)$, such that

$$D(h(b,x),h(b,y)) \le M(b)\|x-y\|, \quad x,y \in K. \tag{18}$$

Hence, passing to the limit in the inequality (17) by the inequality (18) and the definition of h^* we have for all $t \in [a,b]$ that there exists $M(t)$, such that

$$D(h^*(t,x),h^*(t,y)) \le M(t)\|x-y\|, \quad x,y \in K.$$

Let us fix $t, t_0 \in [a,b]$, $n \in \mathbb{N}$ such that $t_0 < t$. Define the partition π_n of the interval $[t_0,t]$ by $\pi_n : a < t_0 < t_1 < \cdots < t_{2n-1} < t_{2n} = t$, where

$$t_i - t_{i-1} = \frac{t-t_0}{2n}, \quad i = 1,2,\ldots,2n.$$

Since the Nemytskii operator H is globally Lipschitzian between $RV_{p,\alpha}([a,b];K)$ and $BW([a,b];cc(Y))$, then there exists a constant M, such that

$$\begin{aligned}\sum_{i=1}^{n} D\Big(&h(t_{2i},f_1(t_{2i})) + h(t_{2i-1},f_2(t_{2i-1})),\\ &h(t_{2i-1},f_1(t_{2i-1})) + h(t_{2i},f_2(t_{2i}))\Big) \le M\|f_1-f_2\|_p,\end{aligned} \tag{19}$$

for $f_1, f_2 \in RV_{p,\alpha}([a,b];K)$.

We define the function $\widetilde{\eta} : [a,b] \to [0,1]$ in the following way

$$\widetilde{\eta}(\tau) := \begin{cases} 0 & \text{for} \quad a \le \tau \le t_0;\\ \dfrac{\alpha(\tau)-\alpha(t_{i-1})}{\alpha(t_i)-\alpha(t_{i-1})} & \text{for} \quad t_{i-1} \le \tau \le t_i,\ i = 1,3,\ldots,2n-1;\\ -\dfrac{\alpha(\tau)-\alpha(t_i)}{\alpha(t_i)-\alpha(t_{i-1})} & \text{for} \quad t_{i-1} \le \tau \le t_i,\ i = 2,4,\ldots,2n;\\ 0 & \text{for} \quad t \le \tau \le b,\end{cases}$$

we have that the function $\widetilde{\eta} \in RV_{p,\alpha}([a,b])$ and $V_{p,\alpha}(\widetilde{\eta};[a,b]) = \dfrac{2^p n^p}{|\alpha(t)-\alpha(t_0)|^{p-1}}$.

Let us fix $x, y \in K$ and define the functions $f_i : [a,b] \to K$ by

$$\begin{cases} f_1(\tau) := \frac{1}{2}\widetilde{\eta}(\tau)x + \Big[1 - \frac{1}{2}\widetilde{\eta}(\tau)y\Big] & \text{for} \quad \tau \in [a,b];\\ f_2(\tau) := \frac{1}{2}\Big[1+\widetilde{\eta}(\tau)\Big]x + \frac{1}{2}\Big[1-\widetilde{\eta}(\tau)\Big]y & \text{for} \quad \tau \in [a,b].\end{cases} \tag{20}$$

The functions $f_i \in RV_{p,\alpha}([a,b];K)$ $(i = 1,2)$ and

$$\|f_1 - f_2\|_p = \frac{\|x-y\|}{2}.$$

Substituting in the inequality (19) the particular functions f_i $(i = 1, 2)$ defined in (20), we obtain

$$\sum_{i=1}^{n} D\Big(h(t_{2i-2}, x) + h(t_{2i}, y), h\Big(t_{2i-1}, \frac{x+y}{2}\Big) + h\Big(t_{2i}, \frac{x+y}{2}\Big)\Big) \le \frac{1}{2}M\|x-y\|, \quad x, y \in K. \tag{21}$$

The Nemytskii operator H maps the spaces $RV_{p,\alpha}([a,b];K)$ into $BW([a,b];cc(Y))$, then for all $z \in K$, the function $h(\cdot, z) \in BW([a,b];cc(Y))$. Letting $t_0 \uparrow t$ in the inequality (21), we get

$$D\Big(h^*(t,x) + h^*(t,y), h^*\Big(t, \frac{x+y}{2}\Big) + h^*\Big(t, \frac{x+y}{2}\Big)\Big) \le \frac{M}{2n}\|x-y\|.$$

Passing to the limit when $n \to \infty$, we get

$$h^*(t,x) + h^*(t,y) + h^*\Big(t, \frac{x+y}{2}\Big) + h^*\Big(t, \frac{x+y}{2}\Big) = 0, \ t \in [a,b], \ x, y \in K.$$

Since $h^*(t,x)$ is a convex set, then

$$h^*\Big(t, \frac{x+y}{2}\Big) = \frac{1}{2}\Big(h^*(t,x) + h^*(t,y)\Big), \quad t \in [a,b], \ x, y \in K.$$

Thus for ever $t \in [a,b]$, set-valued function $h^*(t,\cdot)$ satisfies the Jensen equation. By Lemma 6 and by the property (a) previously established, we get that for all $t \in [a,b]$ there exist an additive set-valued function $A(\cdot) : K \to cc(Y)$ and a set $B(t) \in cc(Y)$, such that

$$h^*(t,x) = A(t)x + B(t), \quad t \in [a,b], \ x \in K.$$

By the same reasoning as in the proof of Theorem 1, we obtain that $A(t)(\cdot) \in L(K, cc(Y))$ and $B \in BW([a,b];cc(Y))$. ■

Acknowledge: The authors of this paper wish to thank the referee for all corrections and suggestions for publishing this article.

References

[1] Chakrabarty M.C., Some results on ω-Derivatives and BV-ω Functions, *J. Aust. Math. Soc.,* October 1967, 345-360.

[2] Chakrabarty M.C., Some results on AC-ω functions, *Fund. Math.,* LXIV (1969), 219–230.

[3] Chistyakov V.V., Lipschitzian superposition operators between spaces of functions of bounded generalized variation with weight, *J. Appl. Anal.,* 6(2) (2000), 173-186.

[4] Jeffery R.L., Generalized integrals with respect to bounded variation, *Can. J. Math.,* 10(1958), 617-628.

[5] Matkowski J., Functional equation and Nemytskij operators, *Funkc. Ekvac.,* 25(1982), 127-132.
[6] Matkowski J., Miś, J., On a characterization of Lipschitzian operators of substitution in the space $BV[a,b]$, *Math. Nachr.,* 117(1984), 155-159.
[7] Merentes N., Nikodem K., On Nemytskij operator and set-valued functions of bounded p-variation, *Radowi Math.,* 8(1992), 139-145.
[8] Merentes N., Rivas S., On Nemytskij operator in the space of set-valued functions of bounded p-variation in the sense of Riesz, *Public. Math. Debr.,* 47(1-2)(1995), 15-27.
[9] Nikodem K., K-convex and K-concave set-valued functions, *Polish. Lódzka. Zeszyty Nauk. Politech. Łódź. Mat. 559,* Rozpr. Nauk. 114, Łódź (1989).
[10] Riesz F., Nagy B.Sz., Functional Analysis, *Ungar,* New York, 1955.
[11] Riesz F., Untersuchugen über systeme integrierbarer funktionen, *Math. Annalen,* 69(1910), 449-497.
[12] Rådström H., An embedding theorem for space of convex sets, *Proc. Amer. Math. Soc.,* 3(1952), 165-169.
[13] Smajdor A., Smajdor W., Jensen equation and Nemytskij operator for set-valued functions, *Radovi. Math.,* 5(1989), 311-319.
[14] Zawadzka G., On Lipschitzian operators of substitution in the space of set-valued functions of bounded variation, *Radovi Math.,* 6(1990), 179-193.

Wadie Aziz
Universidad de los Andes
Departamento de Física y Matemáticas
Trujillo-Venezuela
e-mail: wadie@ula.ve

J. A. Guerrero
Universidad Nacional Experimental del Táchira
Dpto. de Matemática y Física
San Cristóbal-Venezuela
e-mail: jaguerrero4@gmail.com

N. Merentes
Universidad Central de Venezuela
Escuela de Matemáticas
Caracas-Venezuela
e-mail: nmer@ciens.ucv.ve

Received on 20.04.2011 and, in revised form, on 29.03.2012.