

A. A. EL-ATIK* AND ARAFA A. NASEF

**ON ALMOST CO-F-CLOSED TOPOLOGIES
AND THEIR APPLICATIONS**

ABSTRACT. In this paper, we investigate more properties of the notion of co-F-closed topologies which was introduced by Abo Khadra and Nasef [1]. Several results which have appeared in [5] are expanded and improved. Also, we introduce a new concept of almost co-F-closed topologies of a given topological space and give a new class of functions, namely, almost F-continuous functions which properly contains the class of almost F-continuous functions (Chae et al., [5]). Finally, we show that the concept of almost F-continuity can be considered naturally from the point of view of change of topology.

KEY WORDS: near open sets, F-closed spaces, F-continuity, almost F-continuity, co-F-closed spaces, almost co-F-closed spaces.

AMS Mathematics Subject Classification: 45A10, 54C05, 54C10.

1. Introduction

The field of the mathematical science which goes under the name of topology is concerned with all questions directly or indirectly related to continuity. General topologists have introduced and investigated many different generalizations of continuous functions. The most significant of those notions are based on different kinds of compact and closed subsets. The concept of almost C-continuity was defined and studied in [7], [8] and [17], respectively. Gauld [7] investigated the concept of almost H-continuity. Malghan and Hanchinamani [15] have considered the class of almost N-continuous functions between topological spaces. Also other properties and the behaviour of almost N-continuous functions in product spaces are discussed in [11]. Jiang and Reilly [10] defined almost S-continuity. Abo Khadra and Nasef [1] defined the notion co-F-closed topologies and investigate some of its properties.

* Now the address of author is: Department of Mathematics, Faculty of Science, Shaqraa University, El-Dawadmi 11911 M.B. 1040 KSU.

In this paper, some of the basic properties of co-F-closed (τ^*) and almost co-F-closed topologies (τ^{**}) are studied. Also, we investigate certain functions which involve F-closed spaces and we provide improvements of several results due to Chae et al., [5].

2. Terminologies

Throughout this paper X and Y represent topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a space X . The closure and the interior of A in X are denoted by $Cl_X(A)$ and $Int_X(A)$ (or simply $Cl(A)$ and $Int(A)$), respectively. A subset A of X is regular open (resp. α -open [16], semi open [12]) if $A = Int(Cl(A))$ (resp. $A \subset Int(Cl(Int(A)))$, $A \subset Cl(Int(A))$). A subset A of X is said to be feebly open [13] if there is an open set O such that $O \subset A \subset sCl(O)$. It was shown in [[3], Lemma 2.4] and [[9], Proposition 1] that the concept of feebly open set is equivalent to the notion of α -open set. The complement of a feebly open (resp. semi open) set is said to be feebly closed (resp. semi closed). The intersection of all semi closed sets containing A is called the semi-closure of A and denoted by $sCl(A)$. $\tau(X)$, $RO(X)$ and $FO(X)$ will denote the family of all open, regular open and feebly open sets of X , respectively.

A Hausdorff space X is called H-closed [20] if every open cover of X has a finite proximate subcover. A space X is called N-closed [2] (resp. S-closed [19], F-closed [4], quasi H-closed (shortly QHC) [18]) if every open (resp. semi open, feebly open, open) cover of X has a finite subcover such that the interior of the closure (resp. the closure, the closure, the closure) of its members covers X . A subset A of a Hausdorff space X is said to be H-closed relative to X [18] if for any cover $\{U_\alpha : \alpha \in \nabla, U_\alpha \in \tau(X)\}$ of A , there is a finite subset ∇_0 of ∇ such that $A \subset \bigcup\{Cl(U_\alpha) : \alpha \in \nabla_0\}$. A subset A of a space X is called N-closed [2] (resp. S-closed [19], F-closed [4], quasi H-closed [18]) relative to X if for any cover $\{U_\alpha : \alpha \in \nabla, U_\alpha \in \tau(X)\}$ (resp. $U_\alpha \in SO(X)$, $U_\alpha \in FO(X)$, $U_\alpha \in \tau(X)$) of A , there is a finite subset ∇_0 of ∇ such that $A \subset \bigcup_{\alpha \in \nabla_0} \{Int(Cl(\text{resp. } Cl, Cl, Cl)(U_\alpha))\}$.

Definition 1. A function $f : X \rightarrow Y$ from a space X into a space Y is said to be:

(a) almost C-continuous [7] if whenever $U \subset Y$ is a regular open set with compact complement, $f^{-1}(U)$ is open in X .

(b) almost N-continuous [15] if for each point $x \in X$ and each regular open set V containing $f(x)$ and having N-closed complement there is an open set U containing x such that $f(U) \subset V$.

(c) almost H -continuous [7] if for each point $x \in X$ and each regular open set V containing $f(x)$ and having H -closed complement there is an open set U containing x such that $f(U) \subset V$.

(d) almost S -continuous [10] if for each point $x \in X$ and each regular open set V containing $f(x)$ and having S -closed complement there is an open set U containing x such that $f(U) \subset V$.

3. Almost F -continuity

We introduce a new class of functions between topological spaces with the following definition.

Definition 2. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called almost F -continuous at a point $x \in X$ if for each regular open set V in Y containing $f(x)$ and having F -closed complement, there is an open set U in X containing x such that $f(U) \subset V$.

If f is almost F -continuous at each point of its domain, it is said to be almost F -continuous. We can replace an F -closed complement by a QHC complement in Definition 2 from Lemma 2.1 [5].

We recall that a function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called F -continuous [5] if for every point $x \in X$ and each open set V containing $f(x)$ and having F -closed complement, there is an open set U containing x such that $f(U) \subset V$. Observe that the following implications hold and none of them is reversible: Continuity $\Rightarrow F$ -continuity \Rightarrow almost F -continuity.

Theorem 1. For a function $f : (X, \tau) \rightarrow (Y, \sigma)$ the following statements are equivalent:

- (a) f is almost- F -continuous.
- (b) The inverse image of every regular open subset of Y having an F -closed complement is open in X .
- (c) The inverse image of every regular closed F -closed subset of Y is closed in X .
- (d) For each $x \in X$ and each net $\{x_\alpha\}_{\alpha \in \nabla}$ which converges to x , the net $\{f(x_\alpha)\}_{\alpha \in \nabla}$ is eventually in every regular open set containing $f(x)$ and having an F -closed complement.

Proof. (a) \Rightarrow (b): Let V be a regular open set having an F -closed complement and let $x \in f^{-1}(V)$. Then there exists an open set U of X containing x such that $f(U) \subset V$. Thus $x \in U \subset f^{-1}(V)$ and hence $f^{-1}(V)$ is open.

(b) \Rightarrow (c): Let W be regular closed and F -closed in Y , then $Y - W$ is a regular open set having an F -closed complement. By (b) $f^{-1}(Y - W) = X - f^{-1}(W)$ is open and hence $f^{-1}(W)$ is closed in X .

(c) \Rightarrow (a): Let $x \in X$ and $f(x) \in V \in RO(Y, x)$ and having an F -closed complement, then $x \notin f^{-1}(Y - V) = X - f^{-1}(V)$ which is closed by hypothesis. So $x \in f^{-1}(V)$ is open. Setting $U = f^{-1}(V)$, we have $f(U) \subset V$.

(b) \Rightarrow (d): Let $\{x_\alpha\}_{\alpha \in \nabla}$ be a net in X converging to x and let $V \in RO(Y, f(x))$ and having an F -closed complement. Then $x \in f^{-1}(V)$ which is open in X and $\{x_\alpha\}_{\alpha \in \nabla}$ is residually in $f^{-1}(V)$. Hence $\{f(x_\alpha)\}_{\alpha \in \nabla}$ is eventually in V .

(d) \Rightarrow (b): Let $V \in RO(Y)$ having an F -closed complement. To show $f^{-1}(V)$ is open in X , suppose the set $f^{-1}(V)$ containing x is not open. Then there is a net $\{x_\alpha\}_{\alpha \in \nabla}$ converging to x such that it is finitely many in $f^{-1}(V)$ or cofinally in $f^{-1}(V)$. Thus the net $\{f(x_\alpha)\}_{\alpha \in \nabla}$ is finitely many in V or cofinally in V . It contradicts. ■

Theorem 2. *For any function $f : X \rightarrow Y$, the following are true.*

(a) *If f is almost F -continuous and $A \subset X$, then the restriction $f|_A : A \rightarrow Y$ is almost F -continuous.*

(b) *If $\{U_\alpha : \alpha \in \nabla\}$ is an open cover of X and $f_\alpha = f|_{U_\alpha}$ is almost F -continuous for each $\alpha \in \nabla$, then f is almost F -continuous.*

Proof. (a) Let $V \in RO(Y)$ having an F -closed complement. Then we have from Theorem 1 (b), $f^{-1}(V)$ is open in X because $(f|_A)^{-1}(V) = f^{-1}(V) \cap A$ which is open in A .

(b) Let $V \in RO(Y)$ having an F -closed complement. Then since each $f_\alpha = f|_{U_\alpha}$ is almost F -continuous, $f_\alpha^{-1}(V) = (f|_{U_\alpha})^{-1}(V) = f^{-1}(V) \cap U_\alpha$ is open in U_α from Theorem 1, and it is also open in X since U_α is open in X . Thus $\bigcup_{\alpha} (f^{-1}(V) \cap U_\alpha) = f^{-1}(V) \cap \bigcup_{\alpha} U_\alpha$ is open in X . Since $\{U_\alpha : \alpha \in \nabla\}$ is an open cover of X , $f^{-1}(V) \cap \bigcup_{\alpha} U_\alpha = f^{-1}(V) \cap X = f^{-1}(V)$ is open in X . ■

The composition of almost F -continuous functions need not be almost F -continuous. For example, let (X, τ) , (Y, σ) and (Z, θ) be the real line endowed with the cofinite, the discrete and the usual topologies, respectively, and $f_1 : (X, \tau) \rightarrow (Y, \sigma)$ and $f_2 : (Y, \sigma) \rightarrow (Z, \theta)$ be the identities. Then f_1 and f_2 are F -continuous and hence almost F -continuous but $f_2 \circ f_1 : (X, \tau) \rightarrow (Z, \theta)$ is not almost F -continuous, for let $V = \mathbb{R} - [0, 1]$, then V is regular open having an F -closed complement. Since $(f_2 \circ f_1)^{-1}(V) = \mathbb{R} - [0, 1]$ is not open in X . Hence $(f_2 \circ f_1)^{-1}$ is not almost F -continuous from Theorem 1 (b).

Theorem 3. *The following are true where $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are functions.*

(a) *If f is continuous and g is almost F -continuous, then $g \circ f$ is almost F -continuous.*

(b) *If f is surjective (open or closed) and $g \circ f$ is almost F -continuous, then g is almost F -continuous.*

(c) If f is a quotient function, then g is almost F -continuous if and only if $g \circ f$ is almost F -continuous.

Proof. (a) and (b) are obvious.

(c) We prove only the sufficiency, the proof of the necessity being obvious from (a). Let $V \in RO(Z)$ having an F -closed complement. But $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ which is open in X , since f is quotient and $g^{-1}(V)$ is open in Y . Thus g is almost F -continuous. ■

Lemma 1 ([3]). *In an extremally disconnected (shortly e.d.) space X , the following are equivalent for $A \subset X$.*

- (a) A is F -closed relative to X .
- (b) A is S -closed relative to X .
- (c) A is N -closed (or Nearly compact) relative to X .
- (d) A is QHC relative to X .

Theorem 4. *Let $f : X \rightarrow Y$ be a function and Y be extremally disconnected. Then almost F -continuity, almost H -continuity, almost S -continuity and almost N -continuity are equivalent.*

Proof. It follows immediately from Lemma 1. ■

Lemma 2 ([14]). *Let $f : X \rightarrow Y$ be a function, then C -continuity, N -continuity and continuity are equivalent if Y is compact.*

Theorem 5. *Let $f : X \rightarrow Y$ be a function and Y be a compact e.d. space. Then the following are equivalent: almost F -continuity, almost S -continuity, almost C -continuity, almost N -continuity, almost H -continuity and continuity.*

Proof. It follows directly by using Lemma 1 and 2. ■

Lemma 3 ([3]). *The following are equivalent for any subset A of a regular space X .*

- (a) A is F (or QHC)-closed relative to X .
- (b) A is N -closed relative to X .
- (c) A is compact relative to X .

Theorem 6. *Let Y be a regular space. Then the following are equivalent for a function $f : X \rightarrow Y$.*

- (a) f is almost F -continuous.
- (b) f is almost N -continuous.
- (c) f is almost H -continuous.
- (d) f is almost C -continuous.

Proof. It is obvious from Lemma 3. ■

4. Co-F-closed and almost co-F-closed topologies

Let (Y, τ) be a topological space. Let $F'(\tau) = \{U \in \tau : Y - U \text{ is } F\text{-closed relative to } \tau\}$. Since the union of two F-closed sets is F-closed, $F'(\tau)$ is a base for a topology τ^* on Y , called the co-F-closed topology. Let us replace the condition $U \in \tau$ in the definition of $F'(\tau)$ by $U \in RO(Y, \tau)$, the collection of regular open subsets (Y, τ) , to get $F''(\tau)$. Since the intersection of two regular open sets is regular open, $F''(\tau)$ will be a base for another topology τ^{**} on Y , called the almost co-F-closed topology. Note that $\tau^{**} \subset \tau^* \subset \tau$.

The basic relationship between the topology τ^{**} and the concept of almost F -continuity is given by the following result. The topology on X is unchanged, so it is not specified. The proof is immediate from the definitions.

Theorem 7. *For a function $f : X \rightarrow (Y, \tau)$ the following are equivalent:*

- (a) f is F -continuous.
- (b) $f : X \rightarrow (Y, \tau^*)$ is continuous.
- (c) The inverse image of every open set with F -closed complement is open.
- (d) The inverse image of every F -closed and closed set is closed.

Proof. (a) \Leftrightarrow (b): Let $V \in \tau^*$, then $V \in \tau$ and $Y - V$ is F -closed relative to τ . Since f is F -continuous, then for all $x \in f^{-1}(V)$ there exists an open set U such that $x \in U \subset f^{-1}(V)$. Thus $f^{-1}(V) \in \tau$ and f is continuous. The converse is clear, since every continuous function is F -continuous.

(a) \Rightarrow (c), (c) \Rightarrow (d) and (d) \Rightarrow (a). Follows from Theorem 3.1 [5]. \blacksquare

From Theorem 7, we state the following result.

Corollary 1. *A function $f : X \rightarrow (Y, \tau)$ is almost F -continuous if and only if $f : X \rightarrow (Y, \tau^{**})$ is continuous.*

$F''(\tau)$ is a base for a topology denoted by τ^{**} and called an almost Co-F-closed topology.

For any subset A of a space (Y, τ) , we denote the closure of A with respect to τ^* (resp. τ^{**}) by $Cl_{\tau^*}(A)$ (resp. $Cl_{\tau^{**}}(A)$). Similarly for the interior of A .

Lemma 4 ([16]). *For any subset A of a space (Y, τ) we have $A \in \tau^*$ if and only if $A = U - V$, such that $U \in \tau$ and $V \in NO(Y, \tau)$, where $NO(Y, \tau)$ denotes the family of all nowhere dense subsets of (Y, τ) .*

Theorem 8. *For any space (Y, τ) the following hold:*

- (a) $NO(Y, \tau^*) \subset NO(Y, \tau)$.
- (b) $FO(Y, \tau^*) \subset FO(Y, \tau)$.

Proof. (a) Let $A \in NO(Y, \tau^*)$, then $Cl_{\tau^*}(A)$ contains no nonempty set from τ^* . But $Cl_{\tau}(A) \subset Cl_{\tau^*}(A)$, thus $Cl_{\tau}(A)$ contains no nonempty set from τ . Therefore $A \in NO(Y, \tau)$.

(b) Follows from the inclusion $\tau^* \subset \tau$. ■

Theorem 9. *For any topological space (Y, τ) , the space (Y, τ^*) is F -closed.*

Proof. Follows from Theorem 8 (b). ■

Lemma 5. *Let A be a feebly closed subset of an F -closed space (X, τ) , then A is F -closed relative to X .*

Proof. Let $\{V_\alpha : \alpha \in \nabla\}$ be a feebly open cover of A . Then $\{V_\alpha : \alpha \in \nabla\} \cup (X - A)$ is a feebly open cover of X . Since X is F -closed, then there exists a finite subset ∇_0 of ∇ such that $X = \bigcup\{Cl(V_\alpha) : \alpha \in \nabla_0\} \cup (X - A)$. Therefore, $A = \bigcup\{A \cap Cl(V_\alpha) : \alpha \in \nabla_0\} \subset \bigcup\{Cl(V_\alpha) : \alpha \in \nabla_0\}$ and A is F -closed relative to X . ■

Theorem 10. *Let (Y, τ) be a space. Then (Y, τ^*) is feebly Hausdorff if and only if (Y, τ) is F -closed and feebly Hausdorff.*

Proof. The property of being T_1 -space is expansive, that is if (Y, τ) is T_1 and $\tau \subset \tau^*$ then (Y, τ^*) is T_1 but it is not generally contractive. ■

The following result proves the contractivity of T_1 property from (Y, τ) to (Y, τ^*) .

Theorem 11. *If (Y, τ) is T_1 , then (Y, τ^*) is T_1 .*

Proof. Let x be any point of Y , then $\{x\}$ is closed in τ and F -closed in (Y, τ) . Thus $Y - \{x\}$ is open in τ and $\{x\}$ is F -closed. Hence $Y - \{x\}$ is open in τ^* . Therefore (Y, τ^*) is T_1 . ■

5. Conclusion

In this paper we introduce a new class of functions between topological spaces, larger than the class of F -continuous functions of Chae et al.,[5]. We call such functions almost F -continuous. Section relates this class of functions to other classes of functions. In Section we show that if the codomain of an almost F -continuous function f is retopologized in an appropriate way then f is simply a continuous function. The result (Corollary 1) puts the notion of almost F -continuity in its natural context, and enables us to obtain alternative characterizations and standard properties of this class of functions.

References

- [1] ABO KHADRA A.A., NASEF A.A., Co- F -closed topologies and F -closed spaces, *Proc. Math. Phys. Egypt*, 79(2003), 79-90.
- [2] CARNAHAN D., Locally nearly compact spaces, *Bull. Uni Math. Ital.*, 4(6)(1972), 146-153.

- [3] CHAE G.I., LEE D.W., LEE H.W., Feebly irresolute functions, *Sung Shin W. Univ. Report*, 21(1985), 273-280.
- [4] CHAE G.I., LEE D.W., F-closed spaces, *Kyungpook Math. J.*, 27(2)(1987), 127-134.
- [5] CHAE G.I., NASEF A.A., PARK Y.S., F-continuous functions, *East Asian Math Comm.*, 1(1998), 59-67.
- [6] DIMAIO G., S-closed spaces, S-sets and S-continuous functions, *Atti. Accad. Sci. Torino*, 118(1984), 125-134.
- [7] GAULD D.B., Topologies related to notions of near continuity, *Kyungpook Math J.*, 21(1981), 195-204.
- [8] HWANG S.G., Almost C-continuous functions, *J. Korean Math. Soc.*, 14(1978), 229-234.
- [9] JANKOVIC D.S., REILLY I.L., On semi separation properties, *Indian J. Pure Appl. Math.*, 16(9)(1985), 957-964.
- [10] JIANG S., REILLY I.L., On co S-closed and almost co S-closed topologies, *Far East J. Math. Sci.*, 4(2)(1996), 263-268.
- [11] KONSTADILAKI-SAVVOPOULOU CH., REILLY I.L., On almost-N-continuous functions, *J. Austral. Math. Soc. (Series A)*, 58(1995), 1-13.
- [12] LEVINE N., Semi-open sets and semi-continuity in topological spaces, *Amer. Math. Monthly*, 70(1963), 36-41.
- [13] MAHESHWARI S.N., TAPI U.D., Notes on some applications of feebly open sets, *M.B. Jr. Univ. Saugar*, (1978).
- [14] MALGHAN S.R., HANCHINAMANI V.V., N-continuous functions, *Ann. Soc. Sci. Bruxelles*, 98(1984), 69-79.
- [15] MALGHAN S.R., HANCHINAMANI V.V., Almost N-continuous functions, *J. Karnatak Univ. Science*, 31(1986), 93-101.
- [16] NJÁSTAD O., On some classes of nearly open sets, *Pacific J. Math.*, 15(3) (1965), 961-970.
- [17] NOIRI T., Remarks on locally nearly compact spaces, *Boll. Un. Mat. Ital.*, 4(10)(1974), 36-43.
- [18] PORTER J., THOMAS J., On H-closed and minimal Hausdorff spaces, *Trans. Amer. Math. Soc.*, 138(1969), 159-170.
- [19] THOMPSON T., S-closed spaces, *Proc. Amer. Math. Soc.*, 60(1976), 335-338.
- [20] WILLARD S., *General Topology*, Addison-Wesley Publ. Co., Ontario, 1970.

A. A. EL-ATIK

DEPARTMENT OF MATHEMATICS

FACULTY OF SCIENCE

TANTA UNIVERSITY, TANTA, EGYPT

e-mail: aelatik55@yahoo.com

ARAFA A. NASEF

DEPARTMENT OF PHYSICS AND ENGINEERING MATHEMATICS

FACULTY OF ENGINEERING

KAFR-EL-SHEIKH UNIVERSITY, EGYPT

e-mail: nasefa50@yahoo.com

Received on 10.12.2011 and, in revised form, on 25.07.2012.