F A S C I C U L I M A T H E M A T I C I

Nr 50 2013

T. Noiri, J. Bhuvaneswari and N. Rajesh

ON TOTALLY πg-CONTINUOUS FUNCTIONS

Abstract. In this paper, πg-closed sets and πg-open sets are used to define and investigate a new class of functions called, totally πg-continuous functions. Relationships between this new class and other classes of related functions are established.

Key words: topological space, πg-clopen set, totally πg-continuous function.

AMS Mathematics Subject Classification: 54C10.

1. Introduction and preliminaries

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real Analysis concerns the variously modified forms of continuity, separation axioms etc. by utilizing generalized closed sets. In 1970, Levine [7] initiated the study of so-called g-closed sets, that is, a subset A of a topological space (X, τ) is g-closed if the closure of A is included in every open superset of A and defined a $T_{1/2}$ space to be one in which the closed sets and the g-closed sets coincide. Zaitsev [10] defined the concept of π-closed sets and a class of topological spaces called quasi-normal spaces. Recently, Dontchev and Noiri [2] defined the notion of πg-closed sets and used this notion to obtain a characterization and some preservation theorems for quasi-normal spaces. In this paper, we present a new generalization of total continuity called total πg-continuity. We define this class of functions by the requirement that the inverse image of each open set in the codomain is πg-clopen (that is, πg-open and πg-closed) in the domain. The notion of total πg-continuity is a weaker form of total continuity [5]. Also we investigate the fundamental properties of totally πg-continuous functions.

In the present paper, (X, τ) and (Y, σ) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X, τ), $Cl(A)$, $Int(A)$ and A^c denote the closure of A, the interior of A and the complement of A in X, respectively. A subset A of a space (X, τ) is said to be regular open [8] (resp. semiopen [6]) if

$A = Int(Cl(A))$ (resp. $A \subset Cl(Int(A))$). A set is said to be δ-open [9] if it is the union of regular open sets. The complement of a regular open (resp. δ-open) set is said to be regular closed (resp. δ-closed). The finite union of regular open sets is said to be π-open [10]. The complement of a π-open set is said to be π-closed. A subset A of a topological space (X, τ) is said to be πg-closed [2] if $Cl(A) \subset U$ whenever $A \subset U$ and U is π-open. The complement of a πg-closed set is said to be πg-open. The family of all πg-open (resp. πg-closed, πg-clopen) sets of (X, τ) is denoted by $\pi GO(X)$ (resp. $\pi GC(X), \pi GCO(X)$). Let (X, τ) be a topological space and $S \subset X$. Then the set $\bigcup\{A : A \subset S \text{ and } A \in \pi GO(X)\}$ is called the πg-interior of S and is denoted by $\pi g\text{-}Int(S)$ [4]. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be πg-continuous [3] if $f^{-1}(V)$ is πg-open in X for each open set V of Y. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be totally continuous [5] if $f^{-1}(V)$ is a clopen set in X for each open set V of Y.

2. Totally πg-continuous functions

Definition 1. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be totally πg-continuous if $f^{-1}(V)$ is πg-clopen in X for every open set V of Y.*

Remark 1. It is clear that every totally πg-continuous function is πg-continuous. But the converse is false.

Example 1. The identity function on the real line with the usual topology is continuous and hence πg-continuous. The inverse image of (0,1) is not πg-closed and the function is not totally πg-continuous.

Remark 2. It is clear that every totally continuous function is totally πg-continuous. But the converse need not be true as can be seen from the following example.

Example 2. Let $X = \{a, b, c\}$, $Y = \{p, q\}$, $\tau = \{\varnothing, \{a, b\}, X\}$ and $\sigma = \{\varnothing, \{p\}, Y\}$. Define a function $f : (X, \tau) \to (Y, \sigma)$ such that $f(a) = p$, $f(b) = f(c) = q$. Then, clearly f is totally πg-continuous, but not totally continuous.

Definition 2. *A topological space (X, τ) is said to be πg-connected [3] if it cannot be written as the union of two nonempty disjoint πg-open sets.*

Theorem 1. *If f is a totally πg-continuous function from a πg-connected space X onto any space Y, then Y is an indiscrete space.*

Proof. If possible, suppose that Y is not indiscrete. Let A be a proper non-empty open subset of Y. Then $f^{-1}(A)$ is a proper non-empty πg-clopen subset of (X, τ), which is contrary to the fact that X is πg-connected. ∎

Theorem 2. *A topological space (X,τ) is πg-connected if and only if every totally πg-continuous function from a space (X,τ) into any T_0-space (Y,σ) is constant.*

Proof. Suppose that X is πg-connected. Then, every totally πg-continuous function from (X,τ) to a T_0-space (Y,σ) is constant. Conversely, suppose that (X,τ) is not πg-connected. Then, there exists a proper non-empty πg-clopen subset A of X. Let $Y=\{a,b\}$ and $\tau=\{\varnothing,\{a\},\{b\},Y\}$ be a topology for Y. Let $f:(X,\tau)\to(Y,\sigma)$ be a function such that $f(A)=\{a\}$ and $f(Y\backslash A)=\{b\}$. Then f is nonconstant and totally πg-continuous such that Y is T_0, which is a contradiction. Hence X must be πg-connected. ■

Theorem 3. *Let f: $(X,\tau)\to(Y,\sigma)$ be a totally πg-continuous function and Y is a T_1-space. If A is a nonempty πg-connected subset of X, then $f(A)$ is a single point.*

Proof. The proof is clear. ■

Definition 3. *Let (X,τ) be a topological space. We define an equivalence relation on X by setting $x\sim y$ if there is a πg-connected subset of X containing both x and y. The equivalence classes are called the πg-separation of X or πg-component of X.*

Theorem 4. *Let f: $(X,\tau)\to(Y,\sigma)$ be a totally πg-continuous function from a topological space (X,τ) into a T_1-space Y. Then f is constant on each πg-component of X.*

Proof. The proof follows from Definition 3 and Theorem 3. ■

Definition 4 ([3]). *A πg-frontier of a subset A of X is $\pi g\text{-}fr(A)=\pi g\text{-}Cl(A)\cap\pi g\text{-}Cl(X\backslash A)$.*

Theorem 5. *The set of all points $x\in X$ in which a function $f:(X,\tau)\to(Y,\sigma)$ is not totally πg-continuous is the union of πg-frontiers of the inverse images of open sets containing $f(x)$.*

Proof. Suppose that f is not totally πg-continuous at $x\in X$. Then there exists a open set V of Y containing $f(x)$ such that $f(U)$ is not contained in V for each $U\in\pi GO(X)$ containing x and hence $x\in\pi g\text{-}Cl(X\backslash f^{-1}(V))$. On the other hand, $x\in f^{-1}(V)\subset\pi g\text{-}Cl(f^{-1}(V))$ and hence $x\in\pi g\text{-}fr(f^{-1}(V))$. Conversely, suppose that f is totally πg-continuous at $x\in X$ and let V be an open set of Y containing $f(x)$. Then there exists $U\in\pi GO(X)$ containign x such that $U\subset f^{-1}(V)$. Hence $x\in\pi g\text{-}Int(f^{-1}(V))$. Therefore, $x\notin\pi g\text{-}fr(f^{-1}(V))$ for each open set V of Y containing $f(x)$. ■

Theorem 6. *Let $\{X_\lambda : \lambda \in \Lambda\}$ be any family of topological spaces. If $f : X \to \Pi X_\lambda$ is a totally πg-continuous function, then $P_{r\lambda} \circ f : X \to X_\lambda$ is totally πg-continuous for each $\lambda \in \Lambda$, where $P_{r\lambda}$ is the projection of ΠX_λ onto X_λ.*

Proof. We shall consider a fixed $\lambda \in \Lambda$. Suppose U_λ is an arbitrary open set in X_λ. Then $P_{r\lambda}^{-1}(U_\lambda)$ is open in ΠX_λ. Since f is totally πg-continuous, we have $f^{-1}(P_{r\lambda}^{-1}(U_\lambda)) = (P_{r\lambda} \circ f)^{-1}(U_\lambda)$ is πg-clopen in X. Therefore $P_{r\lambda}$ is totally πg-continuous. ■

Definition 5. *A filter base Λ in a space (X, τ) is said to be πg-co-convergent to a point x in X if for any $U \in \pi GCO(X)$ containing x, there exists $B \in \Lambda$ such that $B \subset U$.*

Theorem 7. *If a function $f : (X, \tau) \to (Y, \sigma)$ is totally πg-continuous, then for each point $x \in X$ and each filter base Λ in X πg-co-converging to x, the filter base $f(\Lambda)$ is convergent to $f(x)$.*

Proof. Let $x \in X$ and Λ be any filter base in X πg-co-converging to x. Since f is totally πg-continuous, then for any open set V of Y containing $f(x)$, there exists $U \in \pi GCO(X)$ containing x such that $f(U) \subset V$. Since Λ is πg-co-converging to x, there exists $B \in \Lambda$ such that $B \subset U$. This means that $f(B) \subset V$ and therefore the filter base $f(\Lambda)$ is convergent to $f(x)$. ■

3. Covering properties

Definition 6. *A space (X, τ) is said to be πg-co-compact if every πg-clopen cover of X has a finite subcover.*

A subset A of a space (X, τ) is said to be πg-co-compact relative to X if every cover of A by πg-clopen sets of X has a finite subcover.

A subset A of a space (X, τ) is said to be πg-compact [4] if the subspace A is πg-compact.

Theorem 8. *If a function $f : (X, \tau) \to (Y, \sigma)$ is totally πg-continuous and A is πg-co-compact relative to X, then $f(A)$ is compact in Y.*

Proof. Let $\{H_\alpha : \alpha \in I\}$ be any cover of $f(A)$ by open sets of Y. For each $x \in A$, there exists $\alpha_x \in I$ such that $f(x) \in H_{\alpha_x}$ and there exists $U_x \in \pi GCO(X)$ containing x such that $f(U_x) \subset H_{\alpha_x}$. Since the family $\{U_x : x \in A\}$ is a cover of A by πg-clopen sets of X, there exists a finite subset A_0 of A such that $A \subset \bigcup\{U_x : x \in A_0\}$. Therefore, we obtain $f(A) \subset \bigcup\{f(U_x) : x \in A_0\} \subset \bigcup\{H_{\alpha_x} : x \in A_0\}$ and hence $f(A)$ is compact. ■

Corollary 1. *If $f : (X, \tau) \to (Y, \sigma)$ is a totally πg-continuous surjective function and X is πg-co-compact, then Y is compact.*

Definition 7. *A space (X, τ) is said to be:*
(i) countably πg-co-compact if every countable πg-clopen cover of X has a finite subcover;
(ii) πg-co-Lindelöf if every πg-clopen cover of X has a countable subcover.

Theorem 9. *Let $f : (X, \tau) \to (Y, \sigma)$ be a totally πg-continuous surjective function. Then the following statements hold:*
(i) If X is πg-co-Lindelöf, then Y is Lindelöf;
(ii) If X is countably πg-co-compact, then Y is countably compact.

Proof. (i): Let $\{V_\alpha : \alpha \in I\}$ be an open cover of Y. Since f is totally πg-continuous, then $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is a πg-clopen cover of X. Since X is πg-co-Lindelöf, there exists a countable subset I_0 of I such that $X = \bigcup\{f^{-1}(V_\alpha) : \alpha \in I_0\}$. Thus, $Y = \bigcup\{V_\alpha : \alpha \in I_0\}$ and hence Y is Lindelöf.
(ii): Similar to (i). ■

Definition 8. *A space (X, τ) is said to be πg-co-T_2 if for each pair of distinct points x and y in X, there exist disjoint πg-clopen sets U and V in X such that $x \in U$ and $y \in V$.*

Theorem 10. *If $f : (X, \tau) \to (Y, \sigma)$ is a totally πg-continuous injective function and (Y, σ) is a T_0 space, then (X, τ) is a πg-co-T_2 space.*

Proof. Suppose that Y is a T_0 space. For any distict points x and y in X, there exists an open set V or W of Y such that $f(x) \in V$, $f(y) \notin V$ or $f(x) \notin W$ and $f(y) \in W$. We prove the case that there exists an open set V such that $f(x) \in V$ and $f(y) \notin V$. Since f is totally πg-continuous, $f^{-1}(V)$ is a πg-clopen set of (X, τ) such that $x \in f^{-1}(V)$ and $y \notin f^{-1}(V)$. Therefore, X-$f^{-1}(V)$ is πg-clopen and $y \in X$-$f^{-1}(V)$. This shows that (X, τ) is a πg-co-T_2 space. ■

Definition 9. *A space (X, τ) is said to be πg-co-regular if for each πg-clopen set F and each point $x \notin F$, there exist disjoint open sets U and V such that $F \subset U$ and $x \in V$.*

Definition 10. *A space (X, τ) is said to be πg-co-normal if for any pair of disjoint πg-clopen subsets F_1 and F_2 of X, there exist disjoint open sets U and V such that $F_1 \subset U$ and $F_2 \subset V$.*

Theorem 11. *If f is a totally πg-continuous injective open function from a πg-co-regular space X onto a space Y, then Y is a regular space.*

Proof. Let F be a closed set in Y and $y \notin F$. Take $y = f(x)$. Since f is totally πg-continuous, $f^{-1}(F)$ is a πg-clopen set. Take $G = f^{-1}(F)$.

We have $x \notin G$. Since X is πg-co-regular, there exist disjoint open sets U and V such that $G \subset U$ and $x \in V$. We obtain that $F = f(G) \subset f(U)$ and $y = f(x) \in f(V)$ such that $f(U)$ and $f(V)$ are disjoint open sets. This shows that (Y, σ) is regular. ■

Theorem 12. *If f is a totally πg-continuous injective open function from a πg-co-normal space (X, τ) onto a space (Y, σ), then (Y, σ) is normal.*

Proof. Let F_1 and F_2 be disjoint closed subsets of Y. Since f is totally πg-continuous, $f^{-1}(F_1)$ and $f^{-1}(F_2)$ are πg-clopen sets. Take $U = f^{-1}(F_1)$ and $V = f^{-1}(F_2)$. We have $U \cap V = \varnothing$. Since X is πg-co-normal, there exist disjoint open sets A and B such that $U \subset A$ and $V \subset B$. We obtain that $F_1 = f(U) \subset f(A)$ and $F_2 = f(V) \subset f(B)$ such that $f(A)$ and $f(B)$ are disjoint open sets. Thus, (Y, σ) is normal. ■

Recall that for a function $f : (X, \tau) \to (Y, \sigma)$, the subset $\{(x, f(x)) : x \in X\} \subset X \times Y$ is called the graph of f and is denoted by $G(f)$.

Definition 11. *A graph $G(f)$ of a function $f : (X, \tau) \to (Y, \sigma)$ is said to be strongly πg-co-closed if for each $(x, y) \in (X \times Y)\backslash G(f)$, there exists $U \in \pi GCO(X)$ containing x and an open set V of Y containing y such that $(U \times V) \cap G(f) = \varnothing$.*

Lemma 1. *A graph $G(f)$ of a function $f : (X, \tau) \to (Y, \sigma)$ is strongly πg-co-closed in $X \times Y$ if and only if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist $U \in \pi GCO(X)$ containing x and an open set V of Y containing y such that $f(U) \cap V = \varnothing$.*

Proof. It is an immediate consequence of Definition 11. ■

Theorem 13. *Let $f : (X, \tau) \to (Y, \sigma)$ have a strongly πg-co-closed graph $G(f)$. If f is injective, then X is πg-co-T_2.*

Proof. Let x and y be any two distinct points of X. Then, we have $(x, f(y)) \in (X \times Y)\backslash G(f)$. By Lemma 1, there exist a πg-clopen set U of X and an open set V of Y such that $(x, f(y)) \in U \times V$ and $f(U) \cap V = \varnothing$. Hence $U \cap f^{-1}(V) = \varnothing$ and $y \in X\text{-}U \in \pi GCO(X)$. This implies that (X, τ) is πg-co-T_2. ■

Theorem 14. *If $f : (X, \tau) \to (Y, \sigma)$ is a totally πg-continuous function and (Y, σ) is T_2, then $G(f)$ is strongly πg-co-closed in the product space $X \times Y$.*

Proof. Let $(x, y) \in (X \times Y)\backslash G(f)$. Then $y \neq f(x)$ and there exist open sets V_1 and V_2 of Y such that $f(x) \in V_1$, $y \in V_2$, and $V_1 \cap V_2 = \varnothing$. From hypothesis there exists $U \in \pi GCO(X, x)$ such that $f(U) \subset V_1$. Therefore, we obtain $f(U) \cap V_2 = \varnothing$. ■

Definition 12. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be:*
(a) *totally πg-irresolute if the preimage of a πg-clopen subset of Y is a πg-clopen subset of X.*
(b) *totally pre-πg-clopen if the image of every πg-clopen subset of X is πg-clopen.*

Theorem 15. *Let $f : (X, \tau) \to (Y, \sigma)$ be surjective totally πg-irresolute and totally pre-πg-clopen and g: $(Y, \sigma) \to (Z, \eta)$ be any function. Then $g \circ f$: $(X, \tau) \to (Z, \eta)$ is totally πg-continuous if and only if g is totally πg-continuous.*

Proof. The *'if'* part is obvious. To prove the *'only if'* part, let $g \circ f$: $(X, \tau) \to (Z, \eta)$ be totally πg-continuous and let V be an open subset of Z. Then $(g \circ f)^{-1}$ (V) is a πg-clopen subset of X, that is $f^{-1}(g^{-1}(V))$ is πg-clopen. Since f is totally pre-πg-clopen, $f(f^{-1}(g^{-1}(V)))$ is a πg-clopen subset of Y. So $g^{-1}(V)$ is πg-clopen in Y. Hence g is totally πg-continuous. ■

Theorem 16. *Let $f : (X, \tau) \to (Y, \sigma)$ have a strongly πg-co-closed graph $G(f)$. If f is a surjective totally pre πg-clopen function, then (Y, σ) is a πg-co-T_2 space.*

Proof. Let y_1 and y_2 be any distinct points of Y. Since f is surjective, $f(x) = y_1$ for some $x \in X$ and $(x, y_2) \in (X \times Y) \backslash G(f)$. By Lemma 1, there exist a πg-clopen set U of X and an open set V of Y such that $(x, y_2) \in U \times V$ and $f(U) \cap V = \varnothing$. Since f is totally pre πg-clopen, then $f(U)$ is πg-clopen such that $f(x) = y_1 \in f(U)$. This implies that (Y, σ) is πg-co-T_2. ■

References

[1] Bourbaki N., *General Topology,* Part I, Reading, MA: Addition Wesley, 1996.
[2] Dontchev J., Noiri T., Quasi-normal spaces and πg-closed sets, *Acta Math. Hungar.,* 89(3)(2000), 211-219.
[3] Ekici E., On contra πg-continuous functions, *Chaos, Solitons and Fractals,* 35(2008), 71-81.
[4] Ekici E., Baker C.W., On πg-closed sets and continuity, *Kochi J. Math.,* 2(2007), 35-42.
[5] Jain R.C., *The role of regularly open sets in general topology,* Ph.D. Thesis, Meerut University, Institute of Advanced Studies, Meerut, India, 1980.
[6] Levine N., Semi-open sets and semi-continuity in topological spaces, *Amer. Math. Monthly,* 70(1963), 36-41.
[7] Levine N., Generalized closed sets in topology, *Rend. Circ. Mat. Palermo* (2)*,* 19(1970), 89-96.
[8] Stone M.H., Applications of the theory of boolean rings to general topology, *Trans. Amer. Math. Soc.,* 41(1937), 375-481.

[9] Veličko N.V., *H*-closed topological spaces, *Amer. Math. Soc. Transl.*(2), 78(1968), 103-118.

[10] Zaitsav V., On certain classes of topological spaces and their bicompactification, *Dokl Akad Nauk SSSR,* 178(1968), 778-779.

Takashi Noiri
2949-1 Shiokita-cho, Hinagu, Yatsushiro-shi
Kumamoto-ken, 869-5142, Japan
e-mail: t.noiri@nifty.com

J. Bhuvaneswari
Department of Computer Applications
Rajalakshmi Engineering College
Thandalam, Chennai-602 105
TamilNadu, India
e-mail: sai_jbhuvana@gmail.com

Neelamegarajan Rajesh
Department of Mathematics
Rajah Serfoji Govt. College
Thanjavur-613005, Tamilnadu, India
e-mail: nrajesh_topology@yahoo.co.in

Received on 19.08.2011 and, in revised form, on 28.03.2012.