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**ON THE QUALITATIVE STUDY OF THE NONLINEAR
DIFFERENCE EQUATION** $x_{n+1} = \frac{\alpha x_{n-\sigma}}{\beta + \gamma x_{n-\tau}^p}$

ABSTRACT. In this paper, we investigate the global behavior of the following non-linear difference equation

$$x_{n+1} = \frac{(\alpha x_{n-\sigma})}{(\beta + \gamma x_{n-\tau}^p)}, \quad n = 0, 1, 2, \dots$$

where the coefficients $\alpha, \beta, \gamma, p \in (0, \infty)$ and $\sigma, \tau \in N$ and the initial conditions $x_{-\omega}, \dots, x_{-1}, x_0$ are arbitrary positive real numbers, where $\omega = \max\{\sigma, \tau\}$.

KEY WORDS: difference equations, prime period two solution, boundedness character, semi-cycle analysis, locally asymptotically stable, global stability.

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1. Introduction

The qualitative study of difference equations is a fertile research area and increasingly attracts many mathematicians. This topic draws its importance from the fact that many real life phenomena are modeled using difference equations. Examples from economy, biology, etc. can be found in [2, 5 – 7]. It is known that nonlinear difference equations are capable of producing a complicated behavior regardless its order. This can be easily seen from the family $x_{n+1} = g_\mu(x_n)$, $\mu > 0$, $n \geq 0$. This behavior is ranging according to the value of μ , from the existence of a bounded number of periodic solutions to chaos.

There has been a great interest in studying the global attractivity, the boundedness character and the periodicity nature of nonlinear difference equations. For example, in the articles [1, 6 – 8] closely related global convergence results were obtained which can be applied to nonlinear difference equations in proving that every solution of these equations converges to a period two solution. For other closely related results, (see [3 – 5, 9 – 24])

and the references cited therein. The study of these equations is challenging and rewarding and is still in its infancy. We believe that the nonlinear rational difference equations are of paramount importance in their own right. Furthermore the results about such equations offer prototypes for the development of the basic theory of the global behavior of nonlinear difference equations.

The objective of this article is to investigate some qualitative behavior of the solutions of the nonlinear difference equation

$$(1) \quad x_{n+1} = \frac{(\alpha x_{n-\sigma})}{(\beta + \gamma x_{n-\tau}^p)}, \quad n = 0, 1, 2, \dots .$$

where the coefficients $\alpha, \beta, \gamma, p \in (0, \infty)$ and $\sigma, \tau \in N$ and the initial conditions $x_{-\omega}, \dots, x_{-1}, x_0$ are arbitrary positive real numbers, where $\omega = \max\{\sigma, \tau\}$. Note that the difference equation (1) has been discussed in [7] in the special case when $\sigma = 1$ and $\tau = 2$.

Definition 1. A difference equation of order $(k + 1)$ is of the form

$$(2) \quad x_{n+1} = F(x_n, x_{n-1}, \dots, x_{n-k}), \quad n = 0, 1, 2, \dots .$$

where F is a continuous function which maps some set J^{k+1} into J and J is a set of real numbers. An equilibrium point \tilde{x} of this equation is a point that satisfies the condition $\tilde{x} = F(\tilde{x}, \tilde{x}, \dots, \tilde{x})$. That is, the constant sequence $\{x_n\}_{n=-k}^{\infty}$ with $x_n = \tilde{x}$ for all $n \geq -k$ is a solution of that equation.

Definition 2. We say that a sequence $\{x_n\}_{n=-k}^{\infty}$ is bounded and persisting if there exist positive constants m and M such that

$$m \leq x_n \leq M \quad \text{for all } n \geq -k .$$

Definition 3. A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period r if $x_{n+r} = x_n$ for all $n \geq -k$. A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with prime period r if r is the smallest positive integer having this property.

Definition 4. A positive semi-cycle of $\{x_n\}_{n=-k}^{\infty}$ consists of "a string" of terms $\{x_l, x_{l+1}, \dots, x_m\}$ all greater than or equal to \tilde{x} , with $l \geq -k$ and $m \leq \infty$ such that

$$\text{either } l = -k \quad \text{or} \quad l > -k \quad \text{and} \quad x_{l-1} < \tilde{x},$$

and

$$\text{either } m = \infty \quad \text{or} \quad m < \infty \quad \text{and} \quad x_{m+1} < \tilde{x} .$$

A negative semi-cycle of $\{x_n\}_{n=-k}^{\infty}$ consists of "a string" of terms $\{x_l, x_{l+1}, \dots, x_m\}$ all less than \tilde{x} , with $l \geq -k$ and $m \leq \infty$ such that

$$\text{either } l = -k \quad \text{or} \quad l > -k \quad \text{and} \quad x_{l-1} \geq \tilde{x},$$

and

$$\text{either } m = \infty \quad \text{or} \quad m < \infty \quad \text{and} \quad x_{m+1} \geq \tilde{x}.$$

The linearized equation of (2) about the equilibrium point \tilde{x} is the linear difference equation

$$(3) \quad y_{n+1} = \sum_{i=0}^k \frac{\partial F(\tilde{x}, \tilde{x}, \dots, \tilde{x})}{\partial x_{n-i}} y_{n-i}, \quad n = 0, 1, 2, \dots$$

and its characteristic equation is defined by

$$(4) \quad \lambda^{k+1} = \sum_{i=0}^k \frac{\partial F(\tilde{x}, \tilde{x}, \dots, \tilde{x})}{\partial x_{n-i}} \lambda^{n-i}, \quad n = 0, 1, 2, \dots$$

Theorem 1 ([7]). (i) If all roots of the characteristic equation (4) of the linearized equation (3) have absolute value less than one, then the equilibrium point \tilde{x} is locally asymptotically stable.

(ii) If at least one root of (4) has absolute value greater than one, then the equilibrium point \tilde{x} is unstable.

(iii) If (4) has roots both inside and outside the unit disk, then the equilibrium point \tilde{x} is called a saddle point.

Theorem 2 ([11]). Assume that $p, q \in R$ and $k \in \{0, 1, 2, \dots\}$. Then $|p| + |q| < 1$, is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+1} - px_n + qx_{n-k} = 0, \quad n = 0, 1, 2, \dots$$

2. Change of variables

By using the change of variables $x_n = \left(\frac{\beta}{\gamma}\right)^{\frac{1}{p}} y_n$. Then (1) reduces to the difference equation

$$(5) \quad y_{n+1} = \frac{ry_{n-\sigma}}{1 + y_{n-\tau}^p}, \quad n = 0, 1, 2, \dots$$

where $r = \frac{\alpha}{\beta} > 0$ and the initial conditions $y_{-\omega}, y_{-\omega+1}, \dots, y_{-1}, y_0 \in (0, \infty)$ such that $p > 0$ where $\omega = \max\{\sigma, \tau\}$.

3. The dynamics of equation (5)

In this section, we investigate the dynamics of (5) under the assumptions that all parameters in (5) are positive and the initial conditions are non-negative.

The equilibrium point \tilde{y} of (5) is

$$\tilde{y} = \frac{r\tilde{y}}{1 + \tilde{y}_{n-\tau}^p} .$$

The difference equation (5) can be written in the form

$$(6) \quad y_{n+1} = F(y_{n-\sigma}, \dots, y_{n-\tau}) , \quad n = 0, 1, 2, \dots .$$

where

$$(7) \quad F = \frac{ry_{n-\sigma}}{1 + y_{n-\tau}^p} , \quad n = 0, 1, 2, \dots .$$

The linearized equation of (6) about \tilde{y} is

$$(8) \quad z_{n+1} = \rho_0 y_{n-\sigma} - \rho_1 y_{n-\tau} , \quad n = 0, 1, 2, \dots .$$

where

$$\frac{\partial F(\tilde{y}, \tilde{y})}{\partial y_{n-\sigma}} = \frac{r}{1 + \tilde{y}^p} = \rho_0 \quad \text{and} \quad \frac{\partial F(\tilde{y}, \tilde{y})}{\partial y_{n-\tau}} = - \frac{rp\tilde{y}^p}{(1 + \tilde{y}^p)^2} = \rho_1 .$$

Its characteristic equation is

$$(9) \quad \lambda^{k+1} - \frac{r}{1 + \tilde{y}^p} \lambda^{n-\sigma} + \frac{rp\tilde{y}^p}{(1 + \tilde{y}^p)^2} \lambda^{n-\tau} = 0 .$$

Note that $\tilde{y}_1 = 0$ is always an equilibrium point of (5). When $r > 1$, (5) also possesses the unique positive equilibrium point $\tilde{y}_2 = (r - 1)^{\frac{1}{p}}$.

Theorem 3. *The following statements are true:*

(i) *If $r < 1$, then the equilibrium point $\tilde{y}_1 = 0$ of (5) is locally asymptotically stable.*

(ii) *If $r > 1$, then the equilibrium point $\tilde{y}_1 = 0$ of (5) is a saddle point.*

(iii) *If $r > 1$, $\sigma = \tau$ and $0 < p < \frac{2r}{r-1}$ then the equilibrium point $\tilde{y}_2 = (r - 1)^{\frac{1}{p}}$ of (5) is locally asymptotically stable.*

(iv) *If $r > 1$, $\sigma = \tau$ and $p > \frac{2r}{r-1}$ then the equilibrium point $\tilde{y}_2 = (r - 1)^{\frac{1}{p}}$ of (5) is unstable.*

(v) *If $r > 1$, $\sigma = \tau$ and $p = \frac{2r}{r-1}$ then the linearized stability analysis fails.*

(vi) *If $r > 1$, $\sigma \neq \tau$ then the equilibrium point $\tilde{y}_2 = (r - 1)^{\frac{1}{p}}$ of (5) is unstable.*

Proof. The linearized equation of (5) about the equilibrium point $\tilde{y}_1 = 0$ is

$$(10) \quad z_{n+1} - ry_{n-\sigma} = 0, \quad n = 0, 1, 2, \dots .$$

and its characteristic equation is

$$(11) \quad \lambda^{n-\sigma} (\lambda^{\sigma+1} - r) = 0 .$$

According to Theorems 1 and 2 we deduce that the proofs of (i) and (ii) are completed.

The linearized equation of (5) about the equilibrium point $\tilde{y}_2 = (r-1)^{\frac{1}{p}}$ when $\sigma = \tau$ and $r > 1$ has the form

$$(12) \quad z_{n+1} + \left[\frac{p(r-1)}{r} - 1 \right] z_{n-\sigma} = 0, \quad n = 0, 1, 2, \dots .$$

and its characteristic equation is

$$(13) \quad \lambda^{n+1} + \left[\frac{p(r-1)}{r} - 1 \right] \lambda^{n-\sigma} = 0 .$$

According to Theorems 1 and 2 we deduce that if $0 < p < \frac{2r}{r-1}$ then \tilde{y}_2 is locally asymptotically stable, while if $p > \frac{2r}{r-1}$ then \tilde{y}_2 is unstable and if $p = \frac{2r}{r-1}$ then the linearized stability analysis fails. Now, the proofs of (iii)-(v) are completed. The linearized equation of (5) about the equilibrium point $\tilde{y}_2 = (r-1)^{\frac{1}{p}}$ when $\sigma \neq \tau$ and $r > 1$ has the form

$$(14) \quad z_{n+1} + a_1 z_{n-\sigma} + a_2 z_{n-\tau} = 0, \quad n = 0, 1, 2, \dots .$$

and its characteristic equation is

$$(15) \quad \lambda^{n+1} + a_1 \lambda^{n-\sigma} + a_2 \lambda^{n-\tau} = 0 ,$$

where

$$(16) \quad a_1 = -1, \quad a_2 = \frac{p(r-1)}{r} .$$

It is clear that $|a_1| + |a_2| > 1$, then according to Theorems 1 and 2 the equilibrium point \tilde{y}_2 is unstable. The proof of (vi) is completed. The proof of Theorem 3 is now completed. \blacksquare

Theorem 4. Assume that $r > 1$ and $\omega = \max\{\sigma, \tau\}$. Let $\{y_n\}_{n=-\omega}^\infty$ be a solution of (5) such that σ is odd, τ is even. Assume that the following inequalities hold:

$$(17) \quad \begin{cases} y_{-\sigma}, y_{-\sigma+2}, \dots, y_{-\tau+1}, y_{-\tau+3}, \dots < \tilde{y}_2, \\ y_{-\sigma+1}, y_{-\sigma+3}, \dots, y_{-\tau}, y_{-\tau+2}, \dots \geq \tilde{y}_2, \end{cases}$$

or

$$(18) \quad \begin{cases} y_{-\sigma}, y_{-\sigma+2}, \dots, y_{-\tau+1}, y_{-\tau+3}, \dots > \tilde{y}_2, \\ y_{-\sigma+1}, y_{-\sigma+3}, \dots, y_{-\tau}, y_{-\tau+2}, \dots \leq \tilde{y}_2. \end{cases}$$

Then the solution $\{y_n\}_{n=-\omega}^\infty$ oscillates about the equilibrium point $\tilde{y}_2 = (r - 1)^{\frac{1}{p}}$ with semi-cycles of length one.

Proof. Assume that (17) holds. (The case when (18) holds is similar and will be omitted). Then

$$y_1 = \frac{ry_{-\sigma}}{1 + y_{n-\tau}^p} \leq \frac{r\tilde{y}_2}{1 + \tilde{y}_2^p} = \tilde{y}_2,$$

$$y_2 = \frac{ry_{-\sigma+1}}{1 + y_{n-\tau+1}^p} > \frac{r\tilde{y}_2}{1 + \tilde{y}_2^p} = \tilde{y}_2,$$

$$y_3 = \frac{ry_{-\sigma+2}}{1 + y_{n-\tau+2}^p} < \frac{r\tilde{y}_2}{1 + \tilde{y}_2^p} = \tilde{y}_2,$$

and

$$y_4 = \frac{ry_{-\sigma+3}}{1 + y_{n-\tau+3}^p} > \frac{r\tilde{y}_2}{1 + \tilde{y}_2^p} = \tilde{y}_2,$$

and so on. The proof of Theorem 4 follows by induction. ■

Theorem 5. The following statements are true:

(i) Assume that $r < 1$, then the equilibrium point $\tilde{y}_1 = 0$ of (5) is globally asymptotically stable.

(ii) Assume that $r > 1$, $\sigma = \tau$ and $0 < p < \frac{2r}{r-1}$ then the equilibrium point $\tilde{y}_2 = (r - 1)^{\frac{1}{p}}$ of (5) is globally asymptotically stable.

Proof. We have proved in Theorem 3 that if $r < 1$, then the equilibrium point $\tilde{y}_1 = 0$ of (5) is locally asymptotically stable. So, to prove (i) we must prove that the equilibrium point $\tilde{y}_1 = 0$ of (5) is global attractor. To this end, we note that

$$(19) \quad 0 \leq y_{n+1} = \frac{ry_{n-\sigma}}{1 + y_{n-\tau}^p} \leq ry_{n-\sigma} < y_{n-\sigma}.$$

Consequently, we have $\lim_{n \rightarrow \infty} y_n = 0$. The proof of (i) is now completed.

Also, we have proved in Theorem 3 that if $r > 1$, $\sigma = \tau$ and $0 < p < \frac{2r}{r-1}$, then the equilibrium point $\tilde{y}_2 = (r-1)^{\frac{1}{p}}$ of (5) is locally asymptotically stable. So, to prove (ii) we must prove that the equilibrium point $\tilde{y}_2 = (r-1)^{\frac{1}{p}}$ of (5) is global attractor. To this end, let $S = \limsup_{n \rightarrow \infty} y_n$ and $I = \liminf_{n \rightarrow \infty} y_n$. Then we deduce from (5) that

$$(20) \quad S \leq \frac{rI}{1+I^p} \quad \text{and} \quad I \leq \frac{rS}{1+S^p}.$$

Consequently, we have the inequality

$$(21) \quad \frac{rS^2}{1+S^p} \leq IS \leq \frac{rI^2}{1+I^p},$$

Since

$$(22) \quad S > I,$$

then, it is easy to see that

$$(23) \quad \frac{1}{1+S^p} < \frac{1}{1+I^p}.$$

From (21) and (23), we have

$$(24) \quad S < I.$$

Then from (22) and (24), we deduce that $S = I$. This proves that the equilibrium point \tilde{y}_2 is global attractor. The proof of (ii) is now completed. Thus, the proof of Theorem 5 is now completed. \blacksquare

Theorem 6. Assume that $r > 1$, together with the following cases:

(i) If σ is odd and τ is even,

(ii) If σ and τ are odd.

Then (5) possesses an unbounded solution.

Proof. From Theorem 4, we deduce from case (i) that is

$$y_{2n+1} < \tilde{y}_2 \quad \text{and} \quad y_{2n} \geq \tilde{y}_2.$$

Consequently, we have

$$y_{2n+2} = \frac{ry_{2n+1-\sigma}}{1+y_{2n+1-\tau}^p} > \frac{ry_{2n+1-\sigma}}{1+(r-1)} = y_{2n+1-\sigma},$$

and

$$y_{2n+3} = \frac{ry_{2n+2-\sigma}}{1 + y_{2n+2-\tau}^p} \leq \frac{ry_{2n+2-\sigma}}{1 + (r - 1)} = y_{2n+2-\sigma}.$$

From which it follows that

$$\lim_{n \rightarrow \infty} y_{2n} = \infty \quad \text{and} \quad \lim_{n \rightarrow \infty} y_{2n+1} = 0.$$

This proves that (5) possesses an unbounded solution for case (i).

Similarly, from case (ii) we can show that

$$y_{2n+2} \leq y_{2n+1-\sigma} \quad \text{and} \quad y_{2n+3} > y_{2n+2-\sigma}.$$

From which it follows that

$$\lim_{n \rightarrow \infty} y_{2n} = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} y_{2n+1} = \infty.$$

Thus, the proof of Theorem 6 is now completed. ■

Theorem 7. (i) *If σ is odd and τ is even, then, if $r = 1$ (5) has prime period two solution*

$$(25) \quad \dots, \Phi, 0, \Phi, 0, \dots$$

with $\Phi > 0$. Furthermore, every solution of (5) converges to the prime periodtwo solution (25) with $\Phi > 0$. While if $r > 1$ or $r < 1$ then, (5) has no prime period two solution.

(ii) *If σ is even and τ is odd, then, (5) has no prime period two solution.*

(iii) *If both σ and τ are odd, then, (5) has no prime period two solution.*

(iv) *If both σ and τ are even, then, (5) has prime period two solution if $p > 1$ while if $0 < p < 1$ then, (5) has no prime period two solution.*

Proof. Let

$$\dots, \Phi, \Psi, \Phi, \Psi, \dots$$

be a non-negative distinctive prime period two solution of (5).

Case (i). If σ is odd and τ is even, then, $y_{n+1} = y_{n-\sigma}$ and $y_n = y_{n-\tau}$. Then from (5) we have $\Phi = \frac{r\Phi}{1+\Psi^p}$ and $\Psi = \frac{r\Psi}{1+\Phi^p}$. Consequently, we have

$$(26) \quad 0 \leq \Phi\Psi = \frac{(1-r)(\Phi-\Psi)}{\Phi^{p-1} - \Psi^{p-1}}.$$

From (26) we deduce that if $p > 1$ then $r \leq 1$. If $r < 1$ this implies that $\Phi < 0$ or $\Psi < 0$ which is impossible, since $\Phi > 0$ and $\Psi > 0$. Thus, if $r > 1$ and $r < 1$, then (5) has no prime period two solution. So, $r = 1$. If $p < 1$ then $r \geq 1$. If $r > 1$ then $\Phi = \Psi = (r - 1)^{\frac{1}{p}} \neq 0$ which is impossible, since

$\Phi \neq \Psi$. Thus, if $0 < p < 1$ and $r > 1$ then (5) has no prime period two solution. So, $r = 1$. Assume that $r = 1$ and let $\{x_n\}_{n=-\omega}^{\infty}$ be a solution of (5), where, $\omega = \max\{\sigma, \tau\}$. If $\omega = \tau$, then

$$(27) \quad y_{n+1} - y_{n-\sigma} = \frac{y_{n-\sigma} y_{n-\tau}^p}{1 + y_{n-\tau}^p} \leq 0.$$

From (27) we deduce that, the even terms of this solution decreases to a limit (say $\Psi \geq 0$). Thus, $\Phi = \frac{\Phi}{1+\Psi^p}$ and $\Psi = \frac{\Psi}{1+\Phi^p}$ which implies that $\Phi\Psi^p = 0$ and $\Psi\Phi^p = 0$. This completes the proof of case (i).

Case (ii). If σ is even and τ is odd, then, $y_n = y_{n-\sigma}$ and $y_{n+1} = y_{n-\tau}$. Then from (5) we have $\Phi = \frac{r\Psi}{1+\Phi^p}$ and $\Psi = \frac{r\Phi}{1+\Psi^p}$. Consequently, we have

$$(28) \quad 0 \leq \frac{\Phi^{p+1} - \Psi^{p+1}}{\Phi - \Psi} = - (r + 1).$$

From (28) we deduce that, $r + 1 \leq 0$ and hence $r \leq -1$. This is impossible, since $r > 0$. Therefore (5) has no prime period two solution.

Case (iii). If σ and τ are odd, then, $y_{n+1} = y_{n-\sigma} = y_{n-\tau}$ and from (5) we have $\Phi = \frac{r\Phi}{1+\Phi^p}$ and $\Psi = \frac{r\Psi}{1+\Psi^p}$. Consequently, if $r > 1$ we get $\Phi = \Psi = (r - 1)^{\frac{1}{p}} \neq 0$. This is impossible, since $\Phi \neq \Psi \neq 0$. Therefore (5) has no prime period two solution.

Case (iv). If σ and τ are even, then, $y_n = y_{n-\sigma} = y_{n-\tau}$ and from (5) we have $\Phi = \frac{r\Psi}{1+\Psi^p}$ and $\Psi = \frac{r\Phi}{1+\Phi^p}$. Consequently, we get

$$(29) \quad \Phi\Psi = \frac{(r + 1)(\Phi - \Psi)}{\Phi^{p-1} - \Psi^{p-1}} \geq 0.$$

From (29) we deduce for $p > 1$ and $r > 0$ that (5) has prime period two solution, while if $0 < p < 1$ then $r + 1 \leq 0$. Therefore $r \leq -1$. This is impossible, since $r > 0$. Thus, (5) has no prime period two solution. The proof of Theorem 7 is now completed. \blacksquare

4. Conclusions

In this paper, we have studied the nonlinear difference equation (5) and we have shown under certain conditions on the parameters σ , τ , p and r that its solution is globally asymptotic stable and oscillates about the equilibrium point with semi-cycles of length one. Furthermore, we have shown also under certain conditions on these parameters that (5) possesses an unbounded solution and this equation has (or not) a prime period two solution.

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