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Saïd Abbas and Mouffak Benchohra

# UNIQUENESS RESULTS FOR FREDHOLM TYPE FRACTIONAL ORDER RIEMANN-LIOUVILLE INTEGRAL EQUATIONS

Abstract. In this paper we study the existence and uniqueness of solutions of a certain Fredholm type Riemann-Liouville integral equation of two variables by using Banach contraction principle.

Key words: Fredholm type integral equation, left-sided mixed Riemann-Liouville integral of fractional order, existence and uniqueness, fixed point.

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## 1. Introduction

Fractional integral equations have recently been applied in various areas of engineering, science, finance, applied mathematics, bio-engineering, radiative transfer, neutron transport and the kinetic theory of gases and others [6, 12, 13, 15, 21, 22]. The idea of fractional calculus and fractional order differential equations and inclusions has been a subject of interest not only among mathematicians, but also among physicists and engineers.

Fractional calculus techniques are widely used in rheology, control, porous media, viscoelasticity, electrochemistry, electromagnetism, etc. [18, 20, 24, 25, 27]. There has been a significant development in ordinary and partial fractional differential equations in recent years; see the monographs of Abbas *et al.* [3], Kilbas *et al.* [23], Miller and Ross [26], Samko *et al.* [32], the papers of Abbas *et al.* [1, 2, 4, 5], Belarbi *et al.* [7], Benchohra *et al.* [8, 9, 10], Diethelm [17], Mainardi [24], Podlubny *et al.* [31], and Vityuk and Golushkov [33], and the references therein.

Qualitative properties and structure of the set of solutions of the Darboux problem for hyperbolic partial integer order differential equations have been studied by many authors; see for instance [4, 14, 19]. In [11], Bica *et al.* initiated the study of Fredholm integral equation of the form

$$x(t) = f(t) + \int_0^a g(t, s, x(s), x'(s))ds, \tag{1}$$

in Banach space setting. Recently in [29], Pachpatte studied the qualitative behavior of solutions of equation (1) and its further generalization. Inspired by the results in [11, 29], Pachpatte in [30] studied the Fredholm type integral equation of the form

$$(2)\quad u(x,y)=f(x,y)+\int_0^a\int_0^b g(x,y,s,t,u(s,t),D_1u(s,t),D_2u(s,t))dtds,$$

where $u$ is the unknown function to be found. In this paper we consider the Fredholm type Riemann-Liouville integral equation of the form

$$(3)\qquad u(x,y) = \mu(x,y)+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^a\int_0^b (a-s)^{r_1-1}(b-t)^{r_2-1} \times f(x,y,s,t,u(s,t),({}^cD_\theta^r u)(s,t))dtds,$$

if $(x,y)\in J:=[0,a]\times[0,b]$, where $a,b\in(0,\infty)$, $\theta=(0,0)$, ${}^cD_\theta^r$ is the standard Caputo's fractional derivative of order $r=(r_1,r_2)\in(0,1]\times(0,1]$, $\mu:J\to\mathbb{R}^n$, $f:J\times J\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ are given continuous functions and $\Gamma(\cdot)$ is the (Euler's) Gamma function defined by $\Gamma(\varsigma)=\int_0^\infty t^{\varsigma-1}e^{-t}dt$, $\varsigma>0$.

The present result extends those considered with integer order derivative [11, 19, 28, 29, 30].

## 2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. Let $\varrho>0$, by $L^\varrho(J)$ we denote the space of Lebegue-integrablefunctions $w:J\to\mathbb{R}^n$ with the norm

$$\|w\|_{L^\varrho}=\Big(\int_0^a\int_0^b\|w(x,y)\|^\varrho dydx\Big)^{\frac{1}{\varrho}},$$

where $\|.\|$ denotes a suitable complete norm on $\mathbb{R}^n$.

The case $\varrho=1$ is included and we have

$$\|w\|_{L^1}=\int_0^a\int_0^b\|w(x,y)\|dydx.$$

As usual, by $C(J)$ we denote the Banach space of all continuous functions from $J$ into $\mathbb{R}^n$ with the norm

$$\|w\|_\infty=\sup_{(x,y)\in J}\|w(x,y)\|.$$

**Definition 1** ([33]). *Let $r=(r_1,r_2)\in(0,\infty)\times(0,\infty)$, $\theta=(0,0)$ and $u\in L^1(J)$. The left-sided mixed Riemann-Liouville integral of order $r$ of $u$ is defined by*

$$(I_\theta^r u)(x,y)=\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}u(s,t)dtds.$$

In particular,

$$(I_\theta^\theta u)(x,y)=u(x,y),\ (I_\theta^\sigma u)(x,y)=\int_0^x\int_0^y u(s,t)dtds,$$

for almost all $(x,y)\in J$, where $\sigma=(1,1)$. For instance, $I_\theta^r u$ exists for all $r_1,r_2\in(0,\infty)$, when $u\in L^1(J)$. Note also that when $u\in C(J)$, then $(I_\theta^r u)\in C(J)$, moreover

$$(I_\theta^r u)(x,0)=(I_\theta^r u)(0,y)=0,\ \ x\in[0,a],\ \ y\in[0,b].$$

**Example 1.** Let $\lambda,\omega\in(-1,0)\cup(0,\infty)$ and $r=(r_1,r_2)\in(0,\infty)\times(0,\infty)$, then

$$I_\theta^r x^\lambda y^\omega=\frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda+r_1)\Gamma(1+\omega+r_2)}x^{\lambda+r_1}y^{\omega+r_2},$$

for almost all $(x,y)\in J$.

By $1-r$ we mean $(1-r_1,1-r_2)\in(0,1]\times(0,1]$. Denote by $D_{xy}^2:=\frac{\partial^2}{\partial x\partial y}$, the mixed second order partial derivative.

**Definition 2** ([33]). *Let $r\in(0,1]\times(0,1]$ and $u\in L^1(J)$. The Caputo fractional-order derivative of order $r$ of $u$ is defined by the expression ${}^cD_\theta^r u(x,y)=(I_\theta^{1-r}D_{xy}^2u)(x,y)$.*

The case $\sigma=(1,1)$ is included and we have

$$({}^cD_\theta^\sigma u)(x,y)=(D_{xy}^2u)(x,y),\ \text{for almost all }(x,y)\in J.$$

**Example 2.** Let $\lambda,\omega\in(-1,0)\cup(0,\infty)$ and $r=(r_1,r_2)\in(0,1]\times(0,1]$, then

$${}^cD_\theta^r x^\lambda y^\omega=\frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda-r_1)\Gamma(1+\omega-r_2)}x^{\lambda-r_1}y^{\omega-r_2},$$

for almost all $(x,y)\in J$.

**Definition 3** ([23, 32]). *Let $\alpha\in(0,\infty)$ and $u\in L^1(J)$. The partial Riemann-Liouville integral of order $\alpha$ of $u(x,y)$ with respect to $x$ is defined by the expression*

$$I_{0,x}^\alpha u(x,y)=\frac{1}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}u(s,y)ds,$$

*for almost all $x\in[0,a]$ and all $y\in[0,b]$.*

Analogously, we define the integral

$$I_{0,y}^\alpha u(x,y)=\frac{1}{\Gamma(\alpha)}\int_0^y(y-s)^{\alpha-1}u(x,s)ds,$$

for all $x\in[0,a]$ and almost all $y\in[0,b]$.

**Definition 4** ([23, 32]). *Let $\alpha \in (0,1]$ and $u \in L^1(J)$. The Caputo fractional derivative of order $\alpha$ of $u(x,y)$ with respect to $x$ is defined by the expression*

$$^cD^{\alpha}_{0,x}u(x,y) = I^{1-\alpha}_{0,x}\frac{\partial}{\partial x}u(x,y),$$

*for almost all $x \in [0,a]$ and all $y \in [0,b]$.*

Analogously, we define the derivative

$$^cD^{\alpha}_{0,y}u(x,y) = I^{1-\alpha}_{0,y}\frac{\partial}{\partial y}u(x,y),$$

for all $x \in [0,a]$ and almost all $y \in [0,b]$. For $w, {}^cD^r_\theta w \in C(J)$, denote

$$\|w(x,y)\|_1 = \|w(x,y)\| + \|^cD^r_\theta w(x,y)\|.$$

Let $E$ be the space of functions $w, {}^cD^r_\theta w \in C(J)$, which fulfill the following condition:

$$\exists M \geq 0 : \|w(x,y)\|_1 \leq Me^{\lambda(x+y)}, \text{ for } (x,y) \in J, \tag{4}$$

where $\lambda$ is a positive constant. In the space $E$ we define the norm (see [28])

$$\|w\|_E = \sup_{(x,y)\in J}\{\|w(x,y)\|_1 e^{-\lambda(x+y)}\}.$$

It is easy to see that $(E, \|.\|_E)$ is a Banach space. We note that the condition (4) implies that

$$\|w\|_E \leq M. \tag{5}$$

## 3. Main results

Let us start by defining what we mean by a solution of the equation (3).

**Definition 5.** *We mean by a solution of the equation (6), every function $w \in C(J)$, such that the mixed derivative $D^2_{xy}(w)$ exists and is integrable on $J$, and $w$ satisfies (3) on $J$.*

Further, we present conditions for the uniqueness of the solution of the equation (3).

**Theorem 1.** *Assume*

$(H_1)$ *There exist functions $\rho_1, \rho_2 : J \times J \to \mathbb{R}^+$, such that $f$ and $^cD^r_\theta(f)$ satisfy*

$$\begin{aligned} \|f(x,y,s,t,u,v) \;-\; & f(x,y,s,t,\overline{u},\overline{v})\| \\ & \leq \rho_1(x,y,s,t)(\|u-\overline{u}\| + \|v-\overline{v}\|) \end{aligned} \tag{6}$$

*and*

$$\|({}^cD^r_\theta f)(x,y,s,t,u,v) - ({}^cD^r_\theta f)(x,y,s,t,\overline{u},\overline{v})\| \leq \rho_2(x,y,s,t)(\|u-\overline{u}\|+\|v-\overline{v}\|), \tag{7}$$

*for each* $(x,y),(s,t)\in J$ *and* $u,v,\overline{u},\overline{v}\in\mathbb{R}^n$,

$(H_2)$ *for* $\lambda$ *as in (4), there exist nonnegative constants* $\alpha_1$, $\alpha_2$, $\alpha_3$, $\beta_1$, $\beta_2$ *and* $0<r_3<\min\{r_1,r_2\}$ *such that, for* $(x,y)\in J$, *we have*

$$\begin{cases} \|\mu(x,y)\|_1 \leq \alpha_1 e^{\lambda(x+y)}, \\ \int_0^a\int_0^b \|f(x,y,s,t,0,0)\|^{\frac{1}{r_3}} dtds \leq \alpha_2^{\frac{1}{r_3}} e^{\frac{\lambda}{r_3}(x+y)}, \\ \int_0^a\int_0^b \|{}^cD^r_\theta f(x,y,s,t,0,0)\|^{\frac{1}{r_3}} dtds \leq \alpha_3^{\frac{1}{r_3}} e^{\frac{\lambda}{r_3}(x+y)}, \end{cases} \tag{8}$$

*and*

$$\begin{cases} \int_0^a\int_0^b \rho_1^{\frac{1}{r_3}}(x,y,s,t)e^{\frac{1}{r_3}(s+t)} dtds \leq \beta_1^{\frac{1}{r_3}} e^{\frac{\lambda}{r_3}(x+y)}, \\ \int_0^a\int_0^b \rho_2^{\frac{1}{r_3}}(x,y,s,t)e^{\frac{\lambda}{r_3}(s+t)} dtds \leq \beta_2^{\frac{1}{r_3}} e^{\frac{\lambda}{r_3}(x+y)}. \end{cases} \tag{9}$$

*If*

$$\frac{(\beta_1+\beta_2)a^{(\omega_1+1)(1-r_3)}b^{(\omega_2+1)(1-r_3)}}{(\omega_1+1)^{(1-r_3)}(\omega_2+1)^{(1-r_3)}\Gamma(r_1)\Gamma(r_2)} e^{\lambda(x+y)} < 1, \tag{10}$$

*where* $\omega_1=\frac{r_1-1}{1-r_3}$, $\omega_2=\frac{r_2-1}{1-r_3}$, *then the equation (3) has a unique solution on* $J$ *in* $E$.

**Proof.** Let $u\in E$ and define the operator $N:E\to E$ by

$$(Nu)(x,y) = \mu(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^a\int_0^b (a-s)^{r_1-1}(b-t)^{r_2-1} \times f(x,y,s,t,u(s,t),({}^cD^r_\theta u)(s,t))dtds. \tag{11}$$

Differentiating both sides of (11) by applying the Caputo fractional derivative, we get

$${}^cD^r_\theta(Nu)(x,y) = {}^cD^r_\theta\mu(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^a\int_0^b (a-s)^{r_1-1}(b-t)^{r_2-1} \times {}^cD^r_\theta f(x,y,s,t,u(s,t),({}^cD^r_\theta u)(s,t))dtds. \tag{12}$$

Now, we show that $N$ maps $E$ into itself. Evidently, $N(u)$, ${}^cD^r_\theta(Nu)$ are continuous on $J$. We verify that (4) is fulfilled. From (5), (8), (9) and using the hypotheses, for each $(x,y)\in J$, we have

$$\begin{aligned}
\|(Nu)(x,y)\|_1 \le \|\mu(x,y)\|_1 &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^a\int_0^b (a-s)^{r_1-1}(b-t)^{r_2-1}\\
&\times \|f(x,y,s,t,u(s,t),({}^cD^r_\theta u)(s,t)) - f(x,y,s,t,0,0)\|dtds\\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^a\int_0^b (a-s)^{r_1-1}(b-t)^{r_2-1}\|f(x,y,s,t,0,0)\|dtds\\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^a\int_0^b (a-s)^{r_1-1}(b-t)^{r_2-1}\\
&\times \|{}^cD^r_\theta f(x,y,s,t,u(s,t),({}^cD^r_\theta u)(s,t)) - {}^c D^r_\theta f(x,y,s,t,0,0)\|dtds\\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^a\int_0^b (a-s)^{r_1-1}(b-t)^{r_2-1}\|{}^cD^r_\theta f(x,y,s,t,0,0)\|dtds\\
\le \|\mu(x,y)\|_1 &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\Big(\int_0^a\int_0^b (a-s)^{\frac{r_1-1}{1-r_3}}(b-t)^{\frac{r_2-1}{1-r_3}}dtds\Big)^{1-r_3}\\
&\times \Big(\int_0^a\int_0^b \|f(x,y,s,t,u(s,t),({}^cD^r_\theta u)(s,t)) - f(x,y,s,t,0,0)\|^{\frac{1}{r_3}}dtds\Big)^{r_3}\\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\Big(\int_0^a\int_0^b (a-s)^{\frac{r_1-1}{1-r_3}}(b-t)^{\frac{r_2-1}{1-r_3}}dtds\Big)^{1-r_3}\\
&\times \Big(\int_0^a\int_0^b \|f(x,y,s,t,0,0)\|^{\frac{1}{r_3}}dtds\Big)^{r_3}\\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\Big(\int_0^a\int_0^b (a-s)^{\frac{r_1-1}{1-r_3}}(b-t)^{\frac{r_2-1}{1-r_3}}dtds\Big)^{1-r_3}\\
&\times \Big(\int_0^a\int_0^b \|{}^cD^r_\theta f(x,y,s,t,u(s,t),({}^cD^r_\theta u)(s,t))\\
&\quad - {}^cD^r_\theta f(x,y,s,t,0,0)\|^{\frac{1}{r_3}}dtds\Big)^{r_3}\\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\Big(\int_0^a\int_0^b (a-s)^{\frac{r_1-1}{1-r_3}}(b-t)^{\frac{r_2-1}{1-r_3}}dtds\Big)^{1-r_3}\\
&\times \Big(\int_0^a\int_0^b \|{}^cD^r_\theta f(x,y,s,t,0,0)\|^{\frac{1}{r_3}}dtds\Big)^{r_3}
\end{aligned}$$

Thus, for each $(x,y)\in J$,

$$\begin{aligned}
\|(Nu)(x,y)\|_1 \;\le\; & \|\mu(x,y)\|_1 + \frac{a^{(\omega_1+1)(1-r_3)}b^{(\omega_2+1)(1-r_3)}}{(\omega_1+1)^{(1-r_3)}(\omega_2+1)^{(1-r_3)}\Gamma(r_1)\Gamma(r_2)}\\
&\times\Big[\Big(\int_0^a\int_0^b \|f(x,y,s,t,0,0)\|^{\frac{1}{r_3}}dtds\Big)^{r_3}
\end{aligned}$$

$$+ \Big( \int_0^a \int_0^b \|^c D_\theta^r f(x,y,s,t,0,0)\|^{\frac{1}{r_3}} dtds \Big)^{r_3}$$
$$+ \Big( \int_0^a \int_0^b \rho_1^{\frac{1}{r_3}}(x,y,s,t) \|u(s,t)\|_1^{\frac{1}{r_3}} dtds \Big)^{r_3}$$
$$+ \Big( \int_0^a \int_0^b \rho_2^{\frac{1}{r_3}}(x,y,s,t) \|u(s,t)\|_1^{\frac{1}{r_3}} dtds \Big)^{r_3} \Big]$$
$$\leq \alpha_1 e^{\lambda(x+y)} + \frac{a^{(\omega_1+1)(1-r_3)} b^{(\omega_2+1)(1-r_3)}}{(\omega_1+1)^{(1-r_3)}(\omega_2+1)^{(1-r_3)}\Gamma(r_1)\Gamma(r_2)}$$
$$\times \Big[ \alpha_2 e^{\lambda(x+y)} + \alpha_3 e^{\lambda(x+y)}$$
$$+ \|u|_E \Big( \int_0^a \int_0^b \rho_1^{\frac{1}{r_3}}(x,y,s,t) e^{\frac{\lambda}{r_3}(s+t)} dtds \Big)^{r_3}$$
$$+ \|u|_E \Big( \int_0^a \int_0^b \rho_2^{\frac{1}{r_3}}(x,y,s,t) e^{\frac{\lambda}{r_3}(s+t)} dtds \Big)^{r_3} \Big]$$
$$\leq \Big[ \alpha_1 + \frac{(\alpha_2+\alpha_3+M\beta_1+M\beta_2) a^{(\omega_1+1)(1-r_3)} b^{(\omega_2+1)(1-r_3)}}{(\omega_1+1)^{(1-r_3)}(\omega_2+1)^{(1-r_3)}\Gamma(r_1)\Gamma(r_2)} \Big] e^{\lambda(x+y)}.$$

Hence, for each $(x,y) \in J$, we get

$$(13)\quad \|(Nu)(x,y)\|_1 \leq \Big[ \alpha_1 + \frac{(\alpha_2+\alpha_3+M\beta_1+M\beta_2) a^{(\omega_1+1)(1-r_3)} b^{(\omega_2+1)(1-r_3)}}{(\omega_1+1)^{(1-r_3)}(\omega_2+1)^{(1-r_3)}\Gamma(r_1)\Gamma(r_2)} \Big] e^{\lambda(x+y)}.$$

From (13), it follows that $N(u) \in E$. This proves that the operator $N$ maps $E$ into itself. Next, we verify that the operator $N$ is a contraction map. Let $u, v \in E$. From (11), (12) and using the hypotheses, for each $(x,y) \in J$, we have

$$\|(Nu)(x,y) - (Nv)(x,y)\|_1 \leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^a \int_0^b (a-s)^{r_1-1}(b-t)^{r_2-1}$$
$$\times \|f(x,y,s,t,u(s,t),(^c D_\theta^r u)(s,t))$$
$$- f(x,y,s,t,v(s,t),(^c D_\theta^r v)(s,t))\| dtds$$
$$+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^a \int_0^b (a-s)^{r_1-1}(b-t)^{r_2-1}$$
$$\times \|^c D_\theta^r f(x,y,s,t,u(s,t),(^c D_\theta^r u)(s,t))$$
$$-^c D_\theta^r f(x,y,s,t,v(s,t),(^c D_\theta^r v)(s,t))\| dtds$$
$$\leq \frac{a^{(\omega_1+1)(1-r_3)} b^{(\omega_2+1)(1-r_3)}}{(\omega_1+1)^{(1-r_3)}(\omega_2+1)^{(1-r_3)}\Gamma(r_1)\Gamma(r_2)}$$
$$\times \Big[ \Big( \int_0^a \int_0^b \rho_1^{\frac{1}{r_3}}(x,y,s,t) \|u(s,t)-v(s,t)\|_1^{\frac{1}{r_3}} dtds \Big)^{r_3}$$

$$+\Big(\int_0^a\int_0^b \rho_2^{\frac{1}{r_3}}(x,y,s,t)\|u(s,t)-v(s,t)\|_1^{\frac{1}{r_3}}dtds\Big)^{r_3}\Big]$$
$$\leq \frac{a^{(\omega_1+1)(1-r_3)}b^{(\omega_2+1)(1-r_3)}}{(\omega_1+1)^{(1-r_3)}(\omega_2+1)^{(1-r_3)}\Gamma(r_1)\Gamma(r_2)}$$
$$\times\Big[\Big(\int_0^a\int_0^b \rho_1^{\frac{1}{r_3}}(x,y,s,t)e^{\frac{\lambda}{r_3}(s+t)}dtds\Big)^{r_3}$$
$$+\Big(\int_0^a\int_0^b \rho_2^{\frac{1}{r_3}}(x,y,s,t)e^{\frac{\lambda}{r_3}(s+t)}dtds\Big)^{r_3}\Big]\|u-v\|_E$$
$$\leq \frac{(\beta_1+\beta_2)a^{(\omega_1+1)(1-r_3)}b^{(\omega_2+1)(1-r_3)}}{(\omega_1+1)^{(1-r_3)}(\omega_2+1)^{(1-r_3)}\Gamma(r_1)\Gamma(r_2)}e^{\lambda(x+y)}\|u-v\|_E.$$

Hence

$$\|Nu-Nv\|_E\leq \frac{(\beta_1+\beta_2)a^{(\omega_1+1)(1-r_3)}b^{(\omega_2+1)(1-r_3)}}{(\omega_1+1)^{(1-r_3)}(\omega_2+1)^{(1-r_3)}\Gamma(r_1)\Gamma(r_2)}\|u-v\|_E.$$

From (10), it follows that $N$ has a unique fixed point in $E$ by Banach contraction principle (see [16], p. 37). The fixed point of $N$ is however a solution of equation (3).

The next result deals with the uniqueness of a solution of equation (3) in $E$ when the functions $\rho_i;\ i\in\{1,2\}$ are in the form

$$\rho_i(x,y,s,t)=h_i(s,t)c(x,y)=m_i(s,t)e^{\lambda(x+y-s-t)},\quad i\in\{1,2\}.$$

■

**Theorem 2.** *Assume*

($H_3$) *For* $\lambda$ *as in (4), there exist constants* $\alpha>0$, $0<r_3<\min\{r_1,r_2\}$, *and strictly positive functions* $m_1(x,y),m_2(x,y)\in L^{\frac{1}{r_3}}(J)$ *such that for each* $(x,y),(s,t)\in J$, $\mu$, $f$ *and* ${}^cD^r_\theta(f)$ *satisfy*

$$\|\mu(x,y)\|_1\leq \alpha e^{\lambda(x+y)} \tag{14}$$

*and*

$$\begin{cases}\|f(x,y,s,t,0,0)\|\leq m_1(s,t)e^{\lambda(x+y)},\\ \|{}^cD^r_\theta f(x,y,s,t,0,0)\|\leq m_2(s,t)e^{\lambda(x+y)},\end{cases} \tag{15}$$

($H_4$) *There exist strictly positive functions* $m_3(x,y),m_4(x,y)\in L^{\frac{1}{r_3}}(J)$ *such that for each* $(x,y),(s,t)\in J$ *and* $u,v,\overline{u},\overline{v}\in\mathbb{R}^n$, $f$ *and* ${}^cD^r_\theta(f)$ *satisfy*

$$\|f(x,y,s,t,u,v)-f(x,y,s,t,\overline{u},\overline{v})\|\leq m_3(s,t)\times e^{\lambda(x+y-s-t)}(\|u-\overline{u}\|+\|v-\overline{v}\|) \tag{16}$$

*and*

$$\|({}^cD^r_\theta f)(x,y,s,t,u,v)-({}^cD^r_\theta f)(x,y,s,t,\overline{u},\overline{v})\|\le m_4(s,t) \times e^{\lambda(x+y-s-t)}(\|u-\overline{u}\|+\|v-\overline{v}\|). \tag{17}$$

*If*

$$\frac{(M_3+M_4)a^{(\omega_1+1)(1-r_3)}b^{(\omega_2+1)(1-r_3)}}{(\omega_1+1)^{(1-r_3)}(\omega_2+1)^{(1-r_3)}\Gamma(r_1)\Gamma(r_2)}<1, \tag{18}$$

*where*

$$\omega_1=\frac{r_1-1}{1-r_3},\quad \omega_2=\frac{r_2-1}{1-r_3},\quad M_i=\|m_i\|_{L^{\frac{1}{r_3}}},\quad i\in\{3,4\},$$

*then the equation (3) has a unique solution on $J$ in $E$.*

**Proof.** Consider the operator $N:E\to E$ defined in (11). Now, we show that $N(u)$ maps $E$ into itself. Evidently, $N(u)$, ${}^cD^r_\theta(Nu)$ are continuous on $J$ and $N(u), {}^cD^r_\theta(Nu)\in\mathbb{R}^n$. We verify that (4) is fulfilled. From (5), (16), (17) and using the hypotheses, for each $(x,y)\in J$, we have

$$\begin{aligned}
\|(Nu)(x,y)\|_1 &\le \|\mu(x,y)\|_1+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^a\int_0^b(a-s)^{r_1-1}(b-t)^{r_2-1}\\
&\quad\times\|f(x,y,s,t,u(s,t),({}^cD^r_\theta u)(s,t))-f(x,y,s,t,0,0)\|dtds\\
&\quad+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^a\int_0^b(a-s)^{r_1-1}(b-t)^{r_2-1}\|f(x,y,s,t,0,0)\|dtds\\
&\quad+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^a\int_0^b(a-s)^{r_1-1}(b-t)^{r_2-1}\\
&\quad\times\|{}^cD^r_\theta f(x,y,s,t,u(s,t),({}^cD^r_\theta u)(s,t))-{}^cD^r_\theta f(x,y,s,t,0,0)\|dtds\\
&\quad+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^a\int_0^b(a-s)^{r_1-1}(b-t)^{r_2-1}\|{}^cD^r_\theta f(x,y,s,t,0,0)\|dtds\\
&\le\|\mu(x,y)\|_1+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\Big(\int_0^a\int_0^b(a-s)^{\frac{r_1-1}{1-r_3}}(b-t)^{\frac{r_2-1}{1-r_3}}dtds\Big)^{1-r_3}\\
&\quad\times\Big(\int_0^a\int_0^b\|f(x,y,s,t,u(s,t),({}^cD^r_\theta u)(s,t))-f(x,y,s,t,0,0)\|^{\frac{1}{r_3}}dtds\Big)^{r_3}\\
&\quad+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\Big(\int_0^a\int_0^b(a-s)^{\frac{r_1-1}{1-r_3}}(b-t)^{\frac{r_2-1}{1-r_3}}dtds\Big)^{1-r_3}\\
&\quad\times\Big(\int_0^a\int_0^b\|f(x,y,s,t,0,0)\|^{\frac{1}{r_3}}dtds\Big)^{r_3}\\
&\quad+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\Big(\int_0^a\int_0^b(a-s)^{\frac{r_1-1}{1-r_3}}(b-t)^{\frac{r_2-1}{1-r_3}}dtds\Big)^{1-r_3}
\end{aligned}$$

$$\times\Big(\int_0^a\int_0^b \|^cD_\theta^r f(x,y,s,t,u(s,t),(^cD_\theta^r u)(s,t)) - {}^cD_\theta^r f(x,y,s,t,0,0)\|^{\frac{1}{r_3}}dtds\Big)^{r_3}$$

$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\Big(\int_0^a\int_0^b (a-s)^{\frac{r_1-1}{1-r_3}}(b-t)^{\frac{r_2-1}{1-r_3}}dtds\Big)^{1-r_3}$$

$$\times\Big(\int_0^a\int_0^b \|^cD_\theta^r f(x,y,s,t,0,0)\|^{\frac{1}{r_3}}dtds\Big)^{r_3}$$

$$\leq \|\mu(x,y)\|_1+\frac{a^{(\omega_1+1)(1-r_3)}b^{(\omega_2+1)(1-r_3)}}{(\omega_1+1)^{(1-r_3)}(\omega_2+1)^{(1-r_3)}\Gamma(r_1)\Gamma(r_2)}$$

$$\times\Big[\Big(\int_0^a\int_0^b (m_1(s,t)e^{\lambda(x+y)})^{\frac{1}{r_3}}dtds\Big)^{r_3}$$

$$+\Big(\int_0^a\int_0^b (m_2(s,t)e^{\lambda(x+y)})^{\frac{1}{r_3}}dtds\Big)^{r_3}$$

$$+\Big(\int_0^a\int_0^b (m_3(s,t)\|u(s,t)\|_1e^{\lambda(x+y-s-t)})^{\frac{1}{r_3}}dtds\Big)^{r_3}$$

$$+\Big(\int_0^a\int_0^b (m_4(s,t)\|u(s,t)\|_1e^{\lambda(x+y-s-t)})^{\frac{1}{r_3}}dtds\Big)^{r_3}\Big]$$

$$\leq \alpha e^{\lambda(x+y)}+\frac{a^{(\omega_1+1)(1-r_3)}b^{(\omega_2+1)(1-r_3)}}{(\omega_1+1)^{(1-r_3)}(\omega_2+1)^{(1-r_3)}\Gamma(r_1)\Gamma(r_2)}\Big[M_1e^{\lambda(x+y)}$$

$$+M_2e^{\lambda(x+y)}\|u\|_E\Big(\int_0^a\int_0^b (m_3(s,t)e^{\lambda(x+y)})^{\frac{1}{r_3}}dtds\Big)^{r_3}$$

$$+\|u\|_E\Big(\int_0^a\int_0^b (m_4(s,t)e^{\lambda(x+y)})^{\frac{1}{r_3}}dtds\Big)^{r_3}\Big],$$

where

$$M_i=\|m_i|_{L^{\frac{1}{r_3}}},\quad i\in\{1,2\}.$$

Hence, for each $(x,y)\in J$, we get

$$\|(Nu)(x,y)\|_1 \leq \Big[\alpha+\frac{\Big(M_1+M_2+M(M_3+M_4)\Big)a^{(\omega_1+1)(1-r_3)}b^{(\omega_2+1)(1-r_3)}}{(\omega_1+1)^{(1-r_3)}(\omega_2+1)^{(1-r_3)}\Gamma(r_1)\Gamma(r_2)}\Big]e^{\lambda(x+y)}. \tag{19}$$

From (19), it follows that $N(u)\in E$. This proves that the operator $N$ maps $E$ into itself. Next, we verify that the operator $N$ is a contraction map. Let $u(x,y),v(x,y)\in E$. From (11), (12) and using the hypotheses, for each $(x,y)\in J$, we have

$$\begin{aligned}
\|(Nu)(x,y)-(Nv)(x,y)\|_1 \le{}& \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^a\int_0^b (a-s)^{r_1-1}(b-t)^{r_2-1}\\
&\times \|f(x,y,s,t,u(s,t),({}^cD_\theta^r u)(s,t))\\
&\quad - f(x,y,s,t,v(s,t),({}^cD_\theta^r v)(s,t))\|dtds\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^a\int_0^b (a-s)^{r_1-1}(b-t)^{r_2-1}\\
&\times \|{}^cD_\theta^r f(x,y,s,t,u(s,t),({}^cD_\theta^r u)(s,t))\\
&\quad - {}^cD_\theta^r f(x,y,s,t,v(s,t),({}^cD_\theta^r v)(s,t))\|dtds\\
\le{}& \frac{a^{(\omega_1+1)(1-r_3)}b^{(\omega_2+1)(1-r_3)}}{(\omega_1+1)^{(1-r_3)}(\omega_2+1)^{(1-r_3)}\Gamma(r_1)\Gamma(r_2)}\\
&\times \Big[\Big(\int_0^a\int_0^b (m_3(s,t)\|u(s,t)-v(s,t)\|_1 e^{\lambda(x+y-s-t)})^{\frac{1}{r_3}}dtds\Big)^{r_3}\\
&+\Big(\int_0^a\int_0^b (m_4(s,t)\|u(s,t)-v(s,t)\|_1 e^{\lambda(x+y-s-t)})^{\frac{1}{r_3}}dtds\Big)^{r_3}\Big]\\
\le{}& \frac{(M_3+M_4)a^{(\omega_1+1)(1-r_3)}b^{(\omega_2+1)(1-r_3)}}{(\omega_1+1)^{(1-r_3)}(\omega_2+1)^{(1-r_3)}\Gamma(r_1)\Gamma(r_2)}e^{\lambda(w+y)}\|u-v\|_E.
\end{aligned}$$

Hence

$$\|Nu-Nv\|_E \le \frac{(M_3+M_4)a^{(\omega_1+1)(1-r_3)}b^{(\omega_2+1)(1-r_3)}}{(\omega_1+1)^{(1-r_3)}(\omega_2+1)^{(1-r_3)}\Gamma(r_1)\Gamma(r_2)}\|u-v\|_E.$$

From (18), it follows that $N$ has a unique fixed point in $E$ by Banach contraction principle (see [16], p. 37). The fixed point of $N$ is however a solution of equation (3).

For $w, {}^cD_{0,x}^{r_1}w, {}^cD_{0,y}^{r_2}w \in C(J)$, denote

$$\|w(x,y)\|_1 = \|w(x,y)\| + \|{}^cD_{0,x}^{r_1}w(x,y)\| + \|{}^cD_{0,y}^{r_2}w(x,y)\|.$$

Let $E$ be the space of functions $w, {}^cD_{0,x}^{r_1}w, {}^cD_{0,y}^{r_2}w \in C(J)$, which fulfill condition (4). ■

**Corollary 1.** *Consider the following Fredholm type Riemann-Liouville integral equation of the form*

$$(20)\quad \begin{aligned} u(x,y) ={}& \mu(x,y)+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^a\int_0^b (a-s)^{r_1-1}(b-t)^{r_2-1}\\ &\times f(x,y,s,t,u(s,t),({}^cD_{0,s}^{r_1}u)(s,t),({}^cD_{0,t}^{r_2}u)(s,t))dtds\end{aligned}$$

*if* $(x,y)\in J := [0,a]\times[0,b]$. *Assume*

$(H_1')$ *There exist functions* $\rho_1, \rho_2, \rho_3 : J \times J \to \mathbb{R}^+$, *such that* $f$ ${}^cD^{r_1}_{0,x}(f)$, *and* ${}^cD^{r_2}_{0,y}(f)$ *satisfy*

$$\|f(x,y,s,t,u,v,w) - f(x,y,s,t,\overline{u},\overline{v},\overline{w})\| \leq \rho_1(x,y,s,t)(\|u-\overline{u}\| + \|v-\overline{v}\| + \|w-\overline{w}\|) \tag{21}$$

*and*

$$\|({}^cD^{r_1}_{0,x}f)(x,y,s,t,u,v,w) - ({}^cD^{r_1}_{0,x}f)(x,y,s,t,\overline{u},\overline{v},\overline{w})\| \leq \rho_2(x,y,s,t)(\|u-\overline{u}\| + \|v-\overline{v}\| + \|w-\overline{w}\|), \tag{22}$$

$$\|({}^cD^{r_2}_{0,y}f)(x,y,s,t,u,v,w) - ({}^cD^{r_2}_{0,y}f)(x,y,s,t,\overline{u},\overline{v},\overline{w})\| \leq \rho_3(x,y,s,t)(\|u-\overline{u}\| + \|v-\overline{v}\| + \|w-\overline{w}\|), \tag{23}$$

*for each* $(x,y),(s,t) \in J$ *and* $u,v,w,\overline{u},\overline{v},\overline{w} \in \mathbb{R}^n$,

$(H_2')$ *For* $\lambda$ *as in (4), there exist nonnegative constants* $\alpha_1$, $\alpha_2$, $\alpha_3$, $\alpha_4$, $\beta_1$, $\beta_2$, $\beta_3$
*and* $0 < r_3 < \min\{r_1, r_2\}$ *such that, for* $(x,y) \in J$, *we have*

$$\begin{cases} \|\mu(x,y)\|_1 \leq \alpha_1 e^{\lambda(x+y)}, \\ \int_0^a \int_0^b \|f(x,y,s,t,0,0,0)\|^{\frac{1}{r_3}} dtds \leq \alpha_2^{\frac{1}{r_3}} e^{\frac{\lambda}{r_3}(x+y)}, \\ \int_0^a \int_0^b \|{}^cD^{r_1}_{0,x}f(x,y,s,t,0,0,0)\|^{\frac{1}{r_3}} dtds \leq \alpha_3^{\frac{1}{r_3}} e^{\frac{\lambda}{r_3}(x+y)}, \\ \int_0^a \int_0^b \|{}^cD^{r_2}_{0,y}f(x,y,s,t,0,0,0)\|^{\frac{1}{r_3}} dtds \leq \alpha_4^{\frac{1}{r_3}} e^{\frac{\lambda}{r_3}(x+y)}, \end{cases} \tag{24}$$

*and*

$$\begin{cases} \int_0^a \int_0^b \rho_1^{\frac{1}{r_3}}(x,y,s,t) e^{\frac{\lambda}{r_3}(s+t)} dtds \leq \beta_1^{\frac{1}{r_3}} e^{\frac{\lambda}{r_3}(x+y)}, \\ \int_0^a \int_0^b \rho_2^{\frac{1}{r_3}}(x,y,s,t) e^{\frac{\lambda}{r_3}(s+t)} dtds \leq \beta_2^{\frac{1}{r_3}} e^{\frac{\lambda}{r_3}(x+y)}, \\ \int_0^a \int_0^b \rho_3^{\frac{1}{r_3}}(x,y,s,t) e^{\frac{\lambda}{r_3}(s+t)} dtds \leq \beta_3^2 e^{2\lambda(x+y)}. \end{cases} \tag{25}$$

*If*

$$\frac{(\beta_1+\beta_2+\beta_3)a^{(\omega_1+1)(1-r_3)}b^{(\omega_2+1)(1-r_3)}}{(\omega_1+1)^{(1-r_3)}(\omega_2+1)^{(1-r_3)}\Gamma(r_1)\Gamma(r_2)} e^{\lambda(x+y)} < 1, \tag{26}$$

*where* $\omega_1 = \frac{r_1-1}{1-r_3}$, $\omega_2 = \frac{r_2-1}{1-r_3}$, *then the equation (20) has a unique solution on* $J$ *in* $E$.

**Corollary 2.** *Assume*

$(H_3')$ *For* $\lambda$ *as in (4), there exist constants* $\alpha > 0$, $0 < r_3 < \min\{r_1, r_2\}$, *and strictly positive functions* $m_1(x,y), m_2(x,y) \in L^{\frac{1}{r_3}}(J)$ *such that for each* $(x,y),(s,t) \in J$, $\mu$, $f$ *and* ${}^cD^r_\theta(f)$ *satisfy*

$$\|\mu(x,y)\|_1 \le \alpha e^{\lambda(x+y)} \tag{27}$$

*and*

$$\begin{cases} \|f(x,y,s,t,0,0,0)\| \le m_1(s,t)e^{\lambda(x+y)}, \\ \|{}^cD^r_\theta f(x,y,s,t,0,0,0)\| \le m_2(s,t)e^{\lambda(x+y)}, \end{cases} \tag{28}$$

$(H_4')$ *There exist strictly positive functions* $m_3(x,y), m_4(x,y), m_5(x,y) \in L^{\frac{1}{r_3}}(J)$ *such that for each* $(x,y),(s,t) \in J$ *and* $u, v, \overline{u}, \overline{v} \in \mathbb{R}^n$, $f$ *and* ${}^cD^r_\theta(f)$ *satisfy*

$$\begin{aligned} &\|f(x,y,s,t,u,v,w) - f(x,y,s,t,\overline{u},\overline{v}),\overline{w}\| \\ &\qquad \le m_3(s,t)e^{\lambda(x+y-s-t)}(\|u-\overline{u}\| + \|v-\overline{v}\| + \|w-\overline{w}\|), \end{aligned} \tag{29}$$

$$\begin{aligned} &\|({}^cD^{r_1}_{0,x}f)(x,y,s,t,u,v,w) - ({}^cD^{r_1}_{0,x}f)(x,y,s,t,\overline{u},\overline{v},\overline{w})\| \\ &\qquad \le m_4(s,t)e^{\lambda(x+y-s-t)}(\|u-\overline{u}\| + \|v-\overline{v}\| + \|w-\overline{w}\|) \end{aligned} \tag{30}$$

*and*

$$\begin{aligned} &\|({}^cD^{r_2}_{0,y}f)(x,y,s,t,u,v,w) - ({}^cD^{r_2}_{0,y}f)(x,y,s,t,\overline{u},\overline{v},\overline{w})\| \\ &\qquad \le m_5(s,t)e^{\lambda(x+y-s-t)}(\|u-\overline{u}\| + \|v-\overline{v}\| + \|w-\overline{w}\|). \end{aligned} \tag{31}$$

*If*

$$\frac{(M_3+M_4+M_5)a^{(\omega_1+1)(1-r_3)}b^{(\omega_2+1)(1-r_3)}}{(\omega_1+1)^{(1-r_3)}(\omega_2+1)^{(1-r_3)}\Gamma(r_1)\Gamma(r_2)} < 1, \tag{32}$$

*where*

$$\omega_1 = \frac{r_1-1}{1-r_3}, \quad \omega_2 = \frac{r_2-1}{1-r_3}, \quad M_i = \|m_i\|_{L^{\frac{1}{r_3}}}; \;\; i \in \{3,4,5\},$$

*then the equation (20) has a unique solution on* $J$ *in* $E$.

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SAÏD ABBAS
2320, RUE DE SALABERRY, APT 10
MONTRÉAL, QC H3M 1K9, CANADA
*e-mail:* abbasmsaid@yahoo.fr

MOUFFAK BENCHOHRA
LABORATOIRE DE MATHÉMATIQUES
UNIVERSITÉ DE SIDI BEL-ABBÈS
B.P. 89, 22000, SIDI BEL-ABBÈS, ALGÉRIE
*e-mail:* benchohra@univ-sba.dz

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