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A NOTE ON THE MODIFIED q-GENOCCHI NUMBERS AND POLYNOMIALS WITH WEIGHT (α,β)

Abstract. The purpose of this paper concerns to establish modified q-Genocchi numbers and polynomials with weight (α,β). In this paper we investigate special generalized q-Genocchi polynomials and we apply the method of generating function, which are exploited to derive further classes of q-Genocchi polynomials and develop q-Genocchi numbers and polynomials. By using the Laplace-Mellin transformation integral, we define q-Zeta function with weight (α,β) and by presenting a link between q-Zeta function with weight (α,β) and q-Genocchi numbers with weight (α,β) we obtain an interpolation formula for the q-Genocchi numbers and polynomials with weight (α,β). Also we derive distribution formula (Multiplication Theorem) and Witt's type formula for modified q-Genocchi numbers and polynomials with weight (α,β) which yields a deeper insight into the effectiveness of this type of generalizations for q-Genocchi numbers and polynomials. Our new generating function possess a number of interesting properties which we state in this paper.

Key words: Genocchi numbers and polynomials, q-Genocchi numbers and polynomials, q-Genocchi numbers and polynomials with weight α.

AMS Mathematics Subject Classification: 46A15, 41A65.

1. Introduction, definitions and notations

Recently, q-calculus has served as a bridge between mathematics and physics. Therefore, there is a significant increase of activity in the area of the q-calculus due to applications of the q-calculus in mathematics, statistics and physics. The majority of scientists in the world who use q-calculus today are physicists. q-Calculus is a generalization of many subjects, like hypergeometric series, generating functions, complex analysis, and particle physics. In short, q-calculus is quite a popular subject today. One of important branch of q-calculus in number theory is q-type of special generating functions, for instance q-Bernoulli numbers, q-Euler numbers, and q-Genocchi numbers, here we introduce a new class of q-type generating function. We introduce q-Genocchi numbers with weight (α, β). When we define a new class of generating functions like, q-Genocchi numbers with weight (α, β), then we face to with this question that "can we define a new q-Zeta type function in related of this new class of q-type generating function?". We give a positive answer for our new class of numbers and polynomials. More precisely we show that our q-type generating function is generalization of the Hurwitz Zeta function. Historically many authors have tried to give q-analogues of the Riemann Zeta function $\zeta(s)$, and its related functions. By just following the method of Kaneko et al. [M. Kaneko, N. Kurokawa and M. Wakayama, A variation of Euler's approach to the Riemann Zeta function, Kyushu J. Math. 57 (2003), 175–192], who mainly used Euler-Maclaurin summation formula to present and investigate a q-analogue of the Riemann zeta function $\zeta(s)$, and gave a good and reasonable explanation that their q-analogue may be a best choice. They also commented that q-analogue of $\zeta(s)$ can be achieved by modifying their method. Furthermore it is clear that q-Genocchi polynomials of weight (α, β) are in a class of orthogonal polynomials and we know that most such special functions that are orthogonal are satisfied in multiplication theorem, so in this present paper we show this property is true for q-Genocchi polynomials of weight (α, β). In this introductory section, we present the definitions and notations (and some of the Important properties and characteristics) of the various special functions, polynomials and numbers, which are potentially useful in the remainder of the paper.

Assume that p be a fixed odd prime number. Throughout this paper we use the following notations. By $\mathbb{Z}_p$ we denote the ring of p-adic rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = p^{-1}$. When one speaks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$ or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, we assume that $|1-q|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. We use the following notation:

$$[x]_q = \frac{1-q^x}{1-q}, \quad [x]_{-q} = \frac{1-(-q)^x}{1+q}$$

Note that $\lim_{q \to 1} [x]_q = x$; cf. [1-24].

For a fixed positive integer d with $(d, f) = 1$, we set

$$X = X_d = \varprojlim_{N} \mathbb{Z}/dp^N\mathbb{Z},$$

$$X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} a + dp\mathbb{Z}_p$$

and

$$a + dp^N\mathbb{Z}_p = \left\{x \in X \mid x \equiv a \left(\bmod dp^N \right) \right\},$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \leq a < dp^N$.

By using Koblitz's [N. Koblitz, p-adic Numbers p-adic Analysis and Zeta Functions, Springer-Verlag, New York Inc, 1977] notations, a p-adic distribution μ on X is a $\mathbb{Q}_p$-linear vector space homomorphism from the $\mathbb{Q}_p$-vector space of locally constant functions on X to $\mathbb{Q}_p$. If $f : X \to \mathbb{Q}_p$ is locally constant, instead of writing $\mu(f)$ for the value of μ at f, we usually write $\int f\mu$. Also it is known that we can write μ_q as follows:

$$\mu_q \left(x + p^N\mathbb{Z}_p\right) = \frac{q^x}{[p^N]_q}$$

is a distribution on X for $q \in \mathbb{C}_p$ with $|1-q|_p \leq 1$. For

$$f \in UD(\mathbb{Z}_p) = \{f \mid f : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\},$$

the following fermionic p-adic q-integral on $\mathbb{Z}_p$ is defined by using Kim's measure μ_q:

$$\begin{aligned} I_{-q}(f) &= \int_{\mathbb{Z}_p} f(x)\, d\mu_{-q}(x) \\ &= \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)\, \mu_{-q}\left(x + p^N\mathbb{Z}_p\right) \\ &= \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} (-1)^x f(x)\, q^x. \end{aligned} \tag{1}$$

Let $q \to 1$, then we have fermionic integration on $\mathbb{Z}_p$ as follows:

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x)\, d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} (-1)^x f(x),$$

So by applying $f(x) = e^{xt}$, we get

$$t\int_{\mathbb{Z}_p} e^{tx} d\mu_{-1}(x) = \frac{2t}{e^t+1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}. \tag{2}$$

Where G_n are Genocchi numbers. By using (2), we have

$$\int_{\mathbb{Z}_p} e^{xt} d\mu_{-1}(x) = \sum_{n=0}^{\infty} \frac{G_{n+1}}{n+1}\frac{t^n}{n!}$$

so from above, we obtain

$$\sum_{n=0}^{\infty}\left(\int_{\mathbb{Z}_p} x^n d\mu_{-1}(x)\right)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\left(\frac{G_{n+1}}{n+1}\right)\frac{t^n}{n!}.$$

By comparing coefficients of $\frac{t^n}{n!}$ on both sides of the above equation it is fairly straightforward to deduce,

$$\frac{G_{n+1}}{n+1}=\int_{\mathbb{Z}_p} x^n d\mu_{-1}(x).$$

The definition of modified q-Euler numbers are given by

$$\varepsilon_{0,q}=\frac{[2]_q}{2},\ (q\varepsilon+1)^k-\varepsilon_{k,q}=\begin{cases}[2]_q, & k=0,\\ 0, & k>0,\end{cases} \tag{3}$$

with usual the convention about replacing ε^k by $\varepsilon_{k,q}$ cf. [11], [24]. It was known that the modified q-euler numbers can be represented by p-adic q-integral on $\mathbb{Z}_p$ as follows:

$$\varepsilon_{n,q}=\int_{\mathbb{Z}_p} q^{-t}[t]_q^n d\mu_{-q}(t).$$

In [3, 14, 15, 17], q-Genocchi numbers are defined as follows:

$$G_{0,q}=0, \quad \text{and} \quad q(qG_q+1)^n+G_{n,q}=\begin{cases}[2]_q, & n=1\\ 0, & n>1,\end{cases} \tag{4}$$

with the usual convention of replacing $(G_q)^n$ by $G_{n,q}$.

In [7], (h,q)-Genocchi numbers are indicated as:

$$G_{0,q}^{(h)}=0, \quad \text{and} \quad q^{h-2}\left(qG_q^{(h)}+1\right)^n+G_{n,q}^{(h)}=\begin{cases}[2]_q, & n=1\\ 0, & n>1,\end{cases}$$

with the usual convention about replacing $\left(G_q^{(h)}\right)^n$ by $G_{n,q}^{(h)}$.

Recently, for $n\in\mathbb{Z}_+$, Araci et al. are considered weighted q-Genocchi numbers by

$$\widetilde{G}_{0,q}^{(\alpha)}=0,\ q^{1-\alpha}\left(q\widetilde{G}_q^{(\alpha)}+1\right)^n+\widetilde{G}_{n,q}^{(\alpha)}=\begin{cases}[2]_q, & n=1\\ 0, & n\neq 1,\end{cases} \tag{5}$$

with the usual convention about replacing $(\widetilde{G}_q)^n$ by $\widetilde{G}_{n,q}$ (for more information, see [4]).

For $\alpha, n \in \mathbb{Z}_+$ and $h \in \mathbb{N}$, Araci et al. [5] defined weighted (h,q) -Genocchi numbers as follows:

$$\widetilde{G}_{n+1,q}^{(\alpha,h)} = \int_{\mathbb{Z}_p} q^{(h-1)x} [x]_{q^\alpha}^n \, d\mu_{-q}(x).$$

Taekyun Kim, by using p-adic q-integral on $\mathbb{Z}_p$, introduced a new class of numbers and polynomials. He added a weight on q-Bernoulli numbers and polynomials and defined q-Bernoulli numbers with weight α. He gave some interesting properties concerning q-Bernoulli numbers and polynomials with weight α. After, by using p-adic q-integral on $\mathbb{Z}_p$, several mathematicians started to study on this new branch of generating function theory and extended most of the symmetric properties of q-Bernoulli numbers and polynomials to q-Bernoulli numbers and polynomials with weight α (for more information, see [4], [5], [1], [2], [7], [8], [23], [19], [6], [20], [22]). With the same motivation, we also introduce modified q-Genocchi numbers and polynomials with weight (α,β). Also, we give some interesting properties this type of polynomials. Furthermore, we derive the q-extensions of zeta type functions with weight (α,β) from the Mellin transformation to this generating function which interpolates the q-Genocchi polynomials with weight (α,β) at negative integers.

2. Modified q-Genocchi numbers and polynomials with weight (α,β)

In this section, we derive some interesting properties Modified q-Genocchi numbers and polynomials with weight (α,β).

Lemma 1. *For $n \in \mathbb{Z}_+$, we obtain*

$$I_{-q}^{(\beta)}\left(q^{-\beta x} f_n\right) + (-1)^{n-1} I_{-q}^{(\beta)}\left(q^{-\beta x} f\right) = [2]_{q^\beta} \sum_{l=0}^{n-1} (-1)^{n-l-1} f(l), \tag{6}$$

Proof. Let be $f_n(x) = f(x+n)$ and $I_{-q}^{(\beta)}(f) = \int_{\mathbb{Z}_p} f(x)\, d\mu_{-q^\beta}(x)$ by the (1), we easily get

$$\begin{aligned} -I_{-q}^{(\beta)}\left(q^{-\beta x} f_1\right) &= \lim_{N\to\infty} \frac{1}{[p^N]_{-q^\beta}} \sum_{x=0}^{p^N-1} f(x+1)(-1)^x \\ &= \lim_{N\to\infty} \frac{1}{[p^N]_{-q^\beta}} \sum_{x=0}^{p^N-1} f(x)(-1)^x - [2]_{q^\beta} \lim_{N\to\infty} \frac{f\left(p^N\right) + f(0)}{1+q^{\beta p^N}} \\ &= I_{-q}^{(\beta)}\left(q^{-\beta x} f\right) - [2]_{q^\beta} f(0) \end{aligned} \tag{7}$$

and

$$
\begin{aligned}
I_{-q}^{(\beta)}\left(q^{-\beta x} f_2\right) &= \int_{\mathbb{Z}_p} q^{-\beta x} f(x+2)\, d\mu_{-q^\beta}(x) \\
&= \lim_{N\to\infty} \frac{1}{[p^N]_{-q^\beta}} \sum_{x=0}^{p^N-1} f(x+2)(-1)^x \\
&= I_{-q}^{(\beta)}\left(q^{-\beta x} f\right) + [2]_{q^\beta} \lim_{N\to\infty} \frac{-f(0)+f(1)-f\left(p^N\right)+f\left(p^N+1\right)}{1+q^{\beta p^N}} \\
&= I_{-q}^{(\beta)}\left(q^{-\beta x} f\right) + [2]_{q^\beta} \left(f(1)-f(0)\right).
\end{aligned}
$$

Thus, we have

$$
I_{-q}^{(\beta)}\left(q^{-\beta x} f_2\right) - I_{-q}^{(\beta)}\left(q^{-\beta x} f\right) = [2]_{q^\beta} \sum_{l=0}^{1} (-1)^{1-l} f(l)
$$

By continuing this process, we arrive at the desired result. ■

Definition 1. *Let $\alpha, n, \beta \in \mathbb{Z}_+$. We define modified q-Genocchi numbers with weight (α, β) as follows:*

$$
\frac{g_{n+1,q}^{(\alpha,\beta)}}{n+1} = [2]_{q^\beta} \sum_{m=0}^{\infty} (-1)^m [m]_{q^\alpha}^n . \tag{8}
$$

Theorem 1. *For $\alpha, n, \beta \in \mathbb{Z}_+$, we get*

$$
\frac{g_{n+1,q}^{(\alpha,\beta)}}{n+1} = \frac{[2]_{q^\beta}}{(1-q^\alpha)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{1+q^{\alpha l}}. \tag{9}
$$

Proof. By (8), we develop as follows:

$$
\begin{aligned}
\frac{g_{n+1,q}^{(\alpha,\beta)}}{n+1} &= \frac{[2]_{q^\beta}}{(1-q^\alpha)^n} \sum_{m=0}^{\infty} (-1)^m (1-q^{m\alpha})^n \\
&= \frac{[2]_{q^\beta}}{(1-q^\alpha)^n} \sum_{m=0}^{\infty} (-1)^m \sum_{l=0}^{n} \binom{n}{l} (-1)^l (q^{m\alpha})^l \\
&= \frac{[2]_{q^\beta}}{(1-q^\alpha)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \sum_{m=0}^{\infty} (-1)^m q^{m\alpha l} \\
&= \frac{[2]_{q^\beta}}{(1-q^\alpha)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{1+q^{\alpha l}}.
\end{aligned}
$$

Thus, we complete the proof of Theorem. ■

By the following Theorem, we get Witt's type formula of this type polynomials.

Theorem 2. *For $\beta, \alpha, n \in \mathbb{Z}_+$, we get*

$$\frac{g_{n+1,q}^{(\alpha,\beta)}}{n+1} = \int_{\mathbb{Z}_p} q^{-\beta x} [x]_{q^\alpha}^n \, d\mu_{-q^\beta}(x). \tag{10}$$

Proof. By using p-adic q-integral on $\mathbb{Z}_p$, namely, replace $f(x)$ by $q^{-\beta x} [x]_{q^\alpha}^n$ and $\mu_{-q}(x + p^N \mathbb{Z}_p)$ by $\mu_{-q^\beta}(x + p^N \mathbb{Z}_p)$ into (1), we get

$$\begin{aligned}
&\int_{\mathbb{Z}_p} q^{-\beta x} [x]_{q^\alpha}^n \, d\mu_{-q^\beta}(x) \\
&= \frac{1}{(1-q^\alpha)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l \int_{\mathbb{Z}_p} q^{\alpha l x - \beta x} d\mu_{-q^\beta}(x) \\
&= \frac{1}{(1-q^\alpha)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l \lim_{N\to\infty} \frac{1}{[p^N]_{-q^\beta}} \sum_{x=0}^{p^N-1} \left(-q^{\alpha l}\right)^x \\
&= \frac{1}{(1-q^\alpha)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{[2]_{q^\beta}}{1+q^{\alpha l}} \lim_{N\to\infty} \frac{1+\left(q^{\alpha l}\right)^{p^N}}{1+q^{\beta p^N}} \\
&= \frac{[2]_{q^\beta}}{(1-q^\alpha)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{1}{1+q^{\alpha l}}.
\end{aligned} \tag{11}$$

Use of (9) and (11), we arrive at the desired result. ∎

The Witt's type formula of modified q-Genocchi numbers with weight (α, β) asserted by Theorem 2, do aid in translating the various properties and results involving q-Genocchi numbers with weight (α, β) which we state some of them in this section. We put $\alpha \to 1$ and $\beta \to 1$ into (10), we readily see $\frac{g_{n+1,q}^{(1,1)}}{n+1} = \varepsilon_{n,q}$.

Corollary 1. *Let $C_q^{(\alpha,\beta)}(t) = \sum_{n=0}^\infty g_{n,q}^{(\alpha,\beta)} \frac{t^n}{n!}$. Then we have*

$$C_q^{(\alpha,\beta)}(t) = [2]_{q^\beta} t \sum_{m=0}^\infty (-1)^m e^{t[m]_{q^\alpha}}.$$

Proof. From (8) we easily get,

$$\int_{\mathbb{Z}_p} q^{-\beta x} e^{t[x]_{q^\alpha}} d\mu_{-q^\beta}(x) = [2]_{q^\beta} t \sum_{m=0}^\infty (-1)^m e^{t[m]_{q^\alpha}}. \tag{12}$$

By expression (12), we have

$$\sum_{n=0}^\infty g_{n,q}^{(\alpha,\beta)} \frac{t^n}{n!} = [2]_{q^\beta} t \sum_{m=0}^\infty (-1)^m e^{t[m]_{q^\alpha}}.$$

Thus, we complete the proof of Theorem. ■

Now, we consider the modified q-Genocchi polynomials polynomials with weight α as follows:

$$\frac{g_{n+1,q}^{(\alpha,\beta)}(x)}{n+1} = \int_{\mathbb{Z}_p} q^{-\beta t} \left[x+t\right]_{q^\alpha}^n d\mu_{-q^\beta}(t), \quad n \in \mathbb{N} \text{ and } \alpha \in \mathbb{Z}_+ \tag{13}$$

From expression (13), we see readily

$$\begin{aligned} \frac{g_{n+1,q}^{(\alpha,\beta)}(x)}{n+1} &= \frac{[2]_{q^\beta}}{(1-q^\alpha)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{\alpha l x} \frac{1}{1+q^{\alpha l}} \\ &= [2]_{q^\beta} \sum_{m=0}^{\infty} (-1)^m [m+x]_{q^\alpha}^n . \end{aligned} \tag{14}$$

Let $C_q^{(\alpha,\beta)}(t,x) = \sum_{n=0}^\infty g_{n,q}^{(\alpha,\beta)}(x) \frac{t^n}{n!}$. Then we have

$$C_q^{(\alpha,\beta)}(t,x) = [2]_{q^\beta} t \sum_{m=0}^{\infty} (-1)^m e^{t[m+x]_{q^\alpha}} = \sum_{n=0}^{\infty} g_{n,q}^{(\alpha,\beta)}(x) \frac{t^n}{n!}. \tag{15}$$

By Lemma 1, we get the following Theorem:

Theorem 3. *For $m \in \mathbb{N}$, and $\alpha, \beta, n \in \mathbb{Z}_+$, we get*

$$\frac{g_{m+1,q}^{(\alpha,\beta)}}{m+1} + (-1)^{n-1} \frac{g_{m+1,q}^{(\alpha,\beta)}(n)}{m+1} = [2]_{q^\beta} \sum_{l=0}^{n-1} (-1)^{n-l-1} [l]_{q^\alpha}^m .$$

Proof. By applying Lemma 1 the methodology and techniques used above in getting some identities for the generating functions of the modified q-Genocchi numbers and polynomials with weight (α, β), we arrive at the desired result. ■

Theorem 4. *The following identity holds:*

$$g_{0,q}^{(\alpha,\beta)} = 0, \quad \textit{and} \quad g_{n,q}^{(\alpha,\beta)}(1) + g_{n,q}^{(\alpha,\beta)} = \begin{cases} [2]_{q^\beta}, & \textit{if} \quad n=1, \\ 0, & \textit{if} \quad n>1, \end{cases}$$

Proof. In (7) it is known that

$$I_{-q}^{(\beta)}\left(q^{-\beta x} f_1\right) + I_{-q}^{(\beta)}\left(q^{-\beta x} f\right) = [2]_{q^\beta} f(0).$$

If we take $f(x) = e^{t[x]_{q^\alpha}}$, then we have

$$(16)\quad [2]_{q^\beta} = \int_{\mathbb{Z}_p} q^{-\beta x} e^{t[x+1]_{q^\alpha}} d\mu_{-q^\beta}(x) + \int_{\mathbb{Z}_p} q^{-\beta x} e^{t[x]_{q^\alpha}} d\mu_{-q^{-\beta}}(x)$$
$$= \sum_{n=0}^{\infty} \left(g_{n,q}^{(\alpha,\beta)}(1) + g_{n,q}^{(\alpha,\beta)} \right) \frac{t^{n-1}}{n!}.$$

Therefore, we get the Proof of Theorem. ■

Theorem 5. *For $d \equiv 1 \pmod 2$, $\alpha, \beta \in \mathbb{Z}_+$ and $n \in \mathbb{N}$, we get,*

$$g_{n,q}^{(\alpha,\beta)}(dx) = \frac{[d]_{q^\alpha}^{n-1}}{[d]_{-q^\beta}} \sum_{a=0}^{d-1} (-1)^a g_{n,q^d}^{(\alpha,\beta)} \left(x + \frac{a}{d} \right).$$

Proof. From (13), we can easily derive the following (17)

$$(17)\quad \int_{\mathbb{Z}_p} q^{-\beta t} [x+t]_{q^\alpha}^n d\mu_{-q^\beta}(t)$$
$$= \frac{[d]_{q^\alpha}^n}{[d]_{-q^\beta}} \sum_{a=0}^{d-1} (-1)^a \int_{\mathbb{Z}_p} q^{-\beta t} \left[\frac{x+a}{d} + t \right]_{q^{d\alpha}}^n d\mu_{(-q^d)^\beta}(t)$$
$$= \frac{[d]_{q^\alpha}^n}{[d]_{-q^\beta}} \sum_{a=0}^{d-1} (-1)^a \frac{g_{n+1,q^d}^{(\alpha,\beta)}\left(\frac{x+a}{d}\right)}{n+1}.$$

So, by applying expression (17), we get at the desired result and proof is complete. ■

3. Interpolation function of the polynomials $g_{n,q}^{(\alpha,\beta)}(x)$

In this section, we derive the interpolation function of the generating functions of modified q-Genocchi polynomials with weight α and we give the value of q-extension zeta function with weight (α, β) at negative integers explicitly. For $s \in \mathbb{C}$, by applying the Mellin transformation to (15), we obtain

$$\xi^{(\alpha,\beta)}(s, x \mid q) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-2} \left\{ -C_q^{(\alpha,\beta)}(-t, x) \right\} dt$$
$$= [2]_{q^\beta} \sum_{m=0}^{\infty} (-1)^m \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-t[m+x]_{q^\alpha}} dt$$

where $\Gamma(s)$ is Euler gamma function. We have

$$\xi^{(\alpha,\beta)}(s,x \mid q) = [2]_{q^\beta} \sum_{m=0}^{\infty} \frac{(-1)^m}{[m+x]_{q^\alpha}^s}$$

So, we define q-extension zeta function with weight (α,β) as follows:

Definition 2. *For $s \in \mathbb{C}$ and $\alpha,\beta \in \mathbb{N}$, we have*

$$\xi^{(\alpha,\beta)}(s,x \mid q) = [2]_{q^\beta} \sum_{m=0}^{\infty} \frac{(-1)^m}{[m+x]_{q^\alpha}^s} \tag{18}$$

$\xi^{(\alpha,\beta)}(s,x \mid q)$ can be continued analytically to an entire function.

Observe that, if $q \to 1$, then $\xi^{(\alpha,\beta)}(s,x \mid 1) = \zeta(s,x)$ which is the Hurwitz-Euler zeta functions. Relation between $\xi^{(\alpha,\beta)}(s,x \mid q)$ and $g_{n,q}^{(\alpha,\beta)}(x)$ are given by the following theorem:

Theorem 6. *For $\alpha,\beta \in \mathbb{N}$ and $n \in \mathbb{N}$, we get*

$$\xi^{(\alpha,\beta)}(-n,x \mid q) = \frac{g_{n+1,q}^{(\alpha,\beta)}(x)}{n+1}.$$

Proof. By substituting $s=-n$ into (18), we arrive at the desired result. ■

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References

[1] Araci S., Aslan N., Seo J-J., A note on the weighted Twisted Dirichlet's type q-Euler numbers and polynomials, *Honam Mathematical J.,* 33(3)(2011), 311-320.

[2] Araci S., Acikgoz M., Seo J-J., A study on the weighted q-Genocchi numbers and polynomials with their Interpolation Function, *Honam Mathematical J.,* 34(1)(2012), 11-18.

[3] Araci S., Erdal D., Kang D-J., Some new properties on the q-Genocchi numbers and polynomials associated with q-Bernstein polynomials, *Honam Mathematical J.,* 33(2)(2011), 261-270.

[4] Araci S., Erdal D., Seo J-J., A study on the fermionic p-adic q-integral representation on $\mathbb{Z}_p$ associated with weighted q-Bernstein and q-Genocchi polynomials, *Abstract and Applied Analysis,* Vol. 2011, Article ID 649248, 10 pages.

[5] Araci S., Seo J-J., Erdal D., New construction weighted (h, q)-Genocchi numbers and polynomials related to zeta type functions, *Discrete Dynamics in Nature and Society,* Vol. 2011, Article ID 487490, 7 pages, doi:10.11 55/2011/487490.

[6] Hwang K-W., Dolgy D-V., Lee S.H., Kim T., On the Higher-Order q-Euler numbers and polynomials with weight α, *Discrete Dynamics in Nature and Society,* (2011), Article ID 354329, 12 pages.

[7] Jolany H., Araci S., Acikgoz M., Seo J-J., A note on the generalized q-Genochhi measures with weight alpha, *Bol. Soc. Paran. Mat.,* 31(1)(2013), 17-27.

[8] Kim T., On the weighted q-Bernoulli numbers and polynomials, *Advanced Studies in Contemporary Mathematics,* 21(2)(2011), 207-215, http://arxiv.org/abs/1011.5305.

[9] Kim T., On the q-extension of Euler and Genocchi numbers, *J. Math. Anal. Appl.,* 326(2007), 1458-1465.

[10] Kim T., On the multiple q-Genocchi and Euler numbers, *Russian J. Math. Phys.,* 15(4)(2008), 481-486. arXiv:0801.0978v1[math.NT].

[11] Kim T., The modified q-Euler numbers and polynomials, *Advn. Stud. Contemp. Math.,* 16(2008), 161-170.

[12] Kim T., q-Volkenborn integration, *Russ. J. Math. Phys.,* 9(2002), 288-299.

[13] Kim T., q-Bernoulli numbers and polynomials associated with Gaussian binomial coefficients, *Russ. J. Math. Phys.,* 15(2008), 51-57.

[14] Kim T., An invariant p-adic q-integrals on $\mathbb{Z}_p$, *Applied Mathematics Letters,* 21(2008), 105-108.

[15] Kim T., *A note on the q-Genocchi Numbers and Polynomials,* Journal of Inequalities and Applications, Article ID 71452, 8 pages, doi:10.1155/2007/71452.

[16] Kim T., q-Euler numbers and polynomials associated with p-adic q-integrals, *J. Nonlinear Math. Phys.,* 14(1)(2007), 15-27.

[17] Kim T., New approach to q-Euler polynomials of higher order, *Russ. J. Math. Phys.,* 17(2)(2010).

[18] Kim T., Some identities on the q-Euler polynomials of higher order and q-Stirling numbers by the fermionic p-adic integral on $\mathbb{Z}_p$, *Russ. J. Math. Phys.,* 16(4)(2009), 484-491.

[19] Kim T., Choi J., Kim Y.H., Ryoo C.S., A note on the weighted p-adic q-Euler measure on $\mathbb{Z}_p$, *Advn. Stud. Contemp. Math.,* 21(2011), 35-40.

[20] Kim T., Choi J., On the q-Bernoulli numbers and polynomials with weight α, *Abstract and Applied Analysis,* (2011), Article ID 392025, 14 pages.

[21] Kim T., Choi J., Kim Y.H. and Jang L.C., *On p-Adic Analogue of q-Bernstein Polynomials and Related Integrals,* Discrete Dynamics in Nature and Society, Article ID 179430, 9 pages, doi:10.1155/2010/179430.

[22] Kim T., Dolgy D-V., Lee B., Rim S-H., Identities on the weighted q-Euler

numbers of higher order, *Discrete Dynamics in Nature and Society,* (2011), Article ID 918364, 6 pages.

[23] Kim T., Lee S.H., Dolgy D.V., Ryoo C.S., A note on the generalized q-Bernoulli measures with weight α, *Abstract and Applied Analysis,* Article ID 867217, 9 pages, doi:10.1155/2011/867217.

[24] Ozden H., Simsek Y., Rim S-H., Cangul I-N., A note on p-adic q-Euler measure, *Adv. Stud. Contemp. Math.,* 14(2007), 233-239.

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