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SIGNED STAR $\{k\}$-DOMATIC NUMBER OF A GRAPH

Abstract. Let G be a simple graph without isolated vertices with vertex set $V(G)$ and edge set $E(G)$ and let k be a positive integer. A function $f : E(G) \longrightarrow \{\pm1, \pm2, \ldots, \pm k\}$ is said to be a signed star $\{k\}$-dominating function on G if $\sum_{e\in E(v)} f(e) \geq k$ for every vertex v of G, where $E(v) = \{uv \in E(G) \mid u \in N(v)\}$. The *signed star $\{k\}$-domination number* of a graph G is $\gamma_{\{k\}SS}(G) = \min\{\sum_{e\in E} f(e) \mid f$ is a SS{k}DF on $G\}$. A set $\{f_1, f_2, \ldots, f_d\}$ of distinct signed star $\{k\}$-dominating functions on G with the property that $\sum_{i=1}^{d} f_i(e) \leq k$ for each $e \in E(G)$, is called a signed star $\{k\}$-dominating family (of functions) on G. The maximum number of functions in a signed star $\{k\}$-dominating family on G is the signed star $\{k\}$-domatic number of G, denoted by $d_{\{k\}SS}(G)$. In this paper we study the properties of the signed star $\{k\}$- domination number $\gamma_{\{k\}SS}(G)$ and signed star $\{k\}$-domatic number $d_{\{k\}SS}(G)$. In particular, we determine the signed star $\{k\}$- domination number of some classes of graphs. Some of our results extend these one given by Xu [7] for the signed star domination number and Atapour et al. [1] for the signed star domatic number.

Key words: signed star $\{k\}$-domatic number, signed star domatic number, signed star $\{k\}$-dominating function, signed star dominating functions, signed star $\{k\}$-domination number, signed star domination number, regular graphs.

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1. Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We use [2, 6] for terminology and notation which are not defined here and consider simple graphs without isolated vertices only. For every nonempty subset E' of $E(G)$, the subgraph $G[E']$ induced by E' is the graph whose vertex set consists of those vertices of G incident with at least one edge of E' and whose edge set is E'.

Two edges e_1, e_2 of G are called *adjacent* if they are distinct and have a common vertex. The *open neighborhood* $N_G(e)$ of an edge $e \in E(G)$ is the set of all edges adjacent to e. Its *closed neighborhood* is $N_G[e] = N_G(e) \cup \{e\}$. For a function $f : E(G) \longrightarrow \mathbb{R}$ and a subset S of $E(G)$ we define $f(S) = \sum_{e \in S} f(e)$. The *edge-neighborhood* $E_G(v)$ of a vertex $v \in V(G)$ is the set of all edges incident with the vertex v. For each vertex $v \in V(G)$, we also define $f(v) = \sum_{e \in E_G(v)} f(e)$.

Let k be a positive integer. A function $f : E(G) \longrightarrow \{\pm 1, \pm 2, \ldots, \pm k\}$ is called a *signed star* $\{k\}$*-dominating function* (SS$\{k\}$DF) on G, if $f(v) \geq k$ for every vertex v of G. The *signed star* $\{k\}$*-domination number* of a graph G is $\gamma_{\{k\}SS}(G) = \min\{\sum_{e \in E} f(e) \mid f \text{ is a SS\{k\}DF on } G\}$. The signed star $\{k\}$-dominating function f on G with $f(E(G)) = \gamma_{\{k\}SS}(G)$ is called a $\gamma_{\{k\}SS}(G)$*-function*. The signed star $\{1\}$-domination number of a graph G is the usual signed star domination number $\gamma_{SS}(G)$, which has been introduced by Xu in [7] and has been studied by several authors (see for instance [4, 5, 8, 9]).

A set $\{f_1, f_2, \ldots, f_d\}$ of distinct signed star $\{k\}$-dominating functions on G with the property that $\sum_{i=1}^{d} f_i(e) \leq k$ for each $e \in E(G)$, is called a *signed star* $\{k\}$*-dominating family* (of functions) on G. The maximum number of functions in a signed star $\{k\}$-dominating family on G is the *signed star* $\{k\}$*-domatic number* of G, denoted by $d_{\{k\}SS}(G)$. The signed star $\{k\}$-domatic number is well-defined and $d_{\{k\}SS}(G) \geq 1$ for all graphs G, since the set consisting of any one SS$\{k\}$D function forms a SS$\{k\}$D family on G. A $d_{\{k\}SS}$*-family* of a graph G is a SS$\{k\}$D family containing $d_{\{k\}SS}(D)$ SS$\{k\}$D functions. The signed star $\{1\}$-domatic number $d_{\{1\}SS}(G)$ is the usual signed star domatic number $d_{SS}(G)$ which was introduced by Atapour et al. in [1].

Our purpose in this paper is to initiate the study of signed star $\{k\}$-domination number and signed star $\{k\}$-domatic number in graphs. We first present bounds on signed star $\{k\}$-domination number and then we study basic properties and bounds for the signed star $\{k\}$-domatic number of a graph which some of them are analogous to those of the signed star domatic number $d_{SS}(G)$ in [1]. In addition, we determine the signed star $\{k\}$-domatic number of some classes of graphs.

Observation 1. *Let G be a graph of size m with $\delta(G) \geq 1$. Then $\gamma_{SS}(G) = m$ if and only if each edge $e \in E(G)$ has an endpoint u such that* $\deg(u) = 1$ *or* $\deg(u) = 2$.

Observation 2. *Let G be a graph with $\delta(G) \geq 1$. If v is a vertex of G such that $\delta(G - v) \geq 1$, then*

$$\gamma_{\{k\}SS}(G) \leq \gamma_{\{k\}SS}(G - v) + \max\{k, \deg(v)\}.$$

Proof. Since $\delta(G-v)\ge 1$, there exists a $\gamma_{\{k\}SS}(G-v)$-function f. Let $E_G(v)=\{e_1,e_2,\ldots,e_d\}$. If $\deg(v)=d\ge k$, then define $g:E(G)\to\{\pm1,\pm2,\ldots,\pm k\}$ by $g(e)=f(e)$ for $e\in E(G-v)$ and $g(e_i)=1$ for $i\in\{1,2,\ldots,d\}$. If $d=\deg(v)<k$, then define $g:E(G)\to\{\pm1,\pm2,\ldots,\pm k\}$ by $g(e)=f(e)$ for $e\in E(G-v)$ and $g(e_i)=1$ for $1\le i\le d-1$ and $g(e_d)=k+1-\deg(v)$. In both cases it is easy to see that g is a signed star $\{k\}$-dominating function on G, and therefore we obtain the desired bound $\gamma_{\{k\}SS}(G)\le\gamma_{\{k\}SS}(G-v)+\max\{k,\deg(v)\}$. ■

Observation 3. *Let G be a graph of size m and let $k\ge 2$ be an integer. Then $\gamma_{\{k\}SS}(G)=mk$ if and only if each edge $e\in E(G)$ has an endpoint u such that* $\deg(u)=1$.

Proof. If each edge $e\in E(G)$ has an endpoint u such that $\deg(u)=1$, then trivially $\gamma_{\{k\}SS}(G)=mk$.

Conversely, assume that $\gamma_{\{k\}SS}(G)=mk$. Suppose to the contrary that there exists an edge $e=uv\in E(G)$ such that $\min\{\deg(u),\deg(v)\}\ge 2$. Define $f:E(G)\to\{\pm1,\pm2,\ldots,\pm k\}$ by $f(e)=1$ and $f(e')=k$ for $e'\in E(G)\setminus\{e\}$. Obviously, f is a signed star $\{k\}$-dominating function of G with weight less than mk, a contradiction. This completes the proof. ■

Theorem A. [7] *If G is a graph G with $\delta(G)\ge 3$, then G contains an even cycle.*

Theorem B. [7] *For any graph G of order $n\ge 4$, $\gamma_{SS}(G)\le 2n-4$.*

2. Bounds on the signed star $\{k\}$-domination number

In this section we give some bounds on $\gamma_{\{k\}SS}(G)$.

Theorem 1. *For any graph G of order $n\ge 4$ and any integer $k\ge 2$,*

$$\gamma_{\{k\}SS}(G)\le k(n-1).$$

The bound is sharp for stars.

Proof. We proceed by induction on $m=|E(G)|$. The result is clearly true for $m\le 3$, since $n\ge 4$. Let the statement be true for all graphs of order $n\ge 4$ and size at most $m-1$. Now assume that G is a graph of order $n\ge 4$ and size m.

Assume first that $\delta(G)\ge 3$. By Theorem A, G contains an even cycle C. Let $G'=G-E(C)$, and let f be a $\gamma_{\{k\}SS}(G')$-function. By the induction hypothesis, $\omega(f)\le k(n-1)$. Extending f from G' to G by signing $+1$ and -1 alternating along C, we obtain an SS$\{k\}$DF for G, and hence $\gamma_{\{k\}SS}(G)\le k(n-1)$.

Assume second that $\delta(G)=2$. If v is a vertex of G with $\deg(v)=2$, then $\delta(G-v)\geq 1$ and $|E(G-v)|\leq m-1$. If $|V(G-v)|=3$, then $n=4$, and it is easy to see that $\gamma_{\{k\}SS}(G)\leq k(n-1)$. Let now $|V(G-v)|\geq 4$. Using the induction hypothesis and Observation 2, we obtain

$$\gamma_{\{k\}SS}(G)\leq \gamma_{\{k\}SS}(G-v)+k\leq k(n-2)+k=k(n-1).$$

Assume third that $\delta(G)=1$. If $\Delta(G)=1$, then G is isomorphic to pK_2 with $p=n/2$. We observe that $\gamma_{\{k\}SS}(G)=nk/2\leq k(n-1)$. Let now $\Delta(G)\geq 2$, and let H be a component of G with $\Delta(H)\geq 2$.

If $\delta(H)=1$, then let v be a vertex of H with $\deg_H(v)=1$. Obviously, $\delta(G-v)\geq 1$ and $|E(G-v)|\leq m-1$. If $|V(G-v)|=3$, then $n=4$, and it is easy to see that $\gamma_{\{k\}SS}(G)\leq k(n-1)$. Let now $|V(G-v)|\geq 4$. Using the induction hypothesis and Observation 2, we deduce that

$$\gamma_{\{k\}SS}(G)\leq \gamma_{\{k\}SS}(G-v)+k\leq k(n-2)+k=k(n-1).$$

If $\delta(H)=2$, then let v be a vertex of H with $\deg_H(v)=2$. Obviously, $\delta(G-v)\geq 1$ and $4\leq |E(G-v)|\leq m-1$. Using the induction hypothesis and Observation 2, we deduce that

$$\gamma_{\{k\}SS}(G)\leq \gamma_{\{k\}SS}(G-v)+k\leq k(n-2)+k=k(n-1).$$

Finally assume that $\delta(H)\geq 3$. Using the arguments as in the case $\delta(G)\geq 3$, we obtain the desired result.

Clearly, if G is isomorphic to the star $K_{1,n-1}$, then $\gamma_{\{k\}SS}(G)=k(n-1)$, and the proof is complete. ■

Theorem 2. *For all graphs G of order n and $\delta(G)\geq 1$, $\gamma_{\{k\}SS}(G)\geq \lceil\frac{nk}{2}\rceil$.*

Proof. Suppose that f is a $\gamma_{\{k\}SS}(G)$-function. Then

$$\gamma_{\{k\}SS}(G)=\sum_{e\in E(G)} f(e)=\frac{1}{2}\sum_{v\in V(G)}\sum_{e\in E(v)} f(e)\geq \frac{1}{2}\sum_{v\in V(G)} k=\frac{nk}{2},$$

as desired. ■

Theorem 3. *Let G be an r-regular and 1-factorable graph and let $k\geq 2$ be an integer. Then $\gamma_{\{k\}SS}(G)=\lceil\frac{nk}{2}\rceil$.*

Proof. Let $\{M_1,M_2,\ldots,M_r\}$ be a 1-factorization of G. If r is odd, then the function $f:E(G)\to\{\pm 1,\pm 2,\ldots,\pm k\}$ defined by

$$f(e)=k \text{ if } e\in M_r,\quad f(e)=1 \ \text{ for } \ e\in \cup_{i=1}^{(r-1)/2} M_{2i-1}$$
$$\text{and } f(e)=-1 \ \text{ for } \ e\in \cup_{i=1}^{(r-1)/2} M_{2i},$$

is a SS$\{k\}$DF of G with weight $\frac{nk}{2}$.

Let r be even. Define $f : E(G) \to \{\pm 1, \pm 2, \ldots, \pm k\}$ by

$$f(e) = k - 1 \text{ if } e \in M_r, \ f(e) = 1 \text{ for } e \in M_{r-1} \text{ when } r = 2$$

and by $f(e) = k - 1$ if $e \in M_r$, $f(e) = 1$ for $e \in M_{r-1}$, $f(e) = 1$ for $e \in \cup_{i=1}^{(r-2)/2} M_{2i-1}$ and $f(e) = -1$ for $e \in \cup_{i=1}^{(r-2)/2} M_{2i}$ when $r \geq 4$. Obviously, f is a SS$\{k\}$DF of G with weight $\frac{nk}{2}$. Thus $\gamma_{\{k\}SS}(G) \leq \frac{nk}{2}$. It follows from Theorem 2 that $\gamma_{\{k\}SS}(G) = \frac{nk}{2}$ and the proof is complete. ■

Theorem 4. *Let G be a graph of order n and factorable into r Hamiltonian cycles and let $k \geq 2$ be an integer. Then $\gamma_{\{k\}SS}(G) = \lceil \frac{nk}{2} \rceil$.*

Proof. Let G be a Hamiltonian factorable graph, and let $\{C_1, C_2, \ldots, C_r\}$ be a Hamiltonian factorization of G. If n and k are even, then by signing $k/2$ to each edge C_1 and signing $+1$ and -1 alternating along C_i for $2 \leq i \leq r$, we obtain an SS$\{k\}$DF for G of weight $(nk)/2$. If n is even and k is odd, then by signing $(k-1)/2$ and $(k+1)/2$ alternating along C_1 and signing $+1$ and -1 alternating along C_i for $2 \leq i \leq r$, we obtain an SS$\{k\}$DF for G of weight $(nk)/2$.

Let n be odd. We distinguish four cases.

Case 1. r is odd and k is even.

Then the function $f : E(G) \to \{\pm 1, \pm 2, \ldots, \pm k\}$ defined by

$$f(e) = k/2 \text{ if } e \in C_r, \ f(e) = 1 \text{ for } e \in \cup_{i=1}^{(r-1)/2} C_{2i-1}$$
$$\text{and } f(e) = -1 \text{ for } e \in \cup_{i=1}^{(r-1)/2} C_{2i},$$

is a SS$\{k\}$DF of G with weight $\frac{nk}{2}$.

Case 2. r and k are odd.

Then by signing $(k+1)/2$ and $(k-1)/2$ alternating along C_r, signing $+1$ to the edges in $\cup_{i=1}^{(r-1)/2} C_{2i-1}$ and signing -1 to the edges in $\cup_{i=1}^{(r-1)/2} C_{2i}$, we obtain an SS$\{k\}$DF for G of weight $\lceil (nk)/2 \rceil$.

Case 3. r and k are even.

Then the function $f : E(G) \to \{\pm 1, \pm 2, \ldots, \pm k\}$ defined by

$$f(e) = k \text{ if } e \in C_r, \ f(e) = (-k)/2 \text{ for } e \in C_{r-1}$$

and if $r > 2$

$$f(e) = 1, \ e \in \cup_{i=1}^{(r-2)/2} C_{2i-1} \text{ and } f(e) = -1 \text{ for } e \in \cup_{i=1}^{(r-2)/2} C_{2i},$$

is obviously a SS$\{k\}$DF of G with weight $\frac{nk}{2}$.

Case 4. r is even and k is odd.

Define $f : E(G) \to \{\pm 1, \pm 2, \ldots, \pm k\}$ by assigning k to the edge of C_r, assigning $(1-k)/2$ and $(-1-k)/2$ alternatively along C_{r-1} and if $r > 2$ by

$$f(e) = 1, \;\; e \in \cup_{i=1}^{(r-2)/2} C_{2i-1} \;\text{ and }\; f(e) = -1 \;\text{ for }\; e \in \cup_{i=1}^{(r-2)/2} C_{2i}.$$

It is easy to see that f a SS$\{k\}$DF of G with weight $\lceil \frac{nk}{2} \rceil$.

Thus in all cases $\gamma_{\{k\}SS}(G) \leq \lceil \frac{nk}{2} \rceil$ and the result follows from Theorem 2. ■

According to Theorems 3, 4 and the following three well-known results, we can determine the signed star $\{k\}$-domination number of complete graphs and regular bipartite graphs.

Theorem C. *The complete graph K_{2r} is 1-factorable.*

Theorem D. *For every positive integer r, the complete graph K_{2r+1} is Hamiltonian factorable.*

Theorem E. [König [3] 1916] *Every r-regular bipartite graph is 1-factorable for $r \geq 1$.*

3. Basic properties of the signed star $\{k\}$-domatic number

In this section we study basic properties of $d_{\{k\}SS}(G)$. The special case $k = 1$ of the next result can be found in [1].

Theorem 5. *Let G be a graph of size m, signed star $\{k\}$-domination number $\gamma_{\{k\}SS}(G)$ and signed star $\{k\}$-domatic number $d_{\{k\}SS}(G)$. Then*

$$\gamma_{\{k\}SS}(G) \cdot d_{\{k\}SS}(G) \leq mk.$$

Moreover, if $\gamma_{\{k\}SS}(G) \cdot d_{\{k\}SS}(G) = mk$, then for each $d_{\{k\}SS}$-family $\{f_1, f_2, \cdots, f_d\}$ of G, each function f_i is a $\gamma_{\{k\}SS}$-function and $\sum_{i=1}^{d} f_i(e) = k$ for all $e \in E(G)$.

Proof. If $\{f_1, f_2, \ldots, f_d\}$ is a signed star $\{k\}$-dominating family on G such that $d = d_{\{k\}SS}(G)$, then the definitions imply

$$\begin{aligned} d \cdot \gamma_{\{k\}SS}(G) &= \sum_{i=1}^{d} \gamma_{\{k\}SS}(G) \leq \sum_{i=1}^{d} \sum_{e \in E(G)} f_i(e) \\ &= \sum_{e \in E(G)} \sum_{i=1}^{d} f_i(e) \leq \sum_{e \in E(G)} k = mk \end{aligned}$$

as desired.

If $\gamma_{\{k\}SS}(G) \cdot d_{\{k\}SS}(G) = mk$, then the two inequalities occurring in the proof become equalities. Hence for the $d_{\{k\}SS}$-family $\{f_1, f_2, \cdots, f_d\}$ of G and for each i, $\sum_{e\in E(G)} f_i(e) = \gamma_{\{k\}SS}(G)$, thus each function f_i is a $\gamma_{\{k\}SS}$-function, and $\sum_{i=1}^{d} f_i(e) = k$ for all $e \in E(G)$. ■

Corollary 1. *If G is a graph of size m, then $\gamma_{\{k\}SS}(G) + d_{\{k\}SS}(G) \leq mk + 1$.*

Proof. By Theorem 5,

$$\gamma_{\{k\}SS}(G) + d_{\{k\}SS}(G) \leq d_{\{k\}SS}(G) + \frac{mk}{d_{\{k\}SS}(G)}. \tag{1}$$

Using the fact that the function $g(x) = x + (mk)/x$ is decreasing for $1 \leq x \leq \sqrt{mk}$ and increasing for $\sqrt{mk} \leq x \leq mk$, this inequality leads to the desired bound immediately. ■

Corollary 2. *Let G be a graph of size m. If $2 \leq \gamma_{\{k\}SS}(G) \leq mk - 1$, then*

$$\gamma_{\{k\}SS}(G) + d_{\{k\}SS}(G) \leq mk.$$

Proof. Theorem 5 implies that

$$\gamma_{\{k\}SS}(G) + d_{\{k\}SS}(G) \leq \gamma_{\{k\}SS}(G) + \frac{mk}{\gamma_{\{k\}SS}(G)}. \tag{2}$$

If we define $x = \gamma_{\{k\}SS}(G)$ and $g(x) = x + (mk)/x$ for $x > 0$, then because $2 \leq \gamma_{\{k\}SS}(G) \leq mk-1$, we have to determine the maximum of the function g in the interval $I : 2 \leq x \leq mk - 1$. It is easy to see that

$$\begin{aligned} \max_{x\in I}\{g(x)\} &= \max\{g(2), g(mk-1)\} \\ &= \max\{2 + \frac{mk}{2}, mk - 1 + \frac{mk}{mk-1}\} \\ &= mk - 1 + \frac{mk}{mk-1} < mk + 1, \end{aligned}$$

and we obtain $\gamma_{\{k\}SS}(G) + d_{\{k\}SS}(G) \leq mk$. This completes the proof. ■

Corollary 3. *Let G be a graph of size m. If $\min\{\gamma_{\{k\}SS}(G), d_{\{k\}SS}(G)\} \geq 2$, then*

$$\gamma_{\{k\}SS}(G) + d_{\{k\}SS}(G) \leq \frac{mk}{2} + 2.$$

Proof. Since $\min\{\gamma_{\{k\}SS}(G), d_{\{k\}SS}(G)\} \geq 2$, it follows by Theorem 5 that $2 \leq d_{\{k\}SS}(G) \leq \frac{mk}{2}$. By (1) and the fact that the maximum of $g(x) = x + (mk)/x$ on the interval $2 \leq x \leq (mk)/2$ is $g(2) = g((mk)/2)$, we see that

$$\gamma_{\{k\}SS}(G) + d_{\{k\}SS}(G) \leq d_{\{k\}SS}(G) + \frac{mk}{d_{\{k\}SS}(G)} \leq \frac{mk}{2} + 2.$$

■

Since $\gamma_{\{k\}SS}(K_{1,n}) = nk$ and $d_{\{k\}SS}(K_{1,n}) = 1$, Corollary 3 is no longer true in the case that $\min\{\gamma_{\{k\}SS}(G), d_{\{k\}SS}(G)\} = 1$.

Theorem 6. *Let G be a graph. Then*

$$d_{\{k\}SS}(G) \leq \delta(G).$$

Moreover, if the equality holds, then for each function f_i of a SS$\{k\}$D family $\{f_1, f_2, \cdots, f_d\}$ and for every $e \in E(v)$ where v is a vertex of degree $\delta(G)$, $\sum_{e\in E(v)} f_i(e) = k$ and $\sum_{i=1}^{d} f_i(e) = k$.

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a SS$\{k\}$D family of G such that $d = d_{\{k\}SS}(G)$ and let v be a vertex of degree $\delta(G)$. Then

$$\begin{aligned} dk &= \sum_{i=1}^{d} k \leq \sum_{i=1}^{d} \sum_{e\in E(v)} f_i(e) \\ &= \sum_{e\in E(v)} \sum_{i=1}^{d} f_i(e) \leq \sum_{e\in E(v)} k = k\delta(G). \end{aligned}$$

If $d_{\{k\}SS}(G) = \delta(G)$, then the two inequalities occurring in the proof of each corresponding case become equalities, which gives the properties given in the statement. ■

The special case $k = 1$ of Theorem 6 can be found in [1].

Corollary 4. *Let G be a graph of size m, and let $k \geq 2$ be an integer. Then $\gamma_{\{k\}SS}(G) + d_{\{k\}SS}(G) = mk + 1$ if and only if G is the disjoint union of stars.*

Proof. If G is is the disjoint union of stars, then $\gamma_{\{k\}SS}(G) = mk$ by Observation 3. Hence $d_{\{k\}SS}(G) = 1$ and the result follows.

Conversely, let $\gamma_{\{k\}SS}(G) + d_{\{k\}SS}(G) = mk + 1$. The result is obviously true for $m = 1, 2, 3$. Assume $m \geq 4$. By Corollary 3, we may assume that $\min\{\gamma_{\{k\}SS}(G), d_{\{k\}SS}(G)\} = 1$. If $\gamma_{\{k\}SS}(G) = 1$, then $d_{\{k\}SS}(G) = mk$, which is a contradiction to Theorem 6. If $d_{\{k\}SS}(G) = 1$, then $\gamma_{\{k\}SS}(G) = mk$ and the result follows by Observation 3. ■

Corollary 5. *Let G be a graph of size m. Then $\gamma_{SS}(G)+d_{SS}(G)=m+1$ if and only if each edge $e \in E(G)$ has an endpoint u such that* $\deg(u)=1$ *or* $\deg(u)=2$.

Proof. If G satisfies the condition, i.e., each edge $e \in E(G)$ has an endpoint u such that $\deg(u)=1$ or $\deg(u)=2$, then $\gamma_{SS}(G)=m$ by Observation 1. Theorem 5 implies that $d_{SS}(G)=1$ and so $\gamma_{SS}(G)+d_{SS}(G)=m+1$.

Conversely, assume that $\gamma_{SS}(G)+d_{SS}(G)=m+1$. The result is obviously true for $m=1$ and $m=2$. Assume now that $m \geq 3$. By Corollary 3, we may assume that $\min\{\gamma_{SS}(G), d_{SS}(G)\}=1$. Since $m \geq 3$, we observe that $n \geq 3$ and therefore Theorem 2 implies that $\gamma_{SS}(G) \geq \lceil \frac{n}{2} \rceil > 1$. Thus $d_{SS}(G)=1$ and $\gamma_{SS}(G)=m$, and the result follows by Observation 1. ■

As an application of Theorem 6, we will prove the following Nordhaus-Gaddum type result.

Theorem 7. *For every graph G of order n,*

$$d_{\{k\}SS}(G)+d_{\{k\}SS}(\overline{G}) \leq n-1. \tag{3}$$

If $d_{\{k\}SS}(G)+d_{\{k\}SS}(\overline{G})=n-1$, then G is regular.

Proof. Since $\delta(G)+\delta(\overline{G}) \leq n-1$, Theorem 6 leads to

$$d_{\{k\}SS}(G)+d_{\{k\}SS}(\overline{G}) \leq \delta(G)+\delta(\overline{G}) \leq n-1.$$

If G is not regular, then $\delta(G)+\delta(\overline{G}) \leq n-2$ and hence we obtain the better bound $d_{\{k\}SS}(G)+d_{\{k\}SS}(\overline{G}) \leq n-2$. ■

If C_n is a cycle of length n, then we have shown in [1] that $d_{SS}(C_n)=1$. Next we determine $d_{\{k\}SS}(C_n)$ for $k \geq 2$.

Theorem 8. *If $k \geq 2$ is an integer, and C_n a cycle of length n, then $d_{\{k\}SS}(C_n)=2$ when $k \geq 3$ and n is even and $d_{\{k\}SS}(C_n)=1$ otherwise.*

Proof. Let $C_n=v_1e_1v_2e_2\ldots v_{n-1}e_nv_1$ with $v_j \in V(C_n)$ and $e_j \in E(C_n)$ for $j \in \{1,2,\ldots,n\}$.

First assume that $k \geq 3$ and that n is even. Define $f_i : E(G) \longrightarrow \{\pm1, \pm2, \ldots, \pm k\}$ for $i=1,2$ by $f_1(e_1)=f_1(e_3)=\ldots=f_1(e_{n-1})=1$, $f_1(e_2)=f_1(e_4)=\ldots=f_1(e_n)=k-1$ and $f_2(e_1)=f_2(e_3)=\ldots=f_2(e_{n-1})=k-1$, $f_2(e_2)=f_2(e_4)=\ldots=f_2(e_n)=1$. Then $\{f_1,f_2\}$ is an SS$\{k\}$D family on C_n and thus $d_{\{k\}SS}(C_n) \geq 2$. As Theorem 6 implies that $d_{\{k\}SS}(C_n) \leq 2$, we deduce that $d_{\{k\}SS}(C_n)=2$ in this case.

Assume next that $k=2$ and that n is even. Suppose to the contrary that $d_{\{2\}SS}(C_n)=2$. If $\{f_1,f_2\}$ is an SS$\{k\}$D family on C_n, then Theorem 6 yields to $f_i(e_t)+f_i(e_{t+1})=2$ for $i=1,2$ and $1 \leq t \leq n$, where the indices t

are taken modulo n. Since $f_i(e_t) \in \{-2, -1, 1, 2\}$, this is only possible when $f_1(e_t) = 1$ and $f_2(e_t) = 1$ for each $t \in \{1, 2, \ldots, n\}$. Hence we obtain the contradiction $f_1 \equiv f_2 \equiv 1$ and thus $d_{\{2\}SS}(C_n) = 1$.

Finally assume that n is odd. Suppose to the contrary that $d_{\{k\}SS}(C_n) = 2$. If $\{f_1, f_2\}$ is an SS$\{k\}$D family on C_n, then Theorem 6 implies that $f_i(e_t) + f_i(e_{t+1}) = k$ for $i = 1, 2$ and $1 \le t \le n$, where the indices t are taken modulo n. Since $f_i(e_t) + f_i(e_{t+1}) = f_i(e_{t+1}) + f_i(e_{t+2}) = k$, we conclude that $f_i(e_t) = f_i(e_{t+2})$ for $i = 1, 2$ and $1 \le t \le n$. Therefore $f_i(e_1) = f_i(e_3) = \ldots = f_i(e_n) = a_i$. As e_1 and e_n are adjacent, it follows that $f_i(e_1) + f_i(e_n) = 2a_i = k$. In addition, we have $f_i(e_2) = f_i(e_4) = \ldots = f_i(e_{n-1}) = b_i$. As e_1 and e_2 are adjacent, we find that $f_i(e_1) + f_i(e_2) = a_i + b_i = k$. We deduce that $k = 2a_i = a_i + b_i$ and so $a_i = b_i = k/2$ for $i = 1, 2$. This leads to the contradiction $f_1 \equiv f_2 \equiv k/2$ and thus $d_{\{k\}SS}(C_n) = 1$ when n is odd. ■

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