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STABILITY THEOREM INVOLVING COMPARISON FUNCTIONS AND E−DISTANCE IN HAUSDORFF UNIFORM SPACES

Abstract. In this paper, we establish stability result for a pair of selfmappings in Hausdorff uniform spaces by employing the notion of comparison functions as well as the concept of E-distance introduced by Aamri and El Moutawakil [1]. Our results improve and unify some of the known stability results in literature.

Key words: E-distance, Hausdorff uniform spaces, comparison functions, ψ-contractions, stability results.

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1. Introduction

Several authors such as Berinde [2], Jachymski [8], Kada et al [9], Rhoades [12, 13], Rus [15], Wang et al [17] and Zeidler [18] studied the theory of fixed point or common fixed point for contractive selfmappings in complete metric spaces or Banach spaces in general.

Also, some authors such as Kang [10], Rodríguez-Montes and Charris [14] within the last few decades, established some results on fixed and coincidence points of maps by means of appropriate W-contractive or W-expansive assumptions in uniform spaces.

In the sequel, a uniform space is defined as follows [see Aamri and El Moutawakil [1], Bourbaki [6] and Zeidler [18]]: Let X be a nonempty set and let Φ be a nonempty family of subsets of $X \times X$. The pair (X, Φ) is called a uniform space if it satisfies the following properties:

(i) if G is in Φ, then G contains the diagonal $\{(x,x) | x \in X\}$;

(ii) if G is in Φ and H is a subset of $X \times X$ which contains G, then H is in Φ;

(iii) if G and H are in Φ, then $G \cap H$ is in Φ;

(iv) if G is in Φ, then there exists H in Φ, such that, whenever (x,y) and (y,z) are in H, then (x,z) is in H;

(v) if G is in Φ, then $\{(y,x) | (x,y) \in G\}$ is also in Φ.

Φ is called the uniform structure of X and its elements are called entourages or neighbourhoods or surroundings. If property (v) is omitted, then (X,Φ) is called a quasiuniform space. (For examples, see Bourbaki [6] and Zeidler [18]).

In 2004, Aamri and El Moutawakil [1] proved some common fixed point theorems for some new contractive or expansive maps in uniform spaces by introducing the notions of A-distance and E-distance. Aamri and El Moutawakil [1] introduced and employed the following contractive definition: Let $f,g: X \longrightarrow X$ be selfmappings of X. Then, we have

$$p(f(x),f(y)) \leq \psi(p(g(x),g(y))), \qquad \forall x,y \in X, \tag{1}$$

where p is A-distance and $\psi: \Re^+ \to \Re^+$ is a nondecreasing function satisfying

(i) for each $t \in (0,+\infty)$, $\quad 0 < \psi(t)$,

(ii) $\lim_{n\to\infty} \psi^n(t) = 0, \quad \forall\, t \in (0,+\infty)$.

ψ satisfies also the condition $\psi(t) < t$, for each $t > 0, \quad t \in \Re^+$.

In this paper, we shall establish some stability results for a pair of selfmappings in uniform spaces by employing the concept of E-distance as well as the notion of comparison functions.

2. Preliminaries

The following definitions shall be required in the sequel: Let (X,Φ) be a uniform space and let $(X,\tau(\Phi))$ be a topological space whenever topological concepts are mentioned in the context of a uniform space (X,Φ). Definitions 1-5 are contained in Aamri and El Moutawakil [1].

Definition 1. *If $H \in \Phi$ and $(x,y) \in H, (y,x) \in H$, x and y are said to be H-close. A sequence $\{x_n\}_{n=0}^{\infty} \subset X$ is said to be a Cauchy sequence for Φ if for any $H \in \Phi$, there exists $N \geq 1$ such that x_n and x_m are H-close for $n,m \geq N$.*

Definition 2. *A function $p: X\times X \to \Re^+$ is said to be an A-distance if for any $H \in \Phi$, there exists $\delta > 0$ such that if $p(z,x) \leq \delta$ and $p(z,y) \leq \delta$ for some $z \in X$, then $(x,y) \in H$.*

Definition 3. *A function $p: X\times X \to \Re^+$ is said to be an E-distance if*
(p_1) *p is an A-distance,*
(p_2) $p(x,y) \leq p(x,z) + p(z,y), \quad \forall\, x,y,z \in X$.

Definition 4. *A uniform space (X,Φ) is said to be Hausdorff if and only if the intersection of all $H \in \Phi$ reduces to the diagonal $\{(x,x)|x \in X\}$, i.e. if $(x,y) \in H$ for all $H \in \Phi$ implies $x = y$. This guarantees the uniqueness* *of limits of sequences.* $H \in \Phi$ *is said to be symmetrical if* $H = H^{-1} = \{(y,x)|(x,y) \in H\}$.

Definition 5. *Let* (X, Φ) *be a uniform space and* p *be an* A*-distance on* X.

(*i*) *Sequence* $\{x_n\}_{n=0}^{\infty}$ *is p-Cauchy if given* $\epsilon > 0$, *there exists* N *such that if* $m, n > N$, *then* $p(x_m, x_n) < \epsilon$.

(*ii*) X *is said to be S-complete if for every p-Cauchy sequence* $\{x_n\}_{n=0}^{\infty}$, *there exists* $x \in X$ *with* $\lim_{n\to\infty} p(x_n, x) = 0$.

(*iii*) X *is said to be p-Cauchy complete if for every p-Cauchy sequence* $\{x_n\}_{n=0}^{\infty}$, *there exists* $x \in X$ *with* $\lim_{n\to\infty} x_n = x$ *with respect to* $\tau(\Phi)$.

(*iv*) $f : X \to X$ *is said to be p-continuous if* $\lim_{n\to\infty} p(x_n, x) = 0$ *implies that* $\lim_{n\to\infty} p(f(x_n), f(x)) = 0$.

(*v*) X *is said to be p-bounded if* $\delta_p = \sup\{p(x,y)|x, y \in X\} < \infty$.

The following definition contained in Berinde [2, 3], Rus [15] and Rus et al [16] shall also be required in the sequel.

Definition 6. *A function* $\psi : \Re^+ \to \Re^+$ *is called a comparison function if*

(*i*) ψ *is monotone increasing;*

(*ii*) $\lim_{n\to\infty} \psi^n(t) = 0, \quad \forall\, t \geq 0$.

Many stability results have been obtained within the last few decades by various authors using different contractive definitions. Harder and Hicks [7] considered the following concept to obtain various stability results:

Let (X, d) be a complete metric space, $T : X \to X$ a selfmap of X. Suppose that $F_T = \{u \in X : Tu = u\}$ is the set of fixed points of T in X.

Let $\{x_n\}_{n=0}^{\infty} \subset X$ be the sequence generated by an iteration procedure involving the operator T, that is,

$$x_{n+1} = h(T, x_n), \quad n = 0, 1, 2, \ldots, \tag{2}$$

where $x_0 \in X$ is the initial approximation and h is some function. Suppose $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point u of T. Let $\{y_n\}_{n=0}^{\infty} \subset X$ and set

$$\epsilon_n = d(y_{n+1}, h(T, y_n)), \quad n = 0, 1, 2, \ldots\ . \tag{3}$$

Then, the iteration procedure (2) is said to be $T-$***stable*** or stable with respect to T if and only if $\lim_{n\to\infty} \epsilon_n = 0$ implies $\lim_{n\longrightarrow\infty} y_n = u$.

Throughout this paper, h represents some function, while f and g shall denote two selfmappings of a uniform space (X, Φ).

We shall employ the following definition of stability of iteration process which is a natural extension of Harder and Hicks [7]:

Definition 7. *Let* (X, Φ) *be a uniform space and* $f, g, : X \to X$ *two selfmaps of* X. *Suppose that* $F_f \cap F_g \neq \phi$, *where* $F_f \cap F_g = u$ *is the common fixed point of* f *and* g *in* X; *while* F_f *and* F_g *are the sets of fixed points of* f *and* g *in* X *respectively.*

Let $\{x_n\}_{n=0}^{\infty} \subset X$ *be the sequence generated by an iteration procedure involving the operators* f *and* g, *that is,*

$$x_{n+1} = h(f, g, x_n), \quad n = 0, 1, 2, \ldots, \tag{4}$$

where $x_0 \in X$ *is the initial approximation and* h *is some function. Suppose* $\{x_n\}_{n=0}^{\infty}$ *converges to a common fixed point* u *of* f *and* g *in* X. *Let* $\{y_n\}_{n=0}^{\infty} \subset X$ *and set*

$$\epsilon_n = p(y_{n+1}, h(f, g, y_n)), \quad n = 0, 1, 2, \ldots, \tag{5}$$

where p *is an* A*-distance which replaces the distance function* d *in (3). Then, the iteration procedure (4) is said to be* $(f, g)-$*stable or stable with respect to* f *and* g *if and only if* $\lim_{n\to\infty} \epsilon_n = 0$ *implies* $\lim_{n\to\infty} y_n = u$. *For example, see [5].*

Remark 1. If $f = g = T$ in (4), then we obtain the iteration procedure of Harder and Hicks [7]. Also, if $f = g = T$ and $p = d$ in (5), then we get (2); which was used by Harder and Hicks [10] and many other authors.

In 2007, Olatinwo [11] established some common fixed point theorems by employing the following contractive definition: Let $f, g : X \to X$ be selfmappings of X. There exist $L \geq 0$ and a comparison function $\psi : \Re^+ \to \Re^+$ such that $\forall\, x, y \in X$, we have

$$p(f(x), f(y)) \leq Lp(x, g(x)) + \psi(p(g(x), g(y))), \qquad \forall\, x, y \in X. \tag{6}$$

Recently, Bosede [4] proved some common fixed point theorems by employing the following contractive definition: Let $f, g : X \to X$ be selfmappings of X. There exist comparison functions $\psi_1 : \Re^+ \to \Re^+$ and $\psi_2 : \Re^+ \to \Re^+$ with $\psi_1(0) = 0$ such that $\forall\, x, y \in X$, we have

$$p(f(x), f(y)) \leq \psi_1(p(x, g(x))) + \psi_2(p(g(x), g(y))), \qquad \forall\, x, y \in X \tag{7}$$

where p is A-distance in X.

Our aim in this paper is to establish some stability results for selfmappings in uniform spaces by employing the concept of E-distance as well as the notion of comparison function using the same contractive condition (7), which is more general than (1) and (6).

Remark 2. The contractive condition (7) is more general than (1) and (6) in the sense that if $\psi_1(u) = Lu$ in (7), for $L \geq 0$, $u \in \Re^+$, then we obtain

$$p(f(x), f(y)) \leq Lp(x, g(x)) + \psi_2(p(g(x), g(y))), \qquad \forall\, x, y \in X$$

which is the contractive condition employed by Olatinwo [11] in (6).

Moreover, if $L = 0$ in the above inequality, then we obtain (1), which was employed by Aamri and El Moutawakil [1].

Therefore, contractive condition (7) is a generalization of the contractive definitions (1) and (6) of Aamri and El Moutawakil [1] and Olatinwo [11] respectively.

3. Main result

Theorem. *Let (X, Φ) be a symmetrical Hausdorff uniform space and p an E-distance on X such that X is p-bounded and S-complete. For arbitrary $x_0 \in X$, define a sequence $\{x_n\}_{n=0}^{\infty}$ iteratively by*

$$x_{n+1} = f(x_n), \qquad n = 0, 1, 2, \dots \; . \tag{8}$$

Suppose that f and g are p-continuous selfmappings of X, with a common fixed point u in X, satisfying

(i) $f(X) \subseteq g(X)$;

(ii) $p(f(x_i), f(x_i)) = 0$, $\forall\, x_i \in X$, $i = 0, 1, 2, \dots$.In particular, $p(f(u), f(u)) = 0$;

(iii) $f, g : X \to X$ satisfy the contractive condition (7).

Suppose also that $\psi_1 : \Re^+ \to \Re^+$ and $\psi_2 : \Re^+ \to \Re^+$ are comparison functions such that $\psi_1(0) = 0$.

Then, iteration (8) is $(f, g)-$stable.

Proof. For arbitrary $x_0 \in X$, select $x_1 \in X$ such that $f(x_0) = g(x_1)$. Similarly, for $x_1 \in X$, select $x_2 \in X$ such that $f(x_1) = g(x_2)$.

Continuing this process, we select $x_n \in X$ such that $f(x_{n-1}) = g(x_n)$. Hence, iteration (8) is well-defined.

Let $\{y_n\}_{n=0}^{\infty} \subset X$ and let $\{\epsilon_n\}_{n=0}^{\infty}$ be a sequence defined by $\epsilon_n = p(y_{n+1}, f(y_n))$.

Suppose that $\{x_n\}_{n=0}^{\infty}$ converges to a common fixed point u of f and g in X, that is, $f(u) = g(u) = u$, where u is in X.

Suppose also that

$$\lim_{n\to\infty} \epsilon_n = 0.$$

Then, we shall prove that

$$\lim_{n\to\infty} y_n = u.$$

Since X is p-bounded, we assume that $p(f(u), f(y_0)) \leq \delta_p(X)$, $y_0 \in X$, where $\delta_p(X) = \sup\{p(x, y) | x, y \in X\} < +\infty$. We also observe that $p(x, y)$ is nonnegative, for all $x, y \in X$.

Indeed, since $x_n = f(x_{n-1})$, $n = 1, 2, ...$, then, using the contractive definition (7) and the triangle inequality of E-distance, we obtain

$$
\begin{aligned}
p(y_{n+1}, u) &\leq p(y_{n+1}, f(y_n)) + p(f(y_n), u) \\
&= \epsilon_n + p(f(y_n), f(u)) \\
&= \epsilon_n + p(f(u), f(y_n)) \\
&\leq \epsilon_n + \psi_1(p(u, g(u))) + \psi_2(p(g(u), g(y_n))) \\
&= \epsilon_n + \psi_1(p(f(u), f(u))) + \psi_2(p(f(u), f(y_{n-1}))) \\
&= \epsilon_n + \psi_1(0) + \psi_2(p(f(u), f(y_{n-1}))) \\
&= \epsilon_n + 0 + \psi_2(p(f(u), f(y_{n-1}))) \\
&= \epsilon_n + \psi_2(p(f(u), f(y_{n-1}))) \\
&\leq \epsilon_n + \psi_2\Big(\psi_1(p(u, g(u))) + \psi_2(p(g(u), g(y_{n-1}))\Big) \\
&= \epsilon_n + \psi_2\Big(\psi_1(p(f(u), f(u))) + \psi_2(p(f(u), f(y_{n-2}))\Big) \\
&= \epsilon_n + \psi_2\Big(\psi_1(0) + \psi_2(p(f(u), f(y_{n-2}))\Big) \\
&= \epsilon_n + \psi_2\Big(0 + \psi_2(p(f(u), f(y_{n-2}))\Big) \\
&= \epsilon_n + \psi_2^2(p(f(u), f(y_{n-2}))) \\
&\leq \ldots \leq \epsilon_n + \psi_2^n(p(f(u), f(y_0))) \\
&\leq \epsilon_n + \psi_2^n(\delta_p(X)).
\end{aligned}
\tag{9}
$$

But condition (ii) of Definition 6 of a comparison function gives

$$\lim_{n\to\infty} \psi_2^n(\delta_p(X)) = 0.$$

Hence, taking the limit as $n \to \infty$ of both sides of (9) yields

$$\lim_{n\to\infty} p(y_{n+1}, u) = 0,$$

which implies that

$$\lim_{n\to\infty} y_n = u.$$

Conversely, let $\lim_{n\to\infty} y_n = u$. Then,

$$
\begin{aligned}
\epsilon_n &= p(y_{n+1}, f(y_n)) \\
&\leq p(y_{n+1}, u) + p(u, f(y_n)) \\
&= p(y_{n+1}, u) + p(f(u), f(y_n)) \\
&\leq p(y_{n+1}, u) + +\psi_1(p(u, g(u))) + \psi_2(p(g(u), g(y_n))) \\
&= p(y_{n+1}, u) + +\psi_1(p(f(u), f(u))) + \psi_2(p(f(u), f(y_{n-1}))) \\
&= p(y_{n+1}, u) + \psi_1(0) + \psi_2(p(f(u), f(y_{n-1}))) \\
&= p(y_{n+1}, u) + 0 + \psi_2(p(f(u), f(y_{n-1})))
\end{aligned}
$$

$$
\begin{aligned}
&= p(y_{n+1}, u) + \psi_2(p(f(u), f(y_{n-1}))) \\
&\le p(y_{n+1}, u) + \psi_2\Big(\psi_1(p(u, g(u))) + \psi_2(p(g(u), g(y_{n-1}))\Big) \\
&= p(y_{n+1}, u) + \psi_2\Big(\psi_1(p(f(u), f(u))) + \psi_2(p(f(u), f(y_{n-2}))\Big) \\
&= p(y_{n+1}, u) + \psi_2\Big(\psi_1(0) + \psi_2(p(f(u), f(y_{n-2}))\Big) \\
&= p(y_{n+1}, u) + \psi_2\Big(0 + \psi_2(p(f(u), f(y_{n-2}))\Big) \\
&= p(y_{n+1}, u) + \psi_2^2(p(f(u), f(y_{n-2}))) \\
&\le \ldots \le p(y_{n+1}, u) + \psi_2^n(p(f(u), f(y_0))) \\
&\le p(y_{n+1}, u) + \psi_2^n(\delta_p(X)) \to 0 \ \text{ as } \ n \to \infty.
\end{aligned}
$$

This completes the proof. ■

Remark 3. Our main result in this paper is a generalization of those of Berinde [2], Harder and Hicks [7] and many others; and this is also a further improvement to many existing stability results in literature.

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