F A S C I C U L I M A T H E M A T I C I

Nr 51 2013

H. Bouhadjera and V. Popa

SOME COMMON FIXED POINT THEOREMS FOR OCCASIONALLY WEAKLY COMPATIBLE MAPPINGS SATISFYING IMPLICIT RELATION AND CONTRACTIVE MODULUS

Abstract. In this note, we prove some common fixed point theorems for occasionally weakly compatible mappings satisfying an implicit relation and a contractive modulus.

Key words: weakly compatible mappings, compatible mappings, occasionally weakly compatible mappings, implicit relations, contractive modulus, common fixed point theorems, metric space.

AMS Mathematics Subject Classification: 47H10, 54H25.

1. Introduction

Let S and T be two self mappings of a metric space. Sessa [7] defined S and T to be weakly commuting if $d(STx, TSx) \leq d(Tx, Sx)$ for all x in $\mathcal{X}$. In 1986, Jungck [3] introduced the concept of compatibility as follows: S and T above are compatible if $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$ whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = x$ for some $x \in \mathcal{X}$. Recently, in 2008, Al-Thagafi and Shahzad [1] weakened the above notion by giving the so-called occasionally weak compatibility. Let $\mathcal{X}$ be a set. S and $T : \mathcal{X} \to \mathcal{X}$ are said to be occasionally weakly compatible if and only if, there is a point x in $\mathcal{X}$ which is a coincidence point of S and T at which S and T commute.

Definition 1. *A function $M : [0, \infty) \to [0, \infty)$ is said to be a contractive modulus if $M(0) = 0$ and $M(t) < t$ for $t > 0$.*

Theorem 1 ([2])**.** *Let $\mathcal{X}$ be a set endowed with a symmetric d. Suppose A, B, S and T are four self mappings of $(\mathcal{X}, d)$ satisfying the conditions:*

$$(1)\quad \begin{aligned} d^2(Ax, By) \leq \max\{&M(d(Sx, Ty))M(d(Sx, Ax)), M(d(Sx, Ty)) \\ &M(d(Ty, By)), M(d(Sx, Ax))M(d(Ty, By)), \\ &M(d(Sx, By))M(d(Ty, Ax))\}, \end{aligned}$$

for all x, $y \in \mathcal{X}$, where M is contractive modulus, the pairs (A, S) and (B, T) are owc. Then A, B, S and T have a unique common fixed point.

In [6] and [5] is initiated the study of fixed point for mappings satisfying implicit relations. The purpose of this paper is to prove a general fixed point theorem for four mappings satisfying an implicit relation which generalizes Theorem 1.

2. Implicit relations

Definition 2. *Let (FM) be the set of all functions $F(t_1, t_2, t_3, t_4, t_5, t_6)$ satisfying the following conditions:*
(Fm) : *F is increasing in variable t_1,*
(Fu) : *$F(t, t, 0, 0, t, t) > 0$ for every $t > 0$.*

Example 1. $F = t_1^2 - \max\{M(t_2)M(t_3), M(t_2)M(t_4), M(t_3)M(t_4), M(t_5) M(t_6)\}$, where M is a contractive modulus.

(Fm) : Obviously,

(Fu) : $F(t, t, 0, 0, t, t) = t^2 - M^2(t) > 0$ for every $t > 0$.

Example 2. $F = t_1 - k\max\{M(t_2), M(t_3), M(t_4), \frac{M(t_5)+M(t_6)}{2}\}$, where M is a contractive modulus and $k \in (0, 1)$.

(Fm) : Obviously,

(Fu) : $F(t, t, 0, 0, t, t) = t - kM(t) > 0$ for every $t > 0$.

Example 3.

$$\begin{aligned} F &= t_1^2 - k_1 \max\{M^2(t_2), M^2(t_3), M^2(t_4)\} \\ &\quad -k_2 \max\{M(t_3)M(t_5), M(t_4)M(t_6)\} - k_3 M(t_5)M(t_6), \end{aligned}$$

where M is a contractive modulus, $k_1 > 0$, k_2, $k_3 \geq 0$, and $k_1 + k_3 \leq 1$.

(Fm): Obviously,

(Fu): $F(t, t, 0, 0, t, t) = t^2 - (k_1 + k_3)M^2(t) > 0 \ \forall t > 0$.

Example 4. $F = t_1^2 - aM^2(t_2) - \frac{bM(t_5)M(t_6)}{1+M^2(t_3)+M^2(t_4)}$, where M is a contractive modulus, $a > 0$, $b \geq 0$, and $a + b \leq 1$.

(Fm): Obviously,

(Fu): $F(t, t, 0, 0, t, t) = t^2 - (a + b)M^2(t) > 0 \ \forall t > 0$.

Example 5. $F = t_1 - \max\{M(t_2), M(t_3), M(t_4), M(t_5), M(t_6)\}$, where M is a contractive modulus.

(Fm): Obviously,

(Fu): $F(t, t, 0, 0, t, t) = t - M(t) > 0 \ \forall t > 0$.

Lemma 1 (Jungck and Rhoades [4]). *Let $\mathcal{X}$ be a nonempty set and S and T be occasionally weakly compatible self mappings on $\mathcal{X}$. If S and T have a unique common point of coincidence $w = Sx = Tx$, then w is the unique common fixed point of S and T.*

3. Common fixed point theorems

Theorem 2. *Let $(\mathcal{X}, d)$ be a metric space and A, B, S, $T : (\mathcal{X}, d) \to (\mathcal{X}, d)$ such that*

$$(2)\qquad F(d(Ax,By), M(d(Sx,Ty)), M(d(Sx,Ax)), M(d(Ty,By)), M(d(Sx,By)), M(d(Ty,Ax))) \leq 0,$$

for all $x, y \in \mathcal{X}$, where F satisfies property (Fu) and M is a contractive modulus. If there exist x, y in $\mathcal{X}$ such that $Ax = Sx$ and $By = Ty$, then, A and S have a unique point of coincidence and B and T have a unique point of coincidence (resp. $u = Ax = Sx$ and $v = By = Ty$). Moreover $u = v$.

Proof. First we prove that $Ax = By$. Suppose contrary. By (2) we obtain

$$F(d(Ax,By), M(d(Ax,By)), M(d(Ax,Ax)), M(d(By,By)), M(d(Ax,By)), M(d(By,Ax))) \leq 0.$$

As F is increasing in variable t_1, we have

$$F(M(d(Ax,By)), M(d(Ax,By)), 0, 0, M(d(Ax,By)), M(d(Ax,By))) \leq 0,$$

a contradiction of (Fu). Hence, $M(d(Ax,By)) = 0$ which implies $Ax = By = Sx = Ty = u = v$.

If there exists another point of coincidence for A and S, $w = Az = Sz$ with Az is distinct of Ax, then by (2) we have

$$F(d(Az,By), M(d(Az,By)), M(d(Az,Az)), M(d(By,By)), M(d(Az,By)), M(d(By,Az))) \leq 0.$$

By condition (Fm), we get

$$F(M(d(Az,By)), M(d(Az,By)), 0, 0, M(d(Az,By)), M(d(Az,By))) \leq 0,$$

a contradiction of (Fu). Hence u is the unique point of coincidence for A and S. Similarly, v is the unique point of coincidence of T and B and $u = v$. Therefore $u = v$ is the unique point of coincidence for A and S and B and T. ∎

Theorem 3. *Let A, B, S, T self mappings of a metric space $(\mathcal{X}, d)$ satisfying inequality (2) for all x, y in $\mathcal{X}$ where F is in (FM) and M is a contractive modulus. If the pairs (A, S) and (B, T) are occasionally weakly compatible then, A, B, S and T have a unique common fixed point.*

Proof. Since the pairs (A, S) and (B, T) are occasionally weakly compatible, then, A and S have a point of coincidence $u = Ax = Sx$ and B and T have a point of coincidence $v = By = Ty$. By the above theorem, $u = v$ and it is a unique common point of coincidence for A and S and for B and T. By the above lemma A and S have u as unique common fixed point and B and T have u as the unique common fixed point. Therefore, u is the unique common fixed point of A, B, S and T. ■

Example 6. Let $\mathcal{X} = [0, \infty[$ with the metric $d(x, y) = |x - y|$. Define

$$Ax = Bx = \begin{cases} \frac{3}{4} & \text{if } x \in [0,1[\\ 1 & \text{if } x \in [1, \infty[, \end{cases} \quad Sx = \begin{cases} 2 & \text{if } x \in [0,1[\\ \dfrac{1}{x^2} & \text{if } x \in [1, \infty[, \end{cases}$$

and

$$Tx = \begin{cases} 2 & \text{if } x \in [0,1[\\ \dfrac{1}{x} & \text{if } x \in [1, \infty[. \end{cases}$$

First it is clear to see that A and S are occasionally weakly compatible as well as B and T.

Take $M(t) = \frac{1}{2}t$ and

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \max\{M(t_2), M(t_3), M(t_4), M(t_5), M(t_6)\},$$

we get

(a) For x, $y \in [0, 1[$, we have $Ax = By = \frac{3}{4}$, $Sx = Ty = 2$ and

$$\begin{aligned} F(d(Ax, By), M(d(Sx, Ty)), M(d(Sx, Ax)), \\ M(d(Ty, By)), M(d(Sx, By)), M(d(Ty, Ax))) \\ = F(0, M(0), M(\frac{5}{4}), M(\frac{5}{4}), M(\frac{5}{4}), M(\frac{5}{4})) \\ = F(0, 0, \frac{5}{8}, \frac{5}{8}, \frac{5}{8}, \frac{5}{8}) \\ = 0 - \max\{M(0), M(\frac{5}{8})\} = -\max\{0, \frac{5}{16}\} = -\frac{5}{16} \leq 0 \end{aligned}$$

because that $d(Ax, By) = 0$ and $\max\{M(t_2), M(t_3), M(t_4), M(t_5), M(t_6)\} = \frac{5}{16}$ then $d(Ax, By) \leq \max\{M(t_2), M(t_3), M(t_4), M(t_5), M(t_6)\}$.

(b) For $x,\, y \in [1, \infty[$, we have $Ax = By = 1$, $Sx = \dfrac{1}{x^2}$, $Ty = \dfrac{1}{y}$ and

$$\begin{aligned}
&F(d(Ax, By), M(d(Sx, Ty)), M(d(Sx, Ax)),\\
&\qquad M(d(Ty, By)), M(d(Sx, By)), M(d(Ty, Ax)))\\
&= F(0, M(|\frac{1}{x^2} - \frac{1}{y}|), M(|\frac{1}{x^2} - 1|), M(|\frac{1}{y} - 1|),\\
&\qquad M(|\frac{1}{x^2} - 1|), M(|\frac{1}{y} - 1|))\\
&= F(0, \frac{|\frac{1}{x^2} - \frac{1}{y}|}{2}, \frac{|\frac{1}{x^2} - 1|}{2}, \frac{|\frac{1}{y} - 1|}{2}, \frac{|\frac{1}{x^2} - 1|}{2}, \frac{|\frac{1}{y} - 1|}{2})\\
&= 0 - \max\{M(\frac{|\frac{1}{x^2} - \frac{1}{y}|}{2}), M(\frac{|\frac{1}{x^2} - 1|}{2}), M(\frac{|\frac{1}{y} - 1|}{2}),\\
&\qquad M(\frac{|\frac{1}{x^2} - 1|}{2}), M(\frac{|\frac{1}{y} - 1|}{2})\}\\
&= -\max\{\frac{|\frac{1}{x^2} - \frac{1}{y}|}{4}, \frac{|\frac{1}{x^2} - 1|}{4}, \frac{|\frac{1}{y} - 1|}{4}, \frac{|\frac{1}{x^2} - 1|}{4}, \frac{|\frac{1}{y} - 1|}{4}\} \le 0
\end{aligned}$$

because that $d(Ax, By) = 0$ and $\max\{M(t_2), M(t_3), M(t_4), M(t_5), M(t_6)\} \le 1$ then $d(Ax, By) \le \max\{M(t_2), M(t_3), M(t_4), M(t_5), M(t_6)\}$.

(c) For $x \in [0, 1[$, $y \in [1, \infty[$, we have $Ax = \frac{3}{4}$, $By = 1$, $Sx = 2$, $Ty = \dfrac{1}{y}$ and

$$\begin{aligned}
&F(d(Ax, By), M(d(Sx, Ty)), M(d(Sx, Ax)),\\
&\qquad M(d(Ty, By)), M(d(Sx, By)), M(d(Ty, Ax)))\\
&= F(\frac{1}{4}, M(|2 - \frac{1}{y}|), M(|2 - \frac{3}{4}|), M(|\frac{1}{y} - 1|), M(|2 - 1|), M(|\frac{1}{y} - \frac{3}{4}|))\\
&= F(\frac{1}{4}, M(|2 - \frac{1}{y}|), M(\frac{5}{4}), M(|\frac{1}{y} - 1|), M(1), M(|\frac{1}{y} - \frac{3}{4}|))\\
&= F(\frac{1}{4}, \frac{|2 - \frac{1}{y}|}{2}, \frac{5}{8}, \frac{|\frac{1}{y} - 1|}{2}, \frac{1}{2}, \frac{|\frac{1}{y} - \frac{3}{4}|}{2})\\
&= \frac{1}{4} - \max\{M(\frac{|2 - \frac{1}{y}|}{2}), M(\frac{5}{8}), M(\frac{|\frac{1}{y} - 1|}{2}), M(\frac{1}{2}), M(\frac{|\frac{1}{y} - \frac{3}{4}|}{2})\}\\
&= \frac{1}{4} - \max\{\frac{|2 - \frac{1}{y}|}{4}, \frac{5}{16}, \frac{|\frac{1}{y} - 1|}{4}, \frac{1}{4}, \frac{|\frac{1}{y} - \frac{3}{4}|}{4}\} \le 0
\end{aligned}$$

because that $d(Ax, By) = \frac{1}{4}$ and $M(d(Sx, By)) = \frac{1}{2}$ then

$$d(Ax, By) \le \max\{M(t_2), M(t_3), M(t_4), M(t_5), M(t_6)\}.$$

(d) Finally, for $x \in [1, \infty[$, $y \in [0, 1[$, we have $Ax = 1$, $By = \frac{3}{4}$, $Sx = \dfrac{1}{x^2}$, $Ty = 2$ and

$$\begin{aligned} & F(d(Ax, By), M(d(Sx, Ty)), M(d(Sx, Ax)), \\ & \qquad M(d(Ty, By)), M(d(Sx, By)), M(d(Ty, Ax))) \\ & = F(\frac{1}{4}, M(|\frac{1}{x^2} - 2|), M(|\frac{1}{x^2} - 1|), M(|2 - \frac{3}{4}|), M(|\frac{1}{x^2} - \frac{3}{4}|), M(|2 - 1|)) \\ & = F(\frac{1}{4}, \frac{|\frac{1}{x^2} - 2|}{2}, \frac{|\frac{1}{x^2} - 1|}{2}, \frac{5}{8}, \frac{|\frac{1}{x^2} - \frac{3}{4}|}{2}, \frac{1}{2}) \\ & = \frac{1}{4} - \max\{M(\frac{|\frac{1}{x^2} - 2|}{2}), M(\frac{|\frac{1}{x^2} - 1|}{2}), M(\frac{5}{8}), M(\frac{|\frac{1}{x^2} - \frac{3}{4}|}{2}), M(\frac{1}{2})\} \\ & = \frac{1}{4} - \max\{\frac{|\frac{1}{x^2} - 2|}{4}, \frac{|\frac{1}{x^2} - 1|}{4}, \frac{5}{16}, \frac{|\frac{1}{x^2} - \frac{3}{4}|}{4}, \frac{1}{4}\} \le 0 \end{aligned}$$

because that $d(Ax, By) = \frac{1}{4}$ and $M(d(Ty, Ax)) = \frac{1}{4}$ then

$$d(Ax, By) \le \max\{M(t_2), M(t_3), M(t_4), M(t_5), M(t_6)\}.$$

So, all the hypotheses of the above theorem are satisfied and 1 is the unique common fixed point of mappings A, B, S and T.

Corollary 1. *Theorem 1.*

Proof. The proof follows by Theorem 3 and Example 1. ■

If $A = B$ and $S = T$ by Theorem 3 we obtain:

Theorem 4. *Let A and S be self mappings of a metric space $(\mathcal{X}, d)$ satisfying the inequality*

$$\begin{aligned} & F(d(Ax, Ay), M(d(Sx, Sy)), M(d(Sx, Ax)), \\ & \qquad M(d(Sy, Ay)), M(d(Sx, Ay)), M(d(Sy, Ax))) \le 0, \end{aligned}$$

for all x, $y \in \mathcal{X}$, where F is in $F(M)$ and M is a contractive modulus. If A and S are occasionally weakly compatible, then A and S have a unique common fixed point.

Corollary 2. *Let A and S be self mappings of a metric space $(\mathcal{X}, d)$ satisfying the inequality*

$$d(Ax, Ay) \leq \max\{M(d(Sx, Sy)), M(d(Sx, Ax)), M(d(Sy, Ay)), M(d(Sx, Ay)), M(d(Sy, Ax))\}$$

for all x, $y \in \mathcal{X}$. If A and S are occasionally weakly compatible, then A and S have a unique common fixed point.

Proof. The proof follows by Theorem 4 and Example 5. ■

Example 7. Let $X = [1, \infty)$, $Ax = x$, $Sx = 2x - 1$, $Mx = \frac{1}{2}x$ and $d(x, y) = |x - y|$. It follows that $AS(1) = SA(1) = 1$. Hence A and S are owc. On the other hand $d(Ax, Ay) = |x - y|$, $M(d(Sx, Sy)) = \frac{1}{2}d(Sx, Sy) = |x - y|$. Therefore

$$d(Ax, Ay) \leq \max\{M(d(Sx, Sy)), M(d(Sx, Ax)), M(d(Sy, Ay)), M(d(Sx, Ay)), M(d(Sy, Ax))\}$$

by Theorem 4, A and S have a unique common fixed point which is $x = 1$ because $A(1) = S(1) = 1$.

References

[1] Al-Thagafi M.A., Shahzad N., Generalized I-nonexpansive selfmaps and invariant approximations, *Acta Math. Sin. (Engl. Ser.)*, 24(5)(2008), 867-876.

[2] Bouhadjera H., On common fixed point theorems for three and four self mappings satisfying contractive conditions, *Acta Univ. Palacki. Olomuc., Fac. Rer. Nat., Mathematica* 49, 1(2010), 25-31.

[3] Jungck G., Compatible mappings and common fixed points, *Internat. J. Math. Math. Sci.,* 9(4)(1986), 771-779.

[4] Jungck G., Rhoades B.E., Fixed point theorems for occasionally weakly compatible mappings, *Fixed Point Theory,* 7(2)(2006), 287-296.

[5] Popa V., Some fixed point theorems for compatible mappings satisfying an implicit relation, *Dem. Math.,* 32(1999), 157-163.

[6] Popa V., Some fixed point theorems for implicit contractive mappings, *Stud. Cerc. St. Ser. Matem. Univ. Bacău,* 7(1997), 127-133.

[7] Sessa S., On a weak commutativity condition in fixed point considerations, *Publ. Inst. Math. (Beograd) (N.S.),* 32(46)(1982), 149-153.

H. Bouhadjera
Laboratoire de Mathématiques Appliquées
Université Badji Mokhtar
B.P. 12, 23000, Annaba, Algérie
e-mail: b_hakima2000@yahoo.fr

V. Popa
Department of Mathematics
University Vasile Alecsandri of Bacău
Bacău, Romania
e-mail: vpopa@ub.ro

Received on 16.01.2012 and, in revised form, on 27.02.2013.