F A S C I C U L I   M A T H E M A T I C I

Nr 51 2013

Sumit Chandok* and T. D. Narang**

INVARIANT POINTS OF BEST APPROXIMATION AND BEST SIMULTANEOUS APPROXIMATION

Abstract. In this paper we generalize and extend Brosowski-Meinardus type results on invariant points from the set of best approximation to the set of best simultaneous approximation, which is not necessarily starshaped. As a consequence some results on best approximation are deduced. The proved results extend and generalize some of the results of R. N. Mukherjee and V. Verma [Publ. de l'Inst. Math. 49(1991) 111-116], T.D. Narang and S. Chandok [Selçuk J. Appl. Math. 10(2009) 75-80; Indian J. Math. 51(2009) 293-303], and of G. S. Rao and S. A. Mariadoss [Serdica-Bulgaricae Math. Publ. 9(1983) 244-248].

Key words: approximatively compact set, starshaped set, best approximation, best simultaneous approximation, jointly continuous contractive family, nonexpansive and quasi-nonexpansive mappings.

AMS Mathematics Subject Classification: 41A50, 41A65, 47H10, 54H25.

1. Introduction and preliminaries

The idea of applying fixed point theorems to approximation theory was initiated by G. Meinardus [11]. Meinardus introduced the notion of invariant approximation in normed linear spaces. Generalizing the result of Meinardus, Brosowski [2] proved the following theorem on invariant approximation using fixed point theory:

Theorem 1. *Let T be a linear and nonexpansive operator on a normed linear space E. Let C be a T-invariant subset of E and x a T-invariant point. If the set $P_C(x)$ of best C-approximants to x is non-empty, compact and convex, then it contains a T-invariant point.*

Subsequently, various generalizations of Brosowski's results appeared in the literature. Singh [19] observed that the linearity of the operator T and

* The corresponding author: chansok.s@gmail.com (Sumit Chandok).

** The research has been supported by the U.G.C., India under Emeritus Fellowship.

convexity of the set $P_C(x)$ in Theorem 1 can be relaxed and proved the following:

Theorem 2. *Let $T : E \to E$ be a nonexpansive self mapping on a normed linear space E. Let C be a T-invariant subset of E and x a T-invariant point. If the set $P_C(x)$ is non-empty, compact and starshaped, then it contains a T-invariant point.*

Singh [20] further showed that Theorem 2 remains valid if T is assumed to be nonexpansive only on $P_C(x) \cup \{x\}$.

Since then, many results have been obtained in this direction (see Chandok and Narang [3] [4], Mukherjee and Som [12], Mukherjee and Verma [13], Narang and Chandok [14] [15] [16], Rao and Mariadoss [17] and references cited therein). There have been a number of results on invariant approximation in different abstract spaces. In this paper we prove some similar types of results on T-invariant points for the set of best simultaneous approximation to a pair of points x_1, x_2 in a metric space (X, d) from a set C, which is not necessarily starshaped but has a jointly continuous contractive family. Some results on T-invariant points for the set of best approximation are also deduced. The results proved in the paper generalize and extend some of the results of [13], [14], [15], [17] and of few others.

For a non-empty subset K of a metric space (X, d) and $x \in X$, an element $g_\circ \in K$ is said to be (s.t.b.) a **best approximant** to x or a **best K-approximant** to x if $d(x, g_\circ) = d(x, K) \equiv \inf\{d(x, g) : g \in K\}$. The set of all such $g \in K$ is denoted by $P_K(x)$. Let x_1 and x_2 be two elements of X. An element $g_\circ \in K$ is s.t.b. a best simultaneous approximant to x_1, x_2 if $d(x_1, g_\circ) + d(x_2, g_\circ) = \inf\{d(x_1, g) + d(x_2, g) : g \in K\}$. The set of all such $g_\circ \in K$ is called the set of best simultaneous K-approximant to x_1, x_2.

A sequence $< y_n >$ in K is called a **minimizing sequence** for x if $\lim_{n\to\infty} d(x, y_n) = d(x, K)$. The set K is said to be **approximatively compact** if for each $x \in X$, every minimizing sequence $< y_n >$ in K has a subsequence $< y_{n_i} >$ converging to an element of K.

In the context of best simultaneous approximation, a sequence $< y_n >$ in K is called a **minimizing sequence** for x_1, x_2 if $\lim_{n\to\infty}[d(x_1, y_n) + d(x_2, y_n)] = \inf\{d(x_1, y) + d(x_2, y) : y \in K\}$. The set K is said to be **approximatively compact** if for every pair $x_1, x_2 \in X$, every minimizing sequence $< y_n >$ in K has a subsequence $< y_{n_i} >$ converging to an element of K.

Let (X, d) be a metric space. A mapping $W : X \times X \times [0, 1] \to X$ is said to generate a **convex structure** A on X if for all $x, y \in X$ and $\lambda \in [0, 1]$,

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y),$$

holds for all $u \in A$. The metric space (X, d) together with a convex structure is called a **convex metric space** if $A = X$.

A convex metric space (X, d) is said to satisfy **Property (I)** [6] if for all $x, y, p \in X$ and $\lambda \in [0, 1]$,

$$d(W(x, p, \lambda), W(y, p, \lambda)) \leq \lambda d(x, y).$$

A normed linear space and each of its convex subset are simple examples of convex metric spaces. There are many convex metric spaces which are not normed linear spaces (see [23]). Property (I) is always satisfied in a normed linear space.

A subset K of a convex metric space (X, d) is s.t.b.

(i) a **convex set** [23] if $W(x, y, \lambda) \in K$ for all $x, y \in K$ and $\lambda \in [0, 1]$;

(ii) **starshaped** or **p-starshaped** [9] if there exists $p \in K$ such that $W(x, p, \lambda) \in K$ for all $x \in K$ and $\lambda \in [0, 1]$.

Clearly, each convex set is starshaped but not conversely.

A self map T on a metric space (X, d) is s.t.b.

(i) **nonexpansive** if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in X$;

(ii) **quasi-nonexpansive** if the set $F(T)$ of fixed points of T is non-empty and $d(Tx, p) \leq d(x, p)$ for all $x \in X$ and $p \in F(T)$.

A nonexpansive mapping T on X with $F(T) \neq \emptyset$ is quasi-nonexpansive, but not conversely. A linear quasi-nonexpansive mapping on a Banach space is nonexpansive. But there exist continuous and discontinuous nonlinear quasi-nonexpansive mappings that are not nonexpansive.

Example 1 (see [21], p.27)**.** Consider the metric space $\mathbb{R}$ with usual metric. The mapping $T : \mathbb{R} \to \mathbb{R}$ is defined by

$$T(x) = \begin{cases} \frac{x}{2} \sin \frac{1}{x}, & \text{if } \ x \neq 0 \\ 0, & \text{if } \ x = 0 \end{cases}$$

The only fixed point of T is 0. T is quasi-nonexpansive since for $x \in \mathbb{R}$, we have

$$d(Tx, 0) = |Tx| = \left|\frac{x}{2}\right| \left|\sin \frac{1}{x}\right| \leq \frac{|x|}{2} < |x| = d(x, 0).$$

Hence T is not nonexpansive. Indeed, if $x = \frac{2}{7\pi}$ and $y = \frac{2}{17\pi}$, then $d(x, y) = |x-y| = \frac{20}{119\pi}$, and $d(Tx, Ty) = \frac{1}{2}\left|\frac{2}{7\pi} \sin \frac{7\pi}{2} - \frac{2}{17\pi} \sin \frac{17\pi}{2}\right| = \frac{1}{2}\left|-\frac{2}{7\pi} - \frac{2}{17\pi}\right| = \frac{24}{119\pi}$. Thus $d(Tx, Ty) > d(x, y)$.

Let C be a subset of a metric space (X, d) and $\mathfrak{F} = \{f_\alpha : \alpha \in C\}$ a family of functions from $[0, 1]$ into C, having the property $f_\alpha(1) = \alpha$, for each $\alpha \in C$. Such a family $\mathfrak{F}$ is said to be **contractive** if there exists a function $\Phi : (0, 1) \to (0, 1)$ such that for all $\alpha, \beta \in C$ and for all $t \in (0, 1)$,we have

$$d(f_\alpha(t), f_\beta(t)) \leq \Phi(t)d(\alpha, \beta).$$

Such a family $\mathfrak{F}$ is said to be **jointly continuous** if $t \to t_\circ$ in $[0, 1]$ and $\alpha \to \alpha_\circ$ in C imply $f_\alpha(t) \to f_{\alpha_\circ}(t_\circ)$ in C.

In normed linear spaces these notions were discussed by Dotson [5]. It was observed in [5] that if C is a starshaped subset (of a normed linear space) with star-center p then the family $\mathfrak{F} = \{f_\alpha : \alpha \in C\}$ defined by $f_\alpha(t) = (1 - t)p + t\alpha$ is contractive if we take $\Phi(t) = t$ for $0 < t < 1$, and is jointly continuous. The same is true for starshaped subsets of convex metric spaces with Property (I), by taking $f_\alpha(t) = W(\alpha, p, t)$ and so the class of subsets of X with the property of contractiveness and joint continuity contains the class of starshaped sets which in turn contains the class of convex sets.

2. Main results

The following proposition will be used in the sequel:

Proposition 1 (see, [14]). *Let C be a non-empty approximatively compact subset of a metric space (X, d), $x \in X$ and P_C be the metric projection of X onto C defined by $P_C(x) = \{y \in C : d(x, y) = d(x, C)\}$. Then $P_C(x)$ is a non-empty compact subset of C.*

Note. It can be easily seen (see Singer [22], p. 380) that $P_C(x)$ is always a bounded set and is closed if C is closed.

In the context of best simultaneous approximation, we have:

Proposition 2. *Let C be a non-empty approximatively compact subset of a metric space (X, d), $x_1, x_2 \in X$ and D be the set of best simultaneous C-approximants to x_1, x_2. Then D is a non-empty compact subset of C.*

Proof. By the definition of D, there is a sequence $< y_n >$ in C such that

$$d(x_1, y_n) + d(x_2, y_n) = \inf\{d(x_1, y) + d(x_2, y)\} \tag{1}$$

i.e. $< y_n >$ is a minimizing sequence for the pair x_1, x_2 in X. Since C is approximatively compact, there is a subsequence $< y_{n_i} >$ such that $< y_{n_i} > \to y \in C$. Consider

$$\begin{aligned} d(x_1, y) + d(x_2, y) &= \{d(x_1, \lim y_{n_i}) + d(x_2, \lim y_{n_i})\} \\ &= \lim\{d(x_1, y_{n_i}) + d(x_2, y_{n_i})\} \\ &= \inf\{d(x_1, y) + d(x_2, y)\} \end{aligned}$$

i.e. $y \in D$ and hence D is non-empty.

Now we show that D is compact. Let $< y_n >$ be a sequence in D i.e. $\lim[d(x_1, y_n) + d(x_2, y_n)] = \inf\{d(x_1, y) + d(x_2, y)\}$. Then proceeding as above, we get a subsequence $< y_{n_i} >$ of $< y_n >$ converging to an element $y \in D$. This shows that D is compact. ■

The following result of Hardy and Rogers [7] will be used in the sequel:

Lemma 1. *Let F be a mapping from a complete metric space (X, d) into itself satisfying*

(2) $d(Fx, Fy) \le a[d(x, Fx) + d(y, Fy)] + b[d(y, Fx) + d(x, Fy)] + cd(x, y),$

for any $x, y \in X$ where a, b and c are non-negative numbers such that $2a + 2b + c \le 1$. Then F has a unique fixed point u in X. Infact for any $x \in X$, the sequence $\{F^n x\}$ converges to u.

The following is an example of discontinuous mapping which satisfies inequality (2).

Example 2 (see, [10]). Let $F : [-2, 1] \to [-2, 1]$ be defined as

$$Fx = \begin{cases} \frac{x}{2}, & \text{if } x \in [0, 1) \\ \frac{-1}{8}, & \text{if } x = 1 \\ \frac{-x}{2}, & \text{if } x \in [-2, 0) \end{cases}$$

Theorem 3. *Let T be a continuous self map on a complete metric space (X, d) satisfying (2), C a T-invariant subset of X. Let $Tx_i = x_i$ $(i = 1, 2)$ for some x_1, x_2 not in the norm closure of C. If the set of best simultaneous C-approximant to x_1, x_2 is nonempty, compact and has a contractive jointly continuous family $\mathfrak{F}$, then it contains a T-invariant point.*

Proof. Let D be the set of best simultaneous C-approximants to x_1, x_2. Then

(3) $D = \{z \in C : d(x_1, z) + d(x_2, z) \le d(x_1, y) + d(x_2, y), \text{ for every } y \in C\}.$

Let $z \in D$. Then by (2), we have

$$\begin{aligned} d(x_1, Tz) + d(x_2, Tz) &= d(Tx_1, Tz) + d(Tx_2, Tz) \\ &\le a[d(x_1, Tx_1) + d(z, Tz)] + b[d(z, Tx_1) + d(x_1, Tz)] + cd(x_1, z) \\ &\quad + a[d(x_2, Tx_2) + d(z, Tz)] + b[d(z, Tx_2) + d(x_2, Tz)] + cd(x_2, z) \\ &= 2ad(z, Tz) + (b + c)[d(x_1, z) + d(x_2, z)] + b[d(x_1, Tz) + d(x_2, Tz] \\ &= a[d(z, Tz) - d(x_1, Tz)] + a[d(z, Tz) - d(x_2, Tz)] + a[d(x_1, Tz) \\ &\quad + d(x_2, Tz)] + (b + c)[d(x_1, z) + d(x_2, z)] + b[d(x_1, Tz) + d(x_2, Tz)]. \end{aligned}$$

This gives,

$$(1-a-b)[d(x_1,Tz)+d(x_2,Tz)] \leq (a+b+c)[d(x_1,z)+d(x_2,z)].$$

Hence

$$d(x_1,Tz)+d(x_2,Tz) \leq d(x_1,z)+d(x_2,z) \tag{4}$$

since $2a+2b+c \leq 1$. Also, using (3), we get

$$d(x_1,Tz)+d(x_2,Tz) \leq d(x_1,y)+d(x_2,y) \tag{5}$$

for all $y \in C$. Hence $Tz \in D$. Therefore T is a self map on D. Define $T_n : D \to D$ as $T_n x = f_{Tx}(\lambda_n)$, $x \in D$ where $< \lambda_n >$ is a sequence in $(0,1)$ such that $\lambda_n \to 1$. Also

$$\begin{aligned} d(T_n x, T_n y) &= d(f_{Tx}(\lambda_n), f_{Ty}(\lambda_n)) \\ &\leq \Phi(\lambda_n) d(Tx,Ty) \\ &\leq \Phi(\lambda_n)[a[d(x,Tx)+d(y,Ty)] \\ &\quad + b[d(y,Tx)+d(x,Ty)]+cd(x,y)] \end{aligned}$$

where $\Phi(\lambda_n)[2a+2b+c] \leq 1$. Therefore by Lemma 1 T_n has a unique fixed point in D. Let $T_n z_n = z_n$. Since D is compact, there is a subsequence $< z_{n_i} >$ of $< z_n >$ such that $z_{n_i} \to z_\circ \in D$. We claim that $Tz_\circ = z_\circ$. Consider $z_{n_i} = T_{n_i} z_{n_i} = f_{Tz_{n_i}}(\lambda_{n_i}) \to f_{Tz_\circ}(1)$ as the family $\mathfrak{F}$ is jointly continuous and T is continuous. Thus $z_{n_i} \to Tz_\circ$ and consequently, $Tz_\circ = z_\circ$ i.e. $z_\circ \in D$ is a T-invariant point. ■

Since for an approximatively compact subset C of a metric space (X,d) the set of best simultaneous C-approximant is nonempty and compact (Proposition 2), we have:

Corollary 1. *Let T be a continuous self map on a complete metric space (X,d) satisfying (2), C an approximatively compact and T-invariant subset of X. Let $Tx_i = x_i$ $(i = 1,2)$ for some x_1, x_2 not in the norm closure of C. If the set D of best simultaneous C-approximants to x_1, x_2 is nonempty and has a contractive jointly continuous family $\mathfrak{F}$, then it contains a T-invariant point.*

Corollary 2. *Let T be a continuous self map on a complete convex metric space (X,d) with Property (I) satisfying (2), C an approximatively compact and T-invariant subset of X. Let $Tx_i = x_i$ $(i = 1,2)$ for some x_1, x_2 not in the norm closure of C. If the set D of best simultaneous C-approximants to x_1, x_2 is nonempty and p-starshaped, then it contains a T-invariant point.*

Proof. Define $f_\alpha : [0,1] \to D$ as $f_\alpha(t) = W(\alpha, p, t)$. Then

$$d(f_\alpha(t), f_\beta(t)) = d(W(\alpha, p, t), W(\beta, p, t)) \leq td(\alpha, \beta),$$

$\Phi(t) = t$, $0 < t < 1$, i.e. D is a contractive jointly continuous family. Taking $\lambda_n = \frac{n}{n+1}$ and defining $T_n(x) = f_{Tx}(\lambda_n) = W(Tx, p, \lambda_n)$, we get the result using Theorem 3. ■

Corollary 3 (see [13])**.** *Let T be a continuous self map on a Banach space X satisfying (2), C an approximatively compact and T-invariant subset of X. Let $Tx_i = x_i$ $(i = 1, 2)$ for some x_1, x_2 not in the norm closure of C. If the set of best simultaneous C-approximants to x_1, x_2 is nonempty and starshaped, then it contains a T-invariant point.*

If $a = b = 0$ in Theorem 3 then map T becomes nonexpansive, so we have:

Corollary 4 (see [15])**.** *Let T be a mapping on a metric space (X, d), C a T-invariant subset of X and x a T-invariant point. If $P_C(x)$ is a non-empty, compact set for which there exists a contractive jointly continuous family $\mathfrak{F}$ of functions and T is non-expansive on $P_C(x) \cup \{x\}$ then $P_C(x)$ contains a T-invariant point.*

Corollary 5 (see [12]-Theorem 2, [18]-Theorem 3.4)**.** *Let T be nonexpansive operator on a normed linear space X. Let C be a T-invariant subset of X and x a T-invariant point. If $P_C(x)$ is non-empty, compact and for which there exists a contractive jointly continuous family $\mathfrak{F}$ of functions, then it contains a T-invariant point.*

Since for an approximatively compact subset C of a metric space (X, d) the set $P_C(x)$ is non-empty and compact (Proposition 1), we have:

Corollary 6. *Let T be a mapping on a metric space (X, d), C an approximatively compact, T-invariant subset of X and x a T-invariant point. If there exists a contractive jointly continuous family $\mathfrak{F}$ of functions and T is nonexpansive on $P_C(x) \cup \{x\}$, then $P_C(x)$ contains a T-invariant point.*

Corollary 7. *Let T be a mapping on a convex metric space (X, d) with Property (I), C an approximatively compact, p-starshaped, T-invariant subset of X and x a T-invariant point. If T is nonexpansive on $P_C(x) \cup \{x\}$, then $P_C(x)$ contains a T-invariant point.*

Corollary 8 (see [14]-Theorem 4)**.** *Let T be a quasi-nonexpansive mapping on a convex metric space (X, d) with Property (I), C a T-invariant subset of X and x a T-invariant point. If $P_C(x)$ is nonempty, compact and starshaped, and T is nonexpansive on $P_C(x)$, then $P_C(x)$ contains a T-invariant point.*

Using Proposition 1, we have:

Corollary 9 (see [14]-Theorem 5). *Let T be a quasi-nonexpansive mapping on a convex metric space (X, d) with Property (I), C an approximatively compact, T-invariant subset of X and x a T-invariant point. If $P_C(x)$ is starshaped and T is nonexpansive on $P_C(x)$, then $P_C(x)$ contains a T-invariant point.*

Remark 1. Theorem 3 is a generalization and extension of Theorem 1 of Rao and Mariadoss [17] for a mapping T which maps the set D of best simultaneous C-approximants to $x_1, x_2 \in X$ into itself and the spaces undertaken are metric spaces.

We shall be using the following result of Bose and Mukherjee [1] which is a generalization of a result of Iseki [8].

Lemma 2. *Let $\{F_n\}$ be a sequence of self mappings of complete metric space (X, d) such that*

$$\begin{aligned} d(F_i x, F_j y) \leq\ & a_1 d(x, F_i x) + a_2 d(y, F_j y) + a_3 d(y, F_i x) \\ & + a_4 d(x, F_i y) + a_5 d(x, y), \quad (j > i) \end{aligned} \tag{6}$$

for all $x, y \in X$ where $a_1, a_2, \ldots, a_5$ are non-negative numbers such that $\sum_{k=1}^{5} a_k < 1$ and $a_3 = a_4$. Then the sequence $\{F_n x\}$ has a unique common fixed point.

The following result generalizes Theorem 3.

Theorem 4. *Let T_1 and T_2 be a pair of continuous self maps on a complete metric space (X, d) satisfying $d(T_1 x, T_2 y) \leq a\ d(x, y)$, for $x, y \in X$ $(x \neq y)$, $0 < a \leq 1$. Let for $i = 1, 2$ C be a T_i-invariant subset of X. Suppose that x_1 and x_2 be two common fixed points for the pair T_1 and T_2 not in the norm closure of C. If the set D of best simultaneous C-approximants to x_1, x_2 is nonempty, compact and has a contractive jointly continuous family $\mathfrak{F}$, then it has a point which is both T_1- and T_2-invariant.*

Proof. Since x_1 and x_2 are common fixed points of T_1 and T_2, proceeding as in Theorem 3, we get that $T_1(D) \subseteq D$ and $T_2(D) \subseteq D$. Now we proceed to show that there is a point $z_\circ \in D$ such that $T_i z_\circ = z_\circ$ $(i = 1, 2)$. Define T_{1m} and T_{2n} as $T_{1m} x = f_{T_1 x}(\lambda_m)$, and $T_{2n} x = f_{T_2 x}(\lambda_n)$, $x \in D$ where $< \lambda_m >$ and $< \lambda_n >$ are sequences in $(0, 1)$ such that $< \lambda_m >, < \lambda_n > \to 1$. Then using Lemma 2, we have $T_{1m} z_n = T_{2n} z_n = z_n \in D$. Since D is compact, there is a subsequence $< z_{n_i} >$ of $< z_n >$ such that $z_{n_i} \to z_\circ \in D$. We claim that $T_1 z_\circ = z_\circ = T_2 z_\circ$. Consider $z_{n_i} = T_{1m_i} z_{n_i} = f_{T_1 z_{n_i}}(\lambda_{m_i}) \to f_{T_1 z_\circ}(1)$ as the family $\mathfrak{F}$ is jointly continuous and T_{1m} is continuous. Thus $z_{n_i} \to T_1 z_\circ$ and similarly, $z_{n_i} \to T_2 z_\circ$. Hence the result. ■

Corollary 10. *Let T_1 and T_2 be a pair of continuous self maps on a complete metric space (X, d) satisfying $d(T_1x, T_2y) \leq a\ d(x, y)$, for $x, y \in X$ $(x \neq y)$, $0 < a \leq 1$. Let C be an approximatively compact, T_i-invariant $(i = 1, 2)$ subset of X. Suppose that x_1 and x_2 be two common fixed points for the pair T_1 and T_2 not in the norm closure of C. If the set D of best simultaneous C-approximants to x_1, x_2 is nonempty and has a contractive jointly continuous family $\mathfrak{F}$, then it has a point which is both T_1- and T_2-invariant.*

Corollary 11. *Let T_1 and T_2 be a pair of continuous self maps on a complete convex metric space (X, d) with Property (I) satisfying $d(T_1x, T_2y) \leq a\ d(x, y)$, for $x, y \in X$ $(x \neq y)$, $0 < a \leq 1$. Let C be an approximatively compact, T_i-invariant $(i = 1, 2)$ subset of X. Suppose that x_1 and x_2 be two common fixed points for the pair T_1 and T_2 not in the norm closure of C. If the set D of best simultaneous C-approximants to x_1, x_2 is nonempty and starshaped, then it has a point which is both T_1- and T_2-invariant.*

Corollary 12 (see [13])**.** *Let T_1 and T_2 be a pair of continuous self maps on a Banach space X satisfying $d(T_1x, T_2y) \leq d(x, y)$, for $x, y \in X$ $(x \neq y)$. Let C be an approximatively compact, T_i-invariant $(i = 1, 2)$ subset of X. Suppose that x_1 and x_2 be two common fixed points for the pair T_1 and T_2 not in the norm closure of C. If the set D of best simultaneous C-approximants to x_1, x_2 is nonempty and starshaped, then it has a point which is both T_1- and T_2-invariant.*

Acknowledgements. The authors are thankful to the learned referee for valuable suggestions and comments.

References

[1] Bose R.K., Mukherjee R.N., Stability of fixed point sets and common fixed points of famililes of mappings, *Indian J. Pure Appl. Math.,* 11(1980), 1130-1138.

[2] Brosowski B., Fixpunktsätze in der approximations theorie, *Mathematica* (*Cluj*), 11(1969), 195-220.

[3] Chandok S., Narang T.D., Common fixed points with applications to best simultaneous approximations, *Anal. Theory Appl.,* 28(1)(2012), 1-12.

[4] Chandok S., Narang T.D., Common fixed points of nonexpansive mappings with applications to best and best simultaneous approximation, *J. Appl. Anal.,* 18(1)(2012), 33-46 doi:10.1515/jaa-2012-0002.

[5] Dotson W.G., On fixed points of nonexpansive mappings in nonconvex sets, *Proc. Amer. Math. Soc.,* 38(1973), 155-156.

[6] Guay M.D., Singh K.L., Whitfield J.H.M., Fixed point theorems for nonexpansive mappings in convex metric spaces, *Proc. Conference on nonlinear analysis* (*Ed. S.P. Singh and J.H. Bury*) *Marcel Dekker,* 80(1982), 179-189.

[7] Hardy G.E., Rogers T.D., A generalization of a fixed point theorem of Reich, *Canad. Math. Bull.,* 16(1973), 201-206.

[8] Iseki K., On common fixed points of mappings, *Bull. Austr. Math. Soc.,* 10(1975), 365-370.

[9] ITOH S., Some fixed point theorems in metric spaces, *Fundamenta Mathematicae,* 52(1979), 109-117.
[10] JOSHI M.C., BOSE R.K., *Some Topics in Nonlinear Functional Analysis,* Wiley Eastern, New Delhi, 1985.
[11] MEINARDUS G., Invarianz bei linearen approximationen, *Arch. Rational Mach. Anal.,* 14(1963), 301-303.
[12] MUKHERJEE R.N., SOM T., A note on application of a fixed point theorem in approximation theory, *Indian J. Pure Appl. Math.,* 16(1985), 243-244.
[13] MUKHERJEE R.N., VERMA V., Some fixed point theorems and their applications to best simultaneous approximations, *Publ. de l'Inst. Math.,* 49(1991), 111-116.
[14] NARANG T.D., CHANDOK S., Fixed points of quasi-nonexpansive mappings and best approximation, *Selçuk J. Appl. Math.,* 10(2009), 75-80.
[15] NARANG T.D., CHANDOK S., Fixed points and best approximation in metric spaces, *Indian J. Math.,* 51(2009), 293-303.
[16] NARANG T.D., CHANDOK S., Some fixed point theorems with applications to best simultaneous approximation, *J. Nonlinear Sci. Appl.,* 3(2010), 87-95.
[17] RAO G.S., MARIADOSS S.A., Applications of fixed point theorems to best approximations, *Serdica-Bulgaricae Math. Publ.,* 9(1983), 244-248.
[18] SAHAB S.A., KHAN M.S., Some results on best approximation, *Review of Research,* 17(1987), 143-152.
[19] SINGH S.P., An application of a fixed-point theorem to approximation theory, *J. Approx. Theory,* 25(1979), 89-90.
[20] SINGH S.P., Application of fixed point theorems in approximation theory, *Appl. Nonlinear Anal.(Ed. V. Lakshmikantham), Academic Press,* New York, 1979, 389-397.
[21] SINGH S., WATSON B., SRIVASTAVA P., *Fixed Point Theory and Best Approximation: The KKM-map Principle,* Kluwer Academic Publishers, Dordrecht, 1997.
[22] SINGER I., *Best Apprroximation in Normed Linear Spaces by Elements of Linear Subspaces,* Springer-Verlag, New York, 1970.
[23] TAKAHASHI W., A convexity in metric space and nonexpansive mappings I, *Kodai Math. Sem. Rep.,* 22(1970), 142-149.

SUMIT CHANDOK
DEPARTMENT OF MATHEMATICS
KHALSA COLLEGE OF ENGINEERING & TECHNOLOGY
PUNJAB TECHNICAL UNIVERSITY
RANJIT AVENUE, AMRITSAR-143001, PUNJAB, INDIA
e-mail: chansok.s@gmail.com or chandhok.sumit@gmail.com

T. D. NARANG
DEPARTMENT OF MATHEMATICS
GURU NANAK DEV UNIVERSITY
AMRITSAR-143005, INDIA
e-mail: tdnarang1948@yahoo.co.in

Received on 17.04.2012 and, in revised form, on 20.11.2012.