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SLIGHTLY (μ, λ) -CONTINUOUS FUNCTIONS

ABSTRACT. We introduce a new notion called slightly (μ, λ) -continuous functions on generalized topological spaces. Furthermore, basic properties and preservation theorems of slightly (μ, λ) -continuous functions are investigated and relationships between slightly (μ, λ) -continuous functions and graphs are investigated.

KEY WORDS: generalized topological space, slightly (μ, λ) -continuous, μ -Lindelöf, μ -compact.

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1. Introduction and preliminaries

In [2]-[5], Császár founded the theory of generalized topological spaces, and studied the elementary character of these classes. Especially he introduced the notions of continuous functions on generalized topological spaces, and investigated characterizations of generalized continuous functions (= (μ, λ) -continuous functions in [3]). We recall some notions defined in [3]. Let X be a non empty set and $\exp(X)$ the power set of X . We call a class $\mu \subseteq \exp(X)$ a generalized topology [3]. (briefly, GT) if $\phi \in \mu$ and the arbitrary union of elements of μ belongs to μ . A set X with a GT μ on it is called a generalized topological space (briefly, GTS) and is denoted by (X, μ) .

For a GTS (X, μ) , the elements of μ are called μ -open sets and the complements of μ -open sets are called μ -closed sets. For $A \subseteq X$, we denote by $\mu Cl(A)$ the intersection of all μ -closed sets containing A , i.e., the smallest μ -closed set containing A ; and by $\mu \text{int}(A)$ the union of all μ -open sets contained in A , i.e., the largest μ -open set contained in A (see [3]). Also motivated by the various definitions of continuity based on weak open sets, the following notions of continuity have been defined for functions on GTS.

Definition 1. Let $f : (X, \mu) \rightarrow (Y, \lambda)$ be a function on generalized topological spaces. Then the function f is said to be:

- (i) (μ, λ) -continuous [3] if $V \in \lambda$ implies that $f^{-1}(V) \in \mu$.

(ii) *weakly (μ, λ) -continuous [7]* if for each $x \in X$ and each λ -open set V containing $f(x)$, there exists a μ -open set U containing x such that $f(U) \subseteq \lambda Cl(V)$.

(iii) *contra (μ, λ) -continuous [1]* if $f^{-1}(V)$ is μ -closed in X for each λ -open set V of Y .

(iv) *(μ, λ) -open [8]* if for every μ -open subset A of X , $f(A)$ is λ -open in Y .

In this paper, motivated by the notion of slightly continuous functions introduced in [6], [12], we introduce the generalized version of this notion for GTS. Furthermore, basic properties and preservation theorems of such functions are investigated and relationships between these functions and their graphs are investigated.

2. Slightly (μ, λ) -continuous functions

In this section, the notion of slightly (μ, λ) -continuous functions is introduced. If A is both μ -open and μ -closed, then it is said to be μ -clopen.

Definition 2. Let $f : (X, \mu) \rightarrow (Y, \lambda)$ be a function on generalized topological spaces. Then the function f is said to be:

(i) *slightly (μ, λ) -continuous at point $x \in X$* if for each λ -clopen subset V in Y containing $f(x)$, there exists a μ -open subset U in X containing x such that $f(U) \subseteq V$.

(ii) *slightly (μ, λ) -continuous* if it has this property at each point of X .

Theorem 1. For a function $f : (X, \mu) \rightarrow (Y, \lambda)$, the following properties are equivalent:

(i) *f is slightly (μ, λ) -continuous.*

(ii) *For every λ -clopen set $V \subseteq Y$, $f^{-1}(V)$ is μ -open.*

(iii) *For every λ -clopen set $V \subseteq Y$, $f^{-1}(V)$ is μ -closed.*

(iv) *For every λ -clopen set $V \subseteq Y$, $f^{-1}(V)$ is μ -clopen.*

Proof. (i) \rightarrow (ii): Let V be a λ -clopen subset of Y and let $x \in f^{-1}(V)$, by (i) there exists a μ -open set U_x in X containing x such that $f(U_x) \subseteq V$, hence $x \in U_x \subseteq f^{-1}(V)$. Now take $f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x$. Thus, $f^{-1}(V)$ is μ -open.

(ii) \rightarrow (iii): Let V be a λ -clopen subset of Y , then $Y - V$ is λ -clopen. By (ii) $f^{-1}(Y - V) = X - f^{-1}(V)$ is μ -open, hence $f^{-1}(V)$ is μ -closed.

(iii) \rightarrow (iv): Trivial.

(iv) \rightarrow (i): For $x \in X$ and any λ -clopen set $V \subseteq Y$ containing $f(x)$, by (iv) $f^{-1}(V)$ is a μ -clopen set containing x . Take $U = f^{-1}(V)$ such that $f(U) \subseteq V$. Hence f is slightly (μ, λ) -continuous. \blacksquare

Proposition 1. *If $f : (X, \mu) \rightarrow (Y, \lambda)$ is slightly (μ, λ) -continuous and $g : (Y, \lambda) \rightarrow (Z, \rho)$ is slightly (λ, ρ) -continuous, then $g \circ f : (X, \mu) \rightarrow (Z, \rho)$ is slightly (μ, ρ) -continuous.*

Proof. Let V be any ρ -clopen set in Z . By slight (λ, ρ) -continuity of g , $g^{-1}(V)$ is λ -clopen. Since f is slightly (μ, λ) -continuous, $f^{-1}(g^{-1}(V))$ is μ -open. Therefore, $g \circ f$ is slightly (μ, ρ) -continuous. ■

Corollary 1. *Let $f : (X, \mu) \rightarrow (Y, \lambda)$ be a (μ, λ) -continuous function on GTS and $g : (Y, \lambda) \rightarrow (Z, \rho)$ be a slightly (λ, ρ) -continuous function on GTS, then $g \circ f : (X, \mu) \rightarrow (Z, \rho)$ is slightly (μ, ρ) -continuous.*

Proposition 2. *Let $f : (X, \mu) \rightarrow (Y, \lambda)$ be a slightly (μ, λ) -continuous function on GTS and $g : (Y, \lambda) \rightarrow (Z, \rho)$ be a function on GTS. If f is (μ, λ) -open and surjective and $g \circ f : (X, \mu) \rightarrow (Z, \rho)$ is slightly (μ, ρ) -continuous, then g is slightly (λ, ρ) -continuous.*

Proof. Let V be a ρ -clopen subset of Z . Then $(g \circ f)^{-1}(V)$ is μ -open and since f is (μ, λ) -open and surjective, it follows that $g^{-1}(V)$ is λ -open in Y . ■

Remark 1. The following diagram holds:

$$\begin{array}{c} (\mu, \lambda)\text{-continuous} \rightarrow \text{weakly } (\mu, \lambda)\text{-continuous} \\ \downarrow \\ \text{contra } (\mu, \lambda)\text{-continuous} \rightarrow \text{slightly } (\mu, \lambda)\text{-continuous} \end{array}$$

None of these implications is reversible.

Example 1. Let $X = \{1, 2\}$ with two GT, $\mu = \{\phi, \{2\}\}$ and $\lambda = \{\phi, X, \{1\}\}$. Let $f : (X, \mu) \rightarrow (Y, \lambda)$ be the function defined by

$$f(x) = \begin{cases} 1, & x = 2, \\ 2, & x = 1. \end{cases}$$

Then f is slightly (μ, λ) -continuous but it is not contra (μ, λ) -continuous.

Example 2. Let $X = \{1, 2, 3\}$ with two GT, $\mu = \{\phi, \{1\}, \{2\}, \{1, 2\}\}$ and $\lambda = \{\phi, X, \{1, 3\}, \{2\}\}$. Let $f : (X, \mu) \rightarrow (X, \lambda)$ be the function defined by

$$f(x) = \begin{cases} 1, & x = 2, \\ 2, & x = 1. \end{cases}$$

Then f is slightly (μ, λ) -continuous but it is not weakly (μ, λ) -continuous. The other implication is not reversible as shown in [7].

Recall that a $GTS(X, \mu)$ is said to be:

(i) μ -extremally disconnected [4] if the μ -closure of every μ -open set of X is μ -open.

(ii) μ -Locally indiscrete if every μ -open set of X is μ -closed in X .

(iii) μ -0-dimensional if the GTS has a μ -base consisting of μ -clopen sets.

Proposition 3. *If $f : (X, \mu) \rightarrow (Y, \lambda)$ is slightly (μ, λ) -continuous and Y is λ -extermally disconnected, then f is weakly (μ, λ) -continuous.*

Proof. Let $x \in X$ and let V be a λ -open subset of Y containing $f(x)$. Then $\lambda Cl(V)$ is λ -open and hence λ -clopen. Therefore, there exists a μ -open set $U \subseteq X$ with $x \in U$ and $f(U) \subseteq \lambda Cl(V)$. Thus f is weakly (μ, λ) -continuous. ■

Proposition 4. *If $f : (X, \mu) \rightarrow (Y, \lambda)$ is slightly (μ, λ) -continuous and Y is λ -locally indiscrete, then f is (μ, λ) -continuous and contra (μ, λ) -continuous.*

Proof. Let V be any λ -open set of Y . Since Y is λ -locally indiscrete, V is λ -clopen and hence $f^{-1}(V)$ is μ -clopen in X . Therefore, f is (μ, λ) -continuous and contra (μ, λ) -continuous. ■

Proposition 5. *If $f : (X, \mu) \rightarrow (Y, \lambda)$ is slightly (μ, λ) -continuous and Y is λ -0-dimensional, then f is (μ, λ) -continuous.*

Proof. Let $x \in X$ and $V \subseteq Y$ be any λ -open set containing $f(x)$. Since Y is λ -0-dimensional, there exists a λ -clopen set U containing $f(x)$ such that $U \subseteq V$. But f is slightly (μ, λ) -continuous and there exists a μ -open set G containing x such that $f(x) \in f(G) \subseteq U \subseteq V$. Hence f is (μ, λ) -continuous. ■

Proposition 6. *Let $f : (X, \mu) \rightarrow (Y, \lambda)$ be a function and $g : (X, \mu) \rightarrow (X \times Y, \mu \times \lambda)$ be the graph function of f , defined by $g(x) = (x, f(x))$ for $x \in X$. If g is slightly $(\mu, \mu \times \lambda)$ -continuous, then f is slightly (μ, λ) -continuous.*

Proof. Let V be a λ -clopen in Y , then $X \times V$ is $(\mu \times \lambda)$ -clopen in $X \times Y$. Since g is slightly $(\mu, \mu \times \lambda)$ -continuous, then $f^{-1}(V) = g^{-1}(X \times V)$ is μ -open in X . Thus, f is slightly (μ, λ) -continuous. ■

A $GTS (X, \mu)$ is said to be μ -connected (connected in [10]) if there are not no non-empty disjoint sets $U, V \in \mu$ such that $U \cup V = X$.

Theorem 2. *If $f : (X, \mu) \rightarrow (Y, \lambda)$ is a slightly (μ, λ) -continuous surjection and X is μ -connected, then Y is λ -connected.*

Proof. Suppose Y is not λ -connected. Then, there exist non empty λ -open sets V_1 and V_2 in Y such that $V_1 \cap V_2 = \phi$ and $V_1 \cup V_2 = Y$. Hence we have $f^{-1}(V_1) \cap f^{-1}(V_2) = \phi$ and $f^{-1}(V_1) \cup f^{-1}(V_2) = X$. Since f is surjection, $f^{-1}(V_j) \neq \phi$ for $j = 1, 2$. Since V_j is λ -clopen in Y , then $f^{-1}(V_j)$ is a μ -clopen set for $j = 1, 2$. This implies that X is not μ -connected. This is a contradiction and hence Y is λ -connected. ■

3. Covering properties

Definition 3. A GTS (X, μ) is said to be:

- (i) μ -mildly compact if every μ -clopen cover of X has a finite subcover.
- (ii) μ -compact [11] if every μ -open cover of X has a finite subcover.

A subset A of a GT space (X, μ) is said to be μ -mildly compact relative to X if every cover of A by μ -clopen sets of X has a finite subcover.

Theorem 3. If a function $f : (X, \mu) \rightarrow (Y, \lambda)$ is slightly (μ, λ) -continuous and K is μ -mildly compact relative to X , then $f(K)$ is λ -mildly compact relative to Y .

Proof. Let $\{V_\alpha : \alpha \in \Delta\}$ be any cover of $f(K)$ by λ -clopen sets of the space Y . Since f is slightly (μ, λ) -continuous, then $\{f^{-1}(V_\alpha) : \alpha \in \Delta\}$ is a μ -clopen cover of K . Since K is μ -mildly compact relative to X , there exists a finite subset Δ_\circ of Δ such that $K \subset \cup\{f^{-1}(V_\alpha) : \alpha \in \Delta_\circ\}$. Thus, we have $f(K) \subset \cup\{V_\alpha : \alpha \in \Delta_\circ\}$ and $f(K)$ is λ -mildly compact relative to Y . ■

Corollary 2. Let $f : (X, \mu) \rightarrow (Y, \lambda)$ be a slightly (μ, λ) -continuous surjection. Then

- (i) If X is μ -mildly compact, then Y is λ -mildly compact.
- (ii) If X is μ -compact, then Y is λ -mildly compact.

Definition 4. A GTS (X, μ) is said to be

- (i) μ -mildly Lindelöf if every cover of X by μ -clopen sets has a countable subcover.
- (ii) μ -mildly countably compact if every countably μ -clopen cover of X has a finite subcover.

Theorem 4. Let $f : (X, \mu) \rightarrow (Y, \lambda)$ be a slightly (μ, λ) -continuous surjection on GTS. Then the following statements hold:

- (i) If X is μ -mildly Lindelöf, then Y is λ -mildly Lindelöf.
- (ii) If X is μ -mildly countably compact, then Y is λ -mildly countably compact.

Proof. We prove (ii), the proof of (i) being entirely analogous. Let $\{V_\alpha : \alpha \in \Delta\}$ be any countably λ -clopen cover of Y . Since f is slightly (μ, λ) -continuous, then $\{f^{-1}(V_\alpha) : \alpha \in \Delta\}$ is a countably μ -clopen cover of X . Since X is μ -mildly countably compact, there exists a finite subset Δ_\circ of Δ such that $X = \cup\{f^{-1}(V_\alpha) : \alpha \in \Delta_\circ\}$. Thus, we have $Y = \cup\{V_\alpha : \alpha \in \Delta_\circ\}$ and Y is λ -mildly countably compact. ■

4. Separation axioms

Definition 5. A GTS (X, μ) is said to be:

(i) $\mu - T_1$ [9](resp. μ -clopen T_1) if for each pair of distinct points x and y in X , there exist μ -open (resp. μ -clopen) sets U and V containing x and y , respectively, such that $y \notin U$ and $x \notin V$.

(ii) $\mu - T_2$ [9](resp. μ -clopen T_2) if for each pair of distinct points x and y in X , there exist μ -open (reps. μ -clopen) sets U and V containing x and y , respectively, such that $U \cap V = \phi$.

Theorem 5. If $f : (X, \mu) \rightarrow (Y, \lambda)$ is a slightly (μ, λ) -continuous injection and Y is λ -clopen T_1 , then X is μ -clopen T_1 .

Proof. Suppose that Y is λ -clopen T_1 . For any distinct points x and y in X , there exist two λ -clopen sets V_1 and V_2 such that $f(x) \in V_1$, $f(y) \notin V_1$, $f(x) \notin V_2$, $f(y) \in V_2$. Since f is slightly (μ, λ) -continuous, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are μ -clopen subsets of X such that $x \in f^{-1}(V_1)$, $y \notin f^{-1}(V_1)$, $x \notin f^{-1}(V_2)$, $y \in f^{-1}(V_2)$. This shows that X is μ -clopen T_1 . ■

Theorem 6. If $f : (X, \mu) \rightarrow (Y, \lambda)$ is a slightly (μ, λ) -continuous injection and Y is λ -clopen T_2 , then X is μ -clopen T_2 .

Proof. For any pair of distinct points x and y in X , there exist disjoint λ -clopen sets U and V in Y such that $f(x) \in U$ and $f(y) \in V$. Since f is slightly (μ, λ) -continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are μ -clopen sets in X containing x and y , respectively. Since $U \cap V = \phi$, $f^{-1}(U) \cap f^{-1}(V) = \phi$. This shows that X is μ -clopen T_2 . ■

Theorem 7. If $f, g : (X, \mu) \rightarrow (Y, \lambda)$ are slightly (μ, λ) -continuous functions, μ is a topology and Y is λ -clopen T_2 , then $E = \{x \in X : f(x) = g(x)\}$ is μ -closed in X .

Proof. Let $x \in X - E$, then it follows that $f(x) \neq g(x)$. Since Y is λ -clopen T_2 , there exist λ -clopen sets V_1 and V_2 such that $f(x) \in V_1$, $g(x) \in V_2$ and $V_1 \cap V_2 = \phi$. Since f and g are slightly (μ, λ) -continuous, then $f^{-1}(V_1)$ and $g^{-1}(V_2)$ are μ -open sets in X . Let $U_1 = f^{-1}(V_1)$ and $U_2 = g^{-1}(V_2)$. Then U_1 and U_2 are μ -open sets containing x . Set $A = U_1 \cap U_2$,

then A is μ -open in X . Therefore $f(A) \cap g(A) = f(U_1 \cap U_2) \cap g(U_1 \cap U_2) \subseteq f(U_1) \cap g(U_2) \subset V_1 \cap V_2 = \phi$. Hence $A \cap E = \phi$ and $x \notin \mu Cl(E)$. This shows that E is μ -closed in X . ■

Recall that for a function $f : (X, \mu) \rightarrow (Y, \lambda)$, the subset $\{(x, f(x)) : x \in X\} \subset X \times Y$ is called the graph of f and is denoted by $G(f)$.

Definition 6. A graph $G(f)$ of a function $f : (X, \mu) \rightarrow (Y, \lambda)$ is said to be $(\mu, \lambda)^*$ -closed if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist a μ -clopen set U in X containing x and a λ -clopen set in Y containing y such that $(U \times V) \cap G(f) = \phi$.

Lemma 1. A graph $G(f)$ of a function $f : (X, \mu) \rightarrow (Y, \lambda)$ is $(\mu, \lambda)^*$ -closed in $X \times Y$ if and only if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist a μ -clopen set U in X containing x and a λ -clopen set V in Y containing y such that $f(U) \cap V = \phi$.

Theorem 8. If $f : (X, \mu) \rightarrow (Y, \lambda)$ is slightly (μ, λ) -continuous and Y is λ -clopen T_1 , then $G(f)$ is $(\mu, \lambda)^*$ -closed in $X \times Y$.

Proof. Let $(x, y) \in (X \times Y) \setminus G(f)$, then $f(x) \neq y$. Since Y is λ -clopen T_1 , there exists a λ -clopen set V of Y such that $f(x) \in V$ and $y \notin V$. Since f is slightly (μ, λ) -continuous, then $f^{-1}(V)$ is a μ -clopen set in X containing x . Take $U = f^{-1}(V)$. We have $f(U) \subset V$. Therefore, we obtain $f(U) \cap (Y - V) = \phi$ and $Y - V$ is a λ -clopen set in Y containing y . This shows that $G(f)$ is $(\mu, \lambda)^*$ -closed in $X \times Y$. ■

Corollary 3. If $f : (X, \mu) \rightarrow (Y, \lambda)$ is slightly (μ, λ) -continuous and Y is λ -clopen T_2 , then $G(f)$ is $(\mu, \lambda)^*$ -closed in $X \times Y$.

Theorem 9. Let $f : (X, \mu) \rightarrow (Y, \lambda)$ have a $(\mu, \lambda)^*$ -closed graph $G(f)$. If f is injective, then X is μ -clopen T_1 .

Proof. Let x and y be any two distinct points of X . Then, we have $(x, f(y)) \in (X \times Y) \setminus G(f)$. By Lemma 1, there exist a μ -clopen set U of X and a λ -clopen V of Y such that $(x, f(y)) \in U \times V$ and $f(U) \cap V = \phi$. Hence $U \cap f^{-1}(V) = \phi$ and $y \notin U$. This implies that X is μ -clopen T_1 . ■

Theorem 10. Let $f : (X, \mu) \rightarrow (Y, \lambda)$ have a $(\mu, \lambda)^*$ -closed graph $G(f)$. If f is a (μ, λ) -open surjection, then Y is $\lambda - T_2$.

Proof. Let y_1 and y_2 be any distinct points of Y . Since f is surjective, $f(x) = y_1$ for some $x \in X$ and $(x, y_2) \in (X \times Y) \setminus G(f)$. By Lemma 1, there exist a μ -clopen set U of X and a λ -clopen set V in Y such that $(x, y_2) \in U \times V$ and $f(U) \cap V = \phi$. Since f is (μ, λ) -open, then $f(U)$ is a λ -open set containing y_1 . This implies that Y is $\lambda - T_2$. ■

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