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PROPERTIES OF $ij-$DELTA OPEN SETS

Abstract. The notion of $\delta-$open sets in bitopological spaces was introduced by Banerjee [2]. In this paper we more investigate $ij-\delta-$open sets. We study some sets related to $ij-\delta-$open sets. Also, we study some separation axioms in bitopological spaces based on $ij-\delta-$open sets.

Key words: bitopological spaces, $ij-\delta-$open sets, pairwise $\delta-R_0$, pairwise $\delta-T_2$.

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1. Introduction

The notion of $\delta-$open sets in bitopological spaces was introduced by Banerjee [2], how used this notion to study $\delta-$continuous functions in bitopological spaces. Further investigation of this notion is found in [4]. Separation axioms T_k and R_k in bitopological spaces was introduced and studied in some manners in [1, 3, 5]. In this paper, we introduce more study of $ij-\delta-$open sets. We introduce and study the concepts of $ij-\delta-$derived set, $ij-\delta-$border, $ij-\delta-$interior, $ij-\delta-$exterior of a set. Also, we introduce and study the separation axioms $ij-\delta-R_0$, $ij-\delta-R_1$ and $ij-\delta-T_k$ spaces, $k=0,1,2$, in a version different from that in [1].

Throughout the present paper, (X,τ_1,τ_2) (or briefly X) always mean a bitopological space on which no separation axioms are assumed unless explicitly stated. Also $i,j\in\{1,2\}$ and $i\neq j$. Let A be a subset of X. By $i-Int(A)$ and $i-Cl(A)$, we mean respectively the interior and the closure of A in the topological space (X,τ_i) for $i=1,2$. A point x of X is called an $ij-\delta-$cluster point of A [2] if $i-Int(j-Cl(U))\cap A\neq\phi$ for every τ_i-open set U containing x. The set of all $ij-\delta-$cluster points of A is called the $ij-\delta-$closure of A and is denoted by $ij-Cl_\delta(A)$. A subset A is said to be $ij-\delta-$closed if $ij-Cl_\delta(A)=A$. The complement of an $ij-\delta-$closed set is said to be $ij-\delta-$open. The set of all $ij-\delta-$open (resp. $ij-\delta-$closed) sets of X will be denoted by $ij-\delta O(X)$(resp. $ij-\delta C(X)$).

2. Sets related to $ij-\delta-$open sets

Definition 1. *Let A be a subset of a bitopological space (X,τ_1,τ_2). A point $x\in X$ is called $ij-\delta-$limit point of A if for each $ij-\delta-$open set U containing x, $U\cap(A\backslash\{x\})\neq\phi$. The set of all $ij-\delta-$limit points of A is called the $ij-\delta-$derived set of A and is denoted by $ij-d_\delta(A)$.*

Definition 2. *Let A be a subset of a bitopological space (X,τ_1,τ_2). A point $x\in X$ is called $ij-\delta-$interior point of A if there exists an $ij-\delta-$open set U such that $x\in U\subset A$. The set of all $ij-\delta-$interior points of A is called the $ij-\delta-$interior of A and is denoted by $ij-Int_\delta(A)$.*

Definition 3. *For a subset A of a bitopological space (X,τ_1,τ_2), the $ij-\delta-$border of A, denoted by $ij-b_\delta(A)$, is defined by $ij-b_\delta(A)=A\backslash ij-Int_\delta(A)$.*

Definition 4. *For a subset A of a bitopological space (X,τ_1,τ_2), the $ij-\delta-$frontier of A, denoted by $ij-Fr_\delta(A)$ is defined by $ij-Fr_\delta(A)=ij-Cl_\delta(A)\backslash ij-Int_\delta(A)$.*

Definition 5. *For a subset A of a bitopological space (X,τ_1,τ_2), the $ij-\delta-$exterior of A, denoted by $ij-Ext_\delta(A)$ is defined by $ij-Ext_\delta(A)=ij-Int_\delta(X\backslash A)$.*

Theorem 1. *For subsets A and B of a bitopological space (X,τ_1,τ_2), the following are satisfied.*
(1) *$i-d(A)\subseteq ij-d_\delta(A)$ where $i-d(A)$ is the derived set of A in (X,τ_i).*
(2) *$A\subseteq B$ implies $ij-d_\delta(A)\subseteq ij-d_\delta(B)$.*
(3) *$ij-d_\delta(A)\cup ij-d_\delta(B)=ij-d_\delta(A\cup B)$
and $ij-d_\delta(A\cap B)\subseteq ij-d_\delta(A)\cap ij-d_\delta(B)$.*
(4) *$[ij-d_\delta(ij-d_\delta(A))\backslash A]\subseteq ij-d_\delta(A)$.*
(5) *$ij-d_\delta(A\cup ij-d_\delta(A))\subseteq A\cup ij-d_\delta(A)$.*

Proof. (1) Obvious, since every $ij-\delta-$open set is $i-$open.
(2) Obvious.
(3) Since $A\subseteq A\cup B$ and $B\subseteq A\cup B$, then $ij-d_\delta(A)\subseteq ij-d_\delta(A\cup B)$ and $ij-d_\delta(B)\subseteq ij-d_\delta(A\cup B)$. Therefore $ij-d_\delta(A)\cup ij-d_\delta(B)\subseteq ij-d_\delta(A\cup B)$. Now, let $x\notin ij-d_\delta(A)\cup ij-d_\delta(B)$. Then $x\notin ij-d_\delta(A)$ and $x\notin ij-d_\delta(B)$. Then there exist an $ij-\delta-$open set U containing x and an $ij-\delta-$open set V containing x such that $U\cap(A\backslash\{x\})=\phi$ and $V\cap(B\backslash\{x\})=\phi$. Then $(U\cap V)\cap((A\cup B)\backslash\{x\})=\phi$. Then $x\notin ij-d_\delta(A\cup B)$ and therefore $ij-d_\delta(A\cup B)\subseteq ij-d_\delta(A)\cup ij-d_\delta(B)$. By (2) $ij-d_\delta(A\cap B)\subseteq ij-d_\delta(A)$ and $ij-d_\delta(A\cap B)\subseteq ij-d_\delta(B)$. Therefore, $ij-d_\delta(A\cap B)\subseteq ij-d_\delta(A)\cap ij-d_\delta(B)$.
(4) Let $x\in[ij-d_\delta(ij-d_\delta(A))\backslash A]$ and U be an $ij-\delta-$open set containing x. Then $U\cap(ij-d_\delta(A)\backslash\{x\})\neq\phi$. Let $y\in U\cap(ij-d_\delta(A)\backslash\{x\})$. Then,

since $y \in ij - d_\delta(A)$ and $y \in U$, $U \cap [A \backslash \{y\}] \neq \phi$. Let $z \in U \cap [A \backslash \{y\}]$. Then $z \neq x$ for $z \in A$ and $x \notin A$. Hence $U \cap [A \backslash \{x\}] \neq \phi$. Therefore $x \in ij - d_\delta(A)$.

(5) Let $x \in ij - d_\delta(A \cup ij - d_\delta(A))$. If $x \in A$, the result is obvious. So, let $x \in [ij - d_\delta(A \cup ij - d_\delta(A)) \backslash A]$. Then for an $ij - \delta-$open set U containing x, $U \cap [A \cup ij - d_\delta(A) \backslash \{x\}] \neq \phi$. Thus $U \cap (A \backslash \{x\}) \neq \phi$ or $U \cap (ij - d_\delta(A) \backslash \{x\}) \neq \phi$. It follows from (4) that $U \cap (A \backslash \{x\}) \neq \phi$. Hence $x \in ij - d_\delta(A)$. Therefore, in any case, $ij - d_\delta(A \cup ij - d_\delta(A)) \subseteq A \cup ij - d_\delta(A)$. ■

In general, the reverse inclusions in (a) and (c) above may not be true as shown by the following examples:

Example 1. Let $X = \{a, b, c, d\}$, $\tau_1 = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}\}$, $\tau_2 = \{X, \phi, \{c\}, \{d\}, \{c, d\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$ and $A = \{a, c, d\}$. We can see that $12-\delta O(X) = \{X, \phi, \{a, b, c\}, \{a, b, d\}, \{a, b\}\}$, $12 - d_\delta(A) = \{b, c, d\}$, $1 - d(A) = \{d\}$. Then $1 - d(A) \subseteq 12 - d_\delta(A)$ but $12 - d_\delta(A) \not\subset 1 - d(A)$.

Example 2. Let (X, τ_1, τ_2) as in Example 1 and let $A = \{a, c\}$, $B = \{b, c\}$. Then $A \cap B = \{c\}$. Now $12 - d_\delta(A) = \{b, c, d\}$, $12 - d_\delta(B) = \{a, c, d\}$ and $12 - d_\delta(A \cap B) = \phi$. Then $12 - d_\delta(A \cap B) = \phi \subset \{c, d\} = 12 - d_\delta(A) \cap 12 - d_\delta(B)$. But $12 - d_\delta(A) \cap 12 - d_\delta(B) \not\subset 12 - d_\delta(A \cap B)$.

Theorem 2. *For any subset A of a bitopological space (X, τ_1, τ_2), $ij - Cl_\delta(A) = A \cup ij - d_\delta(A)$.*

Proof. Since $ij - d_\delta(A) \subseteq ij - Cl_\delta(A)$, $A \cup ij - d_\delta(A) \subseteq ij - Cl_\delta(A)$. On the other hand let $x \in ij - Cl_\delta(A)$. If $x \in A$, then the proof is complete. If $x \notin A$, each $ij - \delta-$open set U containing x intersects A at a point distinct from x, so $x \in ij - d_\delta(A)$. Thus $ij - Cl_\delta(A) \subseteq A \cup ij - d_\delta(A)$, which completes the proof. ■

Corollary 1. *A subset A of a bitopological space (X, τ_1, τ_2) is $ij - \delta-$closed if and only if it contains all of its $ij - \delta-$limit points.*

Theorem 3. *For subsets A and B of a bitopological space (X, τ_1, τ_2), the following statements are true:*

(1) *$ij - Int_\delta(A) \subseteq i - Int(A)$, where $i - Int(A)$ is the interior of A in (X, τ_i).*

(2) *$ij - Int_\delta(A)$ is the largest $ij - \delta-$open set contained in A.*

(3) *A is $ij - \delta-$open if and only if $A = ij - Int_\delta(A)$.*

(4) *$ij - Int_\delta(ij - Int_\delta(A)) = ij - Int_\delta(A)$.*

(5) *$ij - Int_\delta(A) = A \backslash ij - d_\delta(X \backslash A)$.*

(6) *$X \backslash ij - Int_\delta(A) = ij - Cl_\delta(X \backslash A)$.*

(7) $X\backslash ij-Cl_\delta(A)=ij-Int_\delta(X\backslash A)$.
(8) $A\subseteq B$ *implies* $ij-Int_\delta(A)\subseteq ij-Int_\delta(B)$.
(9) $ij-Int_\delta(A\cup B)\supseteq ij-Int_\delta(A)\cup ij-Int_\delta(B)$.
(10) $ij-Int_\delta(A\cap B)=ij-Int_\delta(A)\cap ij-Int_\delta(B)$.

Proof. (5) If $x\in(A\backslash ij-d_\delta(X\backslash A))$, then $x\notin ij-d_\delta(X\backslash A)$ and there exists an $ij-\delta-$open set U containing x such that $U\cap(X\backslash A)=\phi$. Then $x\in U\subseteq A$ and hence $x\in ij-Int_\delta(A)$, that is $(A\backslash ij-d_\delta(X\backslash A))\subseteq ij-Int_\delta(A)$. On the other hand if $x\in ij-Int_\delta(A)$, then $x\notin ij-d_\delta(X\backslash A)$ since $ij-Int_\delta(A)\cap(X\backslash A)=\phi$. Hence $ij-Int_\delta(A)=A\backslash ij-d_\delta(X\backslash A)$.

(6) $X\backslash ij-Int_\delta(A)=X\backslash(A\backslash ij-d_\delta(X\backslash A))=(X\backslash A)\cup ij-d_\delta(X\backslash A)=ij-Cl_\delta(X\backslash A)$. ■

Theorem 4. *For a subset A of a bitopological space (X,τ_1,τ_2), the following are satisfied:*
(1) $i-b(A)\subset ij-b_\delta(A)$, *where* $i-b(A)=A\backslash i-Int(A)$.
(2) $A=ij-Int_\delta(A)\cup ij-b_\delta(A)$.
(3) $ij-Int_\delta(A)\cap ij-b_\delta(A)=\phi$.
(4) *A is an* $ij-\delta-$*open set if and only if* $ij-b_\delta(A)=\phi$.
(5) $ij-b_\delta(ij-Int_\delta(A))=\phi$.
(6) $ij-Int_\delta(ij-b_\delta(A))=\phi$.
(7) $ij-b_\delta(ij-b_\delta(A))=ij-b_\delta(A)$.
(8) $ij-b_\delta(A)=A\cap ij-Cl_\delta(X\backslash A)$.
(9) $ij-b_\delta(A)=ij-d_\delta(X\backslash A)$.

Proof. (6) Let $x\in ij-Int_\delta(ij-b_\delta(A))$. Then $x\in ij-b_\delta(A)$. On the other hand since $ij-b_\delta(A)\subset A$, $x\in ij-Int_\delta(ij-b_\delta(A))\subseteq ij-Int_\delta(A)$. Hence $x\in ij-b_\delta(A)\cap ij-Int_\delta(A)$which contradicts (3). Therefore $ij-Int_\delta(ij-b_\delta(A))=\phi$.

(8) $ij-b_\delta(A)=A\backslash ij-Int_\delta(A)=A\backslash(X\backslash ij-Cl_\delta(X\backslash A))=A\cap(ij-Cl_\delta(X\backslash A))$.

(9) $ij-b_\delta(A)=A\backslash ij-Int_\delta(A)=A\backslash(A\backslash ij-d_\delta(X\backslash A))=ij-d_\delta(X\backslash A)$. ■

Theorem 5. *For a subset A of a bitopological space (X,τ_1,τ_2), the following are satisfied:*
(1) $i-Fr(A)\subseteq ij-Fr_\delta(A)$ *where* $i-Fr(A)=i-Cl(A)\backslash i-Int(A)$.
(2) $ij-Cl_\delta(A)=ij-Int_\delta(A)\cup ij-Fr_\delta(A)$.
(3) $ij-Int_\delta(A)\cap ij-Fr_\delta(A)=\phi$.
(4) $ij-b_\delta(A)\subseteq ij-Fr_\delta(A)$.
(5) $ij-Fr_\delta(A)=ij-b_\delta(A)\cup ij-d_\delta(A)$.
(6) *A is* $ij-\delta-$*open if and only if* $ij-Fr_\delta(A)=ij-d_\delta(A)$.
(7) $ij-Fr_\delta(A)=ij-Cl_\delta(A)\cap ij-Cl_\delta(X\backslash A)$.
(8) $ij-Fr_\delta(A)=ij-Fr_\delta(X\backslash A)$.

(9) $ij - Fr_\delta(A)$ *is* $ij - \delta-$*closed.*
(10) $ij - Fr_\delta(ij - Fr_\delta(A)) \subseteq ij - Fr_\delta(A)$.
(11) $ij - Fr_\delta(ij - Int_\delta(A)) \not\subset ij - Fr_\delta(A)$.
($l2$) $ij - Fr_\delta(ij - Cl_\delta(A)) \subseteq ij - Fr_\delta(A)$.
(13) $ij - Int_\delta(A) = A\backslash ij - Fr_\delta(A)$.

Proof. (1) $x \in i-Fr(A) \Rightarrow x \in i-Cl(A)$ and $x \notin i-Int(A) \Rightarrow x \in ij-Cl_\delta(A)$ and $x \notin ij-Int_\delta(A) \Rightarrow x \in ij-Cl_\delta(A)\backslash ij-Int_\delta(A) = ij-Fr_\delta(A)$.

(2) $ij-Int_\delta(A)\cup ij-Fr_\delta(A) = ij-Int_\delta(A)\cup(ij-Cl_\delta(A)\backslash ij-Int_\delta(A) = ij - Cl_\delta(A)$.

(3) $ij - Int_\delta(A) \cap ij - Fr_\delta(A) = ij - Int_\delta(A) \cap (ij - Cl_\delta(A)\backslash ij - Int_\delta(A)) = \phi$.

(4) $ij-b_\delta(A) = A\backslash ij-Int_\delta(A) \subset ij-Cl_\delta(A)\backslash ij-Int_\delta(A) = ij-Fr_\delta(A)$.

(5) $ij-Fr_\delta(A) = ij-Cl_\delta(A)\backslash ij-Int_\delta(A) = A\cup ij-d_\delta(A)\backslash ij-Int(A) = ij - d_\delta(A) \cup A\backslash ij - Int_\delta(A) = ij - d_\delta(A) \cup ij - b_\delta(A)$.

(6) $ij - Fr_\delta(A) = ij - Cl_\delta(A)\backslash ij - Int_\delta(A) = ij - Cl_\delta(A)\cap X\backslash ij - Int_\delta (A) = ij - Cl_\delta(A) \cap ij - Cl_\delta(X\backslash A)$.

(7) $ij - Cl_\delta(ij - Fr_\delta(A)) = ij - Cl_\delta(ij - Cl_\delta(A) \cap ij - Cl_\delta(X\backslash A)) \subseteq ij - Cl_\delta(ij-Cl_\delta(A))\cap ij - Cl_\delta(ij-Cl_\delta(X\backslash A)) = ij-Cl_\delta(A)\cap ij-Cl_\delta(X\backslash A) = ij - Fr_\delta(A)$. Therefore $ij - Fr_\delta(A)$ is $ij - \delta-$closed.

(8) $ij - Fr_\delta(ij - Fr_\delta(A)) = ij - Cl_\delta(ij - Fr_\delta(A)) \cap ij - Cl_\delta(X\backslash ij-Fr_\delta(A)) \subset ij - Cl_\delta(ij - Fr_\delta(A)) = ij - Fr_\delta(A)$.

(9) $ij - Fr_\delta(ij - Cl_\delta(A)) = ij - Cl_\delta(ij - Cl_\delta(A)\backslash ij - Int_\delta(ij - Cl_\delta(A)) = ij - Cl_\delta(A)\backslash ij - Int_\delta(ij - Cl_\delta A)) \subseteq ij - Cl_\delta(A)\backslash ij - Int_\delta(A) = ij - Fr_\delta(A)$.

(10) $A\backslash ij - Fr_\delta(A) = A\backslash(ij - Cl_\delta(A)\backslash(ij - Int_\delta(A)) = ij - Int_\delta(A)$. ■

The converses of Theorem 5(a) and b) above are not true, in general, as shown by the following examples:

Example 3. Let (X, τ_1, τ_2) as in Example 1 and $A = \{b, c, d\}$. Then $1 - Cl(A) = \{b, c, d\}$, $1 - Int(A) = \{b, c\}$. Then $1 - Fr(A) = \{b, c, d\} - \{b, c\} = \{d\}$. Now, $12-Cl_\delta(A) = X$, $12-Int_\delta(A) = \phi$. Then $12-Fr_\delta(A) = X\backslash\phi = X$. Then $1 - Fr(A) \subset 12 - Fr_\delta(A)$ but $12 - Fr_\delta(A) \not\subset 1 - Fr(A)$.

Example 4. Let (X, τ_1, τ_2) and A as in Example 3. Now $12 - b_\delta(A) = A\backslash 12 - Int_\delta(A) = A\backslash\phi = \{b, c, d\}$. Then $12 - b_\delta(A) \subset 12 - Fr_\delta(A)$ but $12 - Fr_\delta(A) \not\subset 12 - b_\delta(A)$.

Remark 1. Let A and B be subsets of a bitopological space (X, τ_1, τ_2). Then $A \subseteq B$ does not imply that either $ij - Fr_\delta(A) \subseteq ij - Fr_\delta(B)$or $ij - Fr_\delta(B) \subseteq ij - Fr_\delta(A)$.

Theorem 6. *For a subset A of a bitopological space (X, τ_1, τ_2), the following are satisfied:*

(1) $ij - Ext_\delta(A) \subseteq i - Ext(A)$ *where* $i - Ext(A) = i - Int(X\backslash A)$.
(2) $ij - Ext_\delta(A)$ *is an* $ij - \delta -$*open set.*
(3) $ij - Ext_\delta(A) = X\backslash ij - Cl_\delta(A)$.
(4) $ij - Ext_\delta(ij - Ext_\delta(A)) = ij - Int_\delta(ij - Cl_\delta(A))$.
(5) $A \subset B$ *implies* $ij - Ext_\delta(A) \supset ij - Ext_\delta(B)$.
(6) $ij - Ext_\delta(A \cup B) = ij - Ext_\delta(A) \cup ij - Ext_\delta(B)$.
(7) $ij - Ext_\delta(A \cap B) \supseteq ij - Ext_\delta(A) \cap ij - Ext_\delta(B)$.
(8) $ij - Ext_\delta(X) = \phi$.
(9) $ij - Ext_\delta(\phi) = X$.
(10) $ij - Ext_\delta(A) = ij - Ext_\delta(X\backslash ij - Ext_\delta(A))$.
(11) $ij - Int_\delta(A) \subseteq ij - Ext_\delta(ij - Ext_\delta(A))$.
(12) $X = ij - Int_\delta(A) \cup ij - Ext_\delta(A) \cup ij - Fr_\delta(A)$.

Proof. (1) $ij-Ext_\delta(A) = ij-Int_\delta(X\backslash A) \subset i-Int(X\backslash A) = i-Ext(A)$.

(4) $ij - Ext_\delta(ij - Ext_\delta(A)) = ij - Int_\delta(X\backslash ij - Ext_\delta(A)) = ij - Int_\delta(X\backslash ij - Int_\delta(X\backslash A)) = ij - Int_\delta(ij - Cl_\delta(A))$.

(7) $ij - Ext_\delta(A) \cap ij - Ext_\delta(B) = ij - Int_\delta(X\backslash A) \cap ij - Int_\delta(X\backslash B) ij - Int_\delta((X\backslash A) \cap (X\backslash B)) = ij - Int_\delta(X\backslash(A \cup B) = ij - Ext_\delta(A \cup B) \subseteq ij - Ext_\delta(A \cap B)$.

(10) $ij-Ext_\delta(X\backslash ij-Ext_\delta(A)) = ij-Int_\delta(ij-Ext_\delta(A)) = ij-Int_\delta(ij-Int_\delta(X\backslash A)) = ij - Int_\delta(X\backslash A) = ij - Ext_\delta(A)$.

(11) $ij-Int_\delta(A) \subseteq ij-Int_\delta(ij-Cl_\delta(A)) = ij-Int_\delta(X\backslash ij-Int_\delta(X\backslash A)) = ij - Int_\delta(X\backslash ij - Ext_\delta(A)) = ij - Ext_\delta(ij - Ext_\delta(A))$.

(12) $ij - Int_\delta(A) \cup ij - Ext_\delta(A) \cup ij - Fr_\delta(A) ij - Int_\delta(A) \cup X\backslash ij - cl_\delta(A) \cup ij - Cl_\delta(A)\backslash ij - Int_\delta(A) = X$. ■

The converses of (e) and (g) above are not true, in general, as shown by the following example:

Example 5. Let (X, τ_1, τ_2) as in Example 3, $A = \{c, d\}$ and $B = \{b, d\}$. Now $12 - Ext_\delta(A) = \{a, b\}$, $12 - Ext_\delta(B) = \phi$ and $12 - Ext_\delta(A \cap B) = \{a, b, c\}$. Then $12 - Ext_\delta(A \cap B) \supseteq 12 - Ext_\delta(A) \cap 12 - Ext_\delta(B)$but $12 - Ext_\delta(A \cap B) \not\subset 12 - Ext_\delta(A) \cap 12 - Ext_\delta(B)$. Also $12 - Ext_\delta(\{b, d\}) \subset 12 - Ext_\delta(\{c\})$ but $\{c\} \not\subset \{b, d\}$.

3. $ij - \delta - T_k$ spaces

Definition 6. *A bitopological space* (X, τ_1, τ_2) *is called*

(1) *Weak pairwise* $\delta - T_0$ *if for any two distinct points of* X, *there exists a subset which either* $ij - \delta -$ *open or* $ji - \delta -$ *open containing one of the points but not the other.*

(2) $ij - \delta - T_0$ *if for any two distinct points of* X, *there exists an* $ij - \delta -$ *open set which containing one of them but not the other. If* (X, τ_1, τ_2) *is* $12 - \delta - T_0$ *and* $21 - \delta - T_0$ *it is called pairwise* $\delta - T_0$.

From the definition, one may deduce that pairwise $\delta - T_0 \Rightarrow ij - \delta - T_0 \Rightarrow$ weak pairwise $\delta - T_0$.

Theorem 7. *A bitopological space (X, τ_1, τ_2) is weak pairwise $\delta - T_0$ if and only if for any two distinct points x and y of X, there exists a subset U which is either $\tau_i - open$ (or $\tau_j - open$) containing one of them (say x) such that $y \notin i - Int(j - Cl(U))$ (or $y \notin j - Int(i - Cl(U))$).*

Proof. Let $x, y \in X$ such that $xneqy$, then there exists a set U which is (say) $ij - \delta - open$ such that $x \in U$ and $y \notin U$. Then there exists a $\tau_i - open$ set G containing x such that $i - Int(j - Cl(G)) \subset U$. Obviously $y \notin i - Int(j - Cl(G))$.

Conversely, let $x, y \in X$ such that $x \neq y$. Then, by assumption, there exists a $\tau_i - open$ set U such that $x \in U$ and $y \notin i - Int(j - Cl(U))$. Now $U \subseteq i - Int(j - Cl(U))$ and $i - Int(j - Cl(U))$ is an $ij - \delta - open$ set. Then X is weak pairwise $\delta - T_0$. ■

Theorem 8. *A bitopological (X, τ_1, τ_2) is strongly pairwise $\delta - T_0$ if and only if for each two distinct points x and y of X, either $ij - Cl_\delta(\{x\}) \neq ij - Cl_\delta(\{y\})$ or $ji - Cl_\delta(\{x\}) \neq ji - Cl_\delta(\{y\})$.*

Proof. Let X be a pairwise strongly $\delta - T_0$ space and $x, y \in X$ such that $x \neq y$ Suppose U is an $ij - \delta - open$ set containing x but not y. Then $y \in ij - Cl_\delta(\{y\}) \subset X \backslash U$ and so $x \notin ij - Cl_\delta(\{y\})$. Hence $ij - Cl_\delta(\{x\}) \neq ij - Cl_\delta(\{y\})$. Conversely, let $x, y \in X$ such that $x \neq y$. Then either $ij - Cl_\delta(\{x\}) \neq ij - Cl_\delta(\{y\})$ or $ji - Cl_\delta(\{x\}) \neq ji - Cl_\delta(\{y\})$. In the former case, let $z \in X$ such that $z \in ij - Cl_\delta(\{y\})$ and $z \notin ij - Cl_\delta(\{x\})$. We assert that $y \notin ij - Cl_\delta(\{x\})$. If $y \in ij - Cl_\delta(\{x\})$, then $ij - Cl_\delta(\{y\}) \subseteq ij - Cl_\delta(\{x\})$ so $z \in ij - Cl_\delta(\{y\}) \subseteq ij - Cl_\delta(\{x\})$ a contradiction. Hence $y \notin ij - Cl_\delta(\{x\})$ and therefore $U = X \backslash ij - Cl_\delta(\{x\})$ is an $ij - \delta - open$ set containing y but not x. The case $ji - Cl_\delta(\{x\}) \neq ji - Cl_\delta(\{y\})$ can be dealt with similarly. ■

Definition 7. *A bitopological space (X, τ_1, τ_2) is called weakly pairwise $\delta - T_1$ if for any two distinct points x and y of X there exist an $ij - \delta - open$ set U and a $ji - \delta - open$ set V such that either $x \in U \backslash V$ and $y \in V \backslash U$ or $y \in U \backslash V$ and $x \in V \backslash U$.*

Theorem 9. *For a bitopological space (X, τ_1, τ_2) the following are equivalent:*

(*i*) (X, τ_1, τ_2) *is weakly pairwise* $\delta - T_1$

(*ii*) $12 - Cl_\delta(\{x\}) \bigcap 21 - Cl_\delta(\{x\}) = \{x\}$ *for every* $x \in X$.

(*iii*) *For every* $x \in X$, *the intersection of all* $12 - \delta - open$ *nbds and all* $21 - \delta - open$ *nbds of* x *is* $x\}$.

Proof. $(i) \Rightarrow (ii)$: Let $x \in X$ and $y \in 12-Cl_\delta(\{x\}) \bigcap 21-Cl_\delta(\{x\})$ where $y \neq x$. Since X is weakly pairwis $\delta - T_1$ there exists a $12-\delta-open$ set U such that $y \in U$, $x \notin U$ or there exists a $21-\delta-open$ set V such that $y \in V$, $x \notin V$. In either case $y \notin 12-Cl_\delta(\{x\}) \bigcap 21-Cl_\delta(\{x\})$. Hence $12-Cl_\delta(\{x\}) \bigcap 21-Cl_\delta(\{x\}) = \{x\}$.

$(ii) \Rightarrow (iii)$: If $x, y \in X$ such that $x \neq y$, then $x \notin 12-Cl_\delta(\{y\}) \bigcap 21-Cl_\delta(\{y\})$, so there is a $12-\delta-open$ set or a $21-\delta-open$ set containing x but not y. Therefore y does not belong to the intersection of all $12-\delta-open$ nbds and all $21-\delta-open$ nbds of x.

$(iii) \Rightarrow (i)$: Let x and y be two distinct points of X. By (iii), y does not belong to a $12-\delta-nbd$ or a $21-\delta-nbd$ of x. Therefore there exists a $12-\delta-open$ or a $21-\delta-open$ set containing x but not y. Hence X is a weakly pairwise $\delta - T_1$ space. ■

Theorem 10. *If a bitopological space (X, τ_1, τ_2) is weakly pairwise $\delta - T_1$, then for any two distinct points x and y of X there exist a $\tau_i - open$ set U and a $\tau_j - open$ set V such that either $x \in U$, $y \notin i-Int(j-Cl(U))$ and $y \in V$, $x \notin j-Int(i-Cl(V))$ or $y \in U$, $x \notin i-In(j-Cl(U))$ and $x \in V$, $y \notin j-Int(i-Cl(V))$.*

Proof. Similar to that of Theorem 7. ■

Definition 8. *A bitopological space (X, τ_1, τ_2) is called $ij-\delta-T_1$ if for any two distinct points x and y of X there exist two $ij-\delta-open$ sets U and V such that $x \in U \backslash V$ and $y \in V \backslash U$. If X is $12-\delta-T_1$ and $21-\delta-T_1$ then it is called pairwise $\delta - T_1$.*

Theorem 11. *A bitopological spaces (X, τ_1, τ_2) is $ij-\delta-T_1$ if and only if $ij-Cl_\delta\{x\} = \{x\}$, for every $x \in X$.*

Proof. Let $ij-\delta-open$, $x \in U$, then $y \neq x$ and there exists an $ij-\delta-open$ set U such that $y \in U$ and $x \notin U$. Therefore $y \notin ij-Cl_\delta\{x\}$ and so $ij-Cl_\delta\{x\} = \{x\}$. Conversely, let $x, y \in X$ such that $x \neq y$. Since $ij-Cl_\delta\{x\} = \{x\}$ and $ij-Cl_\delta\{y\} = \{y\}$, then there exist a $ij-\delta-open$ set U and an $ij-\delta-open$ set V such that $x \in U \backslash V$ and $y \in V \backslash U$. Thus X is $ij-\delta-T_1$. ■

Definition 9. *A bitopological space (X, τ_1, τ_2) is called weakly pairwise $\delta - T_2$ if for any two distinct points x and y of X, there exist an $ij-\delta-$*open *set U and a disjoint $ji-\delta-$*open *set V such that $x \in U$ and $y \in V$ or $x \in V$ and $y \in U$.*

Definition 10. *A bitopological space (X, τ_1, τ_2) is called $ij-\delta-T_2$ if for any two distinct point x and y of X there exist two disjoint $ij-\delta-open$ sets U and V such that $x \in U$ and $y \in V$. If X is $12-\delta-T_2$ and $21-\delta-T_2$, then it is called pairwise $\delta - T_2$.*

Definition 11. *A bitopological space (X, τ_1, τ_2) is called strongly pairwise $\delta - T_2$ if for any distinct points x and y of X, there exist a $ij - \delta -$ open set U and a disjoint $ji - \delta -$ open, set V such that $x \in U$ and $y \in V$.*

From the definitions one may deduce that: strongly pairwise $\delta - T_2 \Rightarrow$ weakly pairwise $\delta - T_2$ and pairwise $\delta - T_2 \Rightarrow ij - \delta - T_2$.

Theorem 12. *A bitopological space (X, τ_1, τ_2) is strongly pairwise $\delta - T_2$ if and only if the intersection of all $ij - \delta -$ closed $ji - \delta - nbd$ of each point of X is reduced to that point.*

Proof. Let X be strongly pairwise $\delta - T_2$ and $x \in X$. To each $y \in X$ with $x \neq y$, there exist an $ij - \delta -$ open set G and a $ji - \delta -$ open set H such that $x \in H$, $y \in G$ and $G \bigcap H = \phi$. Since $x \in H \subset X \backslash G$, hence $X \backslash G$ is $ij - \delta -$ closed $ji - \delta -$ nbd of x to which y does not belong. Consequently the intersection of all $ij - \delta -$ closed $ji - \delta - nbd$ of x is reduced to $\{x\}$. Conversely, let $x, y \in X$ such that $x \neq y$, then by hypothesis, there exists a $ij - \delta -$ closed $ji - \delta - nbd$ U of x such that $y \notin U$. Now, there exists an $ij - \delta -$open set G such that $x \in G \subset U$. Thus G is an $ij - \delta -$ open set, $X \backslash U$ is a $ji - \delta -$open set, $x \in G$, $y \in X \backslash U$ and $G \bigcap X \backslash U = \phi$. Hence X is strongly pairwise $\delta - T_2$. ∎

Theorem 13. *A bitopological space (X, τ_1, τ_2) is $ij - \delta - T_2$ if and only if the intersection of all $ij - \delta -$ closed $ij - \delta - nbd$ of each point of X is reduced to that point.*

Proof. Similar to that of Theorem 11. ∎

Theorem 14. *A bitopological space (X, τ_1, τ_2) is $ij - \delta - T_2$ if and only if given $x \in X$, for each $y \in X$ and $y \neq x$, there exists an $ij - \delta -$ open set U such that $x \in U$ and $y \notin ij - Cl_\delta(U)$.*

Proof. Straightforward. ∎

Recall that a bitopological space (X, τ_1, τ_2) is called $ij -$ nearlycompact [2] if for every $i-$open cover $\{U_\alpha : \alpha \in \Gamma\}$ of X, there exists a finite subset Γ_0 of Γ such that $X \subseteq \cup\{i - Int(j - Cl(U_\alpha)) : \alpha \in \Gamma_0\}$. Equivalently (X, τ_1, τ_2) is $ij -$ nearly$-$compact if every cover of X by $ij - \delta-$open sets have a finite subcover.

Theorem 15. *Let (X, τ_1, τ_2) be a strongly pairwise $\delta - T_2$ space and A be an $ij -$ nearlycompact subset of X. Then A is $ji - \delta -$ closed.*

Proof. If $A = X$, then A obviously $ji - \delta -$ closed. If $A \neq X$, then there is a point $x \in X \backslash A$. Since X is strongly pairwise $\delta - T_2$, for every $y \in A$, there exist a $ji - \delta -$ open set U_y and an $ij - \delta -$ open set V_y such that $x \in U_y$, $y \in V_y$ and $U_y \bigcap V_y = \phi$. Then $\{V_y : y \in A\}$ is a $ij - \delta -$ open cover of A which is $ij -$ nearlycompact then there exists a finite subfamily $V_{y_1}, \dots, V_{y_n}$ such that $A \subset \bigcup_{r=1}^{n} V_{y_r}$. Let $U = \bigcap_{r=1}^{n} U_{y_r}$ and $V = \bigcup_{r=1}^{n} V_{y_r}$. Then U is a $ji - \delta -$ open set, V is an $ij - \delta -$ open set, $x \in U$, $A \subset V$ and $U \bigcap V = \phi$. Thus $x \in U \subset X \backslash A$ and so $X \backslash A$ is $ji - \delta -$ open and so A is $ji - \delta -$ closed. ■

4. $ij - \delta - R_k$ spaces

Definition 12. *Let (X, τ_1, τ_2) be a bitopological space. For a subset A of X we define*

(i) $ij - \ker_\delta(A) = \cap\{U : U \in ij - \delta O(X), A \subseteq U\}$.

(ii) $p - Cl_\delta(\{x\}) = 12 - Cl_\delta(\{x\}) \bigcap 21 - Cl_\delta(\{x\})$.

(iii) $p - \ker_\delta(\{x\}) = 12 - \ker_\delta(\{x\}) \bigcap 21 - \ker_\delta(\{x\})$

Lemma 1. *Let (X, τ_1, τ_2) be a bitopological space and $A \subseteq X$. Then* $ij - \ker_\delta(A) = \{x : x \in X \text{and} ij - Cl\}_\delta(\{\text{x}\}) \cap \text{A} \neq \phi\}$.

Proof. Let $x \in ij - \ker_\delta(A)$ and $ij - Cl_\delta(\{x\}) \cap A = \phi$. Hence $x \notin (X \backslash ij - Cl_\delta(\{x\}))$ which is an $ij - \delta -$ open containing A. This is impossible, since $x \in ij - \ker_\delta(A)$. Consequently, $ij - Cl_\delta(\{x\}) \cap A \neq \phi$. On the other hand let $ij - Cl_\delta(\{x\}) \cap A \neq \phi$ and $x \notin ij - \ker_\delta(A)$. Then there exists an $ij - \delta -$ open set U containing A and $x \notin U$. Let $y \in ij - Cl_\delta(\{x\}) \cap A$. Hence U is an $ij - \delta -$ open neighborhood of y not containing x, which contradicts the fact that $y \in ij - Cl_\delta(\{x\})$. Hence $x \in ij - \ker_\delta(A)$. ■

Lemma 2. *Let (X, τ_1, τ_2) be a bitopological space and $x \in X$. Then $y \in ij - \ker_\delta(\{x\})$ if and only if $x \in ij - Cl_\delta(\{y\})$.*

Proof. Suppose that $y \notin ij - \ker_\delta(\{x\})$. Then there exists an $ij - \delta -$open set V such that $x \in V$ and $y \notin V$. Therefore we have $x \notin ij - Cl_\delta(\{y\})$. The proof of the converse can be done similarly. ■

Lemma 3. *For two points x and y of a bitopological space (X, τ_1, τ_2), the following are equivalent:*

(1) $ij - \ker_\delta(\{x\}) \neq ij - \ker_\delta(\{y\})$.

(2) $ij - Cl_\delta(\{x\}) \neq ij - Cl_\delta(\{y\})$.

Proof. (1) $\Rightarrow$ (2): Suppose that $ij - \ker_\delta(\{x\}) \neq ij - \ker_\delta(\{y\})$. Then there exists a point $z \in X$ such that $z \in ij - \ker_\delta(\{x\})$ and $z \notin ij - \ker_\delta(\{y\})$. It follows from $z \in ij - \ker_\delta(\{x\})$ that $x \in ij - Cl_\delta(\{z\})$. By $z \notin ij - \ker_\delta(\{y\})$, we have $\{y\} \cap ij - Cl_\delta(\{z\}) = \phi$. Since $x \in ij - Cl_\delta(\{z\})$, $ij - Cl_\delta(\{x\}) \subseteq ij - Cl_\delta(\{z\})$. Hence $\{y\} \cap ij - Cl_\delta(\{x\}) = \phi$. Therefore, $ij - Cl_\delta(\{x\}) \neq ij - Cl_\delta(\{y\})$.

(2) $\Rightarrow$ (1): Suppose that $ij-Cl_\delta(\{x\}) \neq ij-Cl_\delta(\{y\})$. Then there exists a point $z \in X$ such that $z \in ij-Cl_\delta(\{x\})$ and $z \notin ij-Cl_\delta(\{y\})$. Then there exists an $ij-\delta-$open set containing z and therefore x but not y, i.e., $y \notin ij-\ker_\delta(\{x\})$. Hence $ij-\ker_\delta(\{x\}) \neq ij-\ker_\delta(\{y\})$. ■

Definition 13. *A bitopological space (X, τ_1, τ_2) is called an $ij-\delta-R_0$ space if for every $ij-\delta-$open set U, $x \in U$ implies $ij-Cl_\delta(\{x\}) \subseteq U$.*

Theorem 16. *A bitopological space (X, τ_1, τ_2) is $ij-\delta-R_0$ if and only if for every two points x and y in X, $ij-Cl_\delta(\{x\}) \neq ij-Cl_\delta(\{y\})$ implies $ij-Cl_\delta(\{x\}) \cap ij-Cl_\delta(\{y\}) = \phi$.*

Proof. Suppose that (X, τ_1, τ_2) is $ij-\delta-R_0$ and $x, y \in X$ such that $ij-Cl_\delta(\{x\}) \neq ij-Cl_\delta(\{y\})$. Then there exists $z \in ij-Cl_\delta(\{x\})$ such that $z \notin ij-Cl_\delta(\{y\})$. Therefore, there exists an $ij-\delta-$open set V such that $y \notin V$ and $z \in V$, hence $x \in V$. Therefore, we have $x \notin ij-Cl_\delta(\{y\})$. Thus $x \in (X \backslash ij-Cl_\delta(\{y\})) \in ij-\delta O(X)$, which implies $ij-Cl_\delta(\{x\}) \subseteq$ $\subseteq (X \backslash ij-Cl_\delta(\{y\}))$ and $ij-Cl_\delta(\{x\}) \cap ij-Cl_\delta(\{y\}) = \phi$.

Conversely, let V be an $ij-\delta-$open set and $x \in V$. We will show that $ij-Cl_\delta(\{x\}) \subseteq V$. Let $y \notin V$, i.e., $y \in X \backslash V$. Then $x \neq y$ and $x \notin ij-Cl_\delta(\{y\})$. This shows that $ij-Cl_\delta(\{x\}) \neq ij-Cl_\delta(\{y\})$. By assumption, $ij-Cl_\delta(\{x\}) \cap ij-Cl_\delta(\{y\}) = \phi$. Hence $y \notin ij-Cl_\delta(\{x\})$ and therefore $ij-Cl_\delta(\{x\}) \subseteq V$. ■

Theorem 17. *A bitopological space (X, τ_1, τ_2) is $ij-\delta-R_0$ if and only if for every two points x and y in X, $ij-\ker_\delta(\{x\}) \neq ij-\ker_\delta(\{y\})$ implies $ij-\ker_\delta(\{x\}) \cap ij-\ker_\delta(\{y\}) = \phi$.*

Proof. Suppose that (X, τ_1, τ_2) is $ij-\delta-R_0$. By Lemma 3, for any points $x, y \in X$ if $ij-\ker_\delta(\{x\}) \neq ij-\ker_\delta(\{y\})$, then $ij-Cl_\delta(\{x\}) \neq ij-Cl_\delta(\{y\})$. Now we prove that $ij-\ker_\delta(\{x\}) \cap ij-\ker_\delta(\{y\}) = \phi$. Assume that $z \in ij-\ker_\delta(\{x\}) \cap ij-\ker_\delta(\{y\})$. By $z \in ij-\ker_\delta(\{x\})$ and Lemma 2, it follows that $x \in ij-Cl_\delta(\{z\})$. Since $x \in ij-Cl_\delta(\{x\})$, by Theorem 16, $ij-Cl_\delta(\{x\}) = ij-Cl_\delta(\{z\})$. Similarly, we have $ij-Cl_\delta(\{y\}) = ij-Cl_\delta(\{z\}) = ij-Cl_\delta(\{x\})$. This is a contradiction. Therefore, we have $ij-\ker_\delta(\{x\}) \cap ij-\ker_\delta(\{y\}) = \phi$.

Conversely, let (X, τ_1, τ_2) be a bitopological space such that for any two points x and y in X, $ij-\ker_\delta(\{x\}) \neq ij-\ker_\delta(\{y\})$ implies $ij-\ker_\delta(\{x\}) \cap ij-\ker_\delta(\{y\}) = \phi$. If $ij-Cl_\delta(\{x\}) \neq ij-Cl_\delta(\{y\})$, then by Lemma 3 $ij-\ker_\delta(\{x\}) \neq ij-\ker_\delta(\{y\})$. Hence $ij-\ker_\delta(\{x\}) \cap ij-\ker_\delta(\{y\}) = \phi$ which implies $ij-Cl_\delta(\{x\}) \cap ij-Cl_\delta(\{y\}) = \phi$. Because $z \in ij-\ker_\delta(\{x\})$ implies that $x \in ij-Cl_\delta(\{z\})$. Therefore $ij-\ker_\delta(\{x\}) \cap ij-\ker_\delta(\{y\}) \neq \phi$. By hypothesis, we have $ij-\ker_\delta(\{x\}) \neq ij-\ker_\delta(\{z\})$. Then $z \in ij-$

$\ker_\delta(\{x\}) \cap ij - \ker_\delta(\{y\})$ implies that $ij - \ker_\delta(\{x\}) = ij - \ker_\delta(\{z\}) = ij - \ker_\delta(\{y\})$, a contradiction. Hence $ij - Cl_\delta(\{x\}) \cap ij - Cl_\delta(\{y\}) = \phi$. By Theorem 16, (X, τ_1, τ_2) is an $ij - \delta - R_0$ space. ∎

Theorem 18. *For a bitopological space (X, τ_1, τ_2), the following are equivalent:*

(1) *(X, τ_1, τ_2) is an $ij - \delta - R_0$ space.*

(2) *For any subset $A \neq \phi$ and $G \in ij - \delta O(X)$ such that $A \cap G \neq \phi$, there exists $F \in ij - \delta C(X)$ such that that $A \cap F \neq \phi$ and $F \subseteq G$.*

(3) *For any $G \in ij - \delta O(X)$, $G = \cup\{F \in ij - \delta C(X) : F \subseteq G\}$.*

(4) *For any $F \in ij - \delta C(X)$, $F = \cap\{G \in ij - \delta O(X) : F \subseteq G\}$.*

(5) *For any $x \in X$, $ij - Cl_\delta(\{x\}) \subseteq ij - \ker_\delta(\{x\})$.*

Proof. (1) $\Rightarrow$ (2) Let A be a nonempty subset of X and $G \in ij - \delta O(X)$ such that $A \cap G \neq \phi$. Then there exists $x \in A \cap G$. Since $x \in G \in ij - \delta O(X)$, $ij - Cl_\delta(\{x\}) \subseteq G$. Set $F = ij - Cl_\delta(\{x\})$. Then F is an $ij - \delta-$closed subset of X such that $F \subseteq G$ and $A \cap F \neq \phi$.

(2) $\Rightarrow$ (3): Let $G \in ij - \delta O(X)$. Then $\cup\{F \in ij - \delta C(X) : F \subseteq G\} \subseteq G$. Let x be any point of G. There exists $F \in ij - \delta C(X)$ such that $x \in F$ and $F \subseteq G$. Therefore, we have $x \in F \subseteq \cup\{F \in ij - \delta C(X) : F \subseteq G\}$ and hence $G = \cup\{F \in ij - \delta C(X) : F \subseteq G\}$.

(3) $\Rightarrow$ (4) Obvious.

(4) $\Rightarrow$ (5) Let x be any point of X and $y \notin ij - \ker_\delta(\{x\})$. There exists $ij - \delta-$open set V such that $x \in V$and $y \notin V$. Hence $ij - Cl_\delta(\{y\}) \cap V = \phi$. By (d) $(\cap\{G \in ij - \delta O(X) : ij - Cl_\delta(\{y\}) \subseteq G\}) \cap V = \phi$. There exists $G \in ij - \delta O(X)$ such that $x \notin G$ and $ij - Cl_\delta(\{y\}) \subseteq G$. Therefore $ij - Cl_\delta(\{x\}) \cap G = \phi$ and $y \notin ij - Cl_\delta(\{x\})$. Consequently, we obtain $ij - Cl_\delta(\{x\}) \subseteq ij - \ker_\delta(\{x\})$.

(5) $\Rightarrow$ (1) Let $G \in ij - \delta O(X)$ and $x \in G$. Suppose $y \in ij - \ker_\delta(\{x\})$. Then $x \in ij - Cl_\delta(\{y\})$ and $y \in G$. This implies that $ij - Cl_\delta(\{x\}) \subseteq ij - \ker_\delta(\{x\}) \subseteq G$. Therefore (X, τ_1, τ_2) is an $ij - \delta - R_0$ space. ∎

Corollary 2. *For a bitopological space (X, τ_1, τ_2), the following are equivalent:*

(1) *(X, τ_1, τ_2) is an $ij - \delta - R_0$ space.*

(2) *$ij - Cl_\delta(\{x\}) = ij - \ker_\delta(\{x\})$ for all $x \in X$.*

Proof. (1) $\Rightarrow$ (2) Suppose that (X, τ_1, τ_2) is an $ij - \delta - R_0$ space. By Theorem 18 $ij - Cl_\delta(\{x\}) \subseteq ij - \ker_\delta(\{x\})$ for each $x \in X$. Let $y \in ij - \ker_\delta(\{x\})$. Then $x \in ij - Cl_\delta(\{y\})$ and so $ij - Cl_\delta(\{x\}) = ij - Cl_\delta(\{y\})$. Therefore $y \in ij - Cl_\delta(\{x\})$ and hence $ij - \ker_\delta(\{x\}) \subseteq ij - Cl_\delta(\{x\})$. This shows that $ij - Cl_\delta(\{x\}) = ij - \ker_\delta(\{x\})$.

(1) $\Rightarrow$ (2) This is obvious by Theorem 18. ∎

Theorem 19. *For a bitopological space (X,τ_1,τ_2), the following are equivalent:*

(1) (X,τ_1,τ_2) *is an* $ij-\delta-R_0$ *space.*

(2) $x\in ij-Cl_\delta(\{y\})$ *if and only if* $y\in ij-Cl_\delta(\{x\})$ *for any points* $x,y\in X$.

Proof. (1) $\Rightarrow$ (2) Assume that (X,τ_1,τ_2) is $ij-\delta-R_0$. Let $x\in ij-Cl_\delta(\{y\})$, then by Theorem 18, $x\in ij-\ker_\delta(\{y\})$. Therefore, every $ij-\delta-$open set containing y contains x. Hence $y\in ij-Cl_\delta(\{x\})$.

(2) $\Rightarrow$ (1) Let U be an $ij-\delta-$open set and $x\in U$. If $y\notin U$, then $x\notin ij-Cl_\delta(\{y\})$ and hence $y\notin ij-Cl_\delta(\{x\})$. This implies that $ij-Cl_\delta(\{x\})\subseteq U$. Hence (X,τ_1,τ_2) is $ij-\delta-R_0$. ■

Theorem 20. *For a bitopological space (X,τ_1,τ_2), the following are equivalent:*

(1) (X,τ_1,τ_2) *is an* $ij-\delta-R_0$ *space.*

(2) *If* F *is an* $ij-\delta-$*closed set, then* $F=ij-\ker_\delta(F)$.

(3) *If* F *is an* $ij-\delta-$*closed set and* $x\in F$, *then* $ij-\ker_\delta(\{x\})\subseteq F$.

(4) *If* $x\in X$, *then* $ij-\ker_\delta(\{x\})\subseteq ij-Cl_\delta(\{x\})$.

Proof. (1) $\Rightarrow$ (2) Let F be an $ij-\delta-$closed set and $x\notin F$. Then $X\backslash F$ is an $ij-\delta-$open set containing x. Then, by (1), $ij-Cl_\delta(\{x\})\subseteq X\backslash F$. Thus $ij-Cl_\delta(\{x\})\cap F=\phi$and by Lemma 2, $x\notin ij-\ker_\delta(F)$. Therefore $ij-\ker_\delta(F)=F$.

(2) $\Rightarrow$ (3) In general, $A\subseteq B$ implies $ij-\ker_\delta(A)\subseteq ij-\ker_\delta(B)$. Therefore, it follows from (b) that $ij-\ker_\delta(\{x\})\subseteq ij-\ker_\delta(F)=F$.

(3) $\Rightarrow$ (4) Since $x\in ij-Cl_\delta(\{x\})$ and $ij-Cl_\delta(\{x\})$ is an $ij-\delta-$closed set, by (c) $ij-\ker_\delta(\{x\})\subseteq ij-Cl_\delta(\{x\})$.

(4) $\Rightarrow$ (1) We show the implication using Theorem 19 Let $x\in ij-Cl_\delta(\{y\})$. Then by Lemma 2, $y\in ij-\ker_\delta(\{x\})$. Since $x\in ij-Cl_\delta(\{x\})$ which $ij-\delta-$closed, by (d) we have $y\in ij-\ker_\delta(\{x\})\subseteq ij-Cl_\delta(\{x\})$. Therefore $x\in ij-Cl_\delta(\{y\})$ implies $y\in ij-Cl_\delta(\{x\})$. The converse is obvious and (X,τ_1,τ_2) is $ij-\delta-R_0$.

By Theorem 18((1) $\Leftrightarrow$ (5)) and Theorem 20((1) $\Leftrightarrow$ (4)| one can conclude that a bitopological space (X,τ_1,τ_2) is $ij-\delta-R_0$ if and only if for any $x\in X$, we have $ij-Cl_\delta(\{x\})=ij-\ker_\delta(\{x\})$. ■

Definition 14. *A bitopological space (X,τ_1,τ_2) is called a pairwise $\delta-R_0$ space if for each $ij-\delta-$ open set U, $x\in U$ implies $ji-Cl_\delta(\{x\})\subset U$.*

Theorem 21. *A bitopological space(X,τ_1,τ_2) is pairwise $\delta-R_0$ if and only if for each $x,y\in X$ either $ij-Cl_\delta(\{x\})\bigcap ji-Cl_\delta(\{y\})=\phi$ or $\{x,y\}\subseteq ij-Cl_\delta(\{x\})\bigcap ji-Cl_\delta(\{y\})$.*

Proof. Let $ij-Cl_\delta(\{x\})\bigcap ji-Cl_\delta(\{y\})\neq\phi$ and $\{x,y\}\not\subset ij-Cl_\delta(\{x\})\bigcap ji-Cl_\delta(\{y\})$. Let $z\in ij-Cl_\delta(\{x\})\bigcap ji-Cl_\delta(\{y\})$ and $x\notin ij-Cl_\delta(\{x\})\bigcap ji-Cl_\delta(\{y\})$. Then $x\notin ji-Cl_\delta(\{y\})$ which implies that $x\in X\backslash ji-Cl_\delta(\{y\})$ which is a $ji-\delta-$open set, but $ij-Cl_\delta(\{x\})\not\subset X\backslash ji-Cl_\delta(\{y\})$ because $z\in ji-Cl_\delta(\{y\})$ so (X,τ_1,τ_2) is not pairwise $\delta-R_0$. Conversely, let G be an $ij-\delta-$open set containing a point x of X. Suppose that $ji-Cl_\delta(\{x\})\not\subset G$, then there is a point $y\in ji-Cl_\delta(\{x\})$ such that $y\notin G$ and so $ij-Cl_\delta(\{y\})\cap G=\phi$, since $X\backslash G$ is $ij-\delta-$closed and $y\in X\backslash G$. Hence $\{x,y\}\not\subset ij-Cl_\delta(\{x\})\bigcap ji-Cl_\delta(\{y\})$ and $ij-Cl_\delta(\{x\})\bigcap ji-Cl_\delta(\{y\})\neq\phi$ ■

Theorem 22. *For a bitopological space*(X,τ_1,τ_2)*, the following are equivalent*

(1) (X,τ_1,τ_2) *is pairwise* $\delta-R_0$.

(2) *For any* $x\in X, ij-Cl_\delta(\{x\})\subset ji-\ker_\delta(\{x\})$.

(3) *For any* $x,y\in X$, $y\in ij-\ker_\delta(\{x\})$ *if and only if* $x\in ji-\ker_\delta(\{y\})$.

(4) *For any* $x,y\in X$, $y\in ij-Cl_\delta(\{x\})$ *if and only if* $x\in ji-Cl_\delta(\{y\})$.

(5) *For any* $ij-\delta-$*closed set* F *and* $x\in X\backslash F$, *there exists a* $ji-\delta-$*open set* G *such that* $x\notin G$ *and* $F\subset G$.

(6) *Each* $ij-\delta-$*closed set* F *can be expressed as* $F=\bigcap\{G:G\in ji-\delta O(X),\ F\subset G\}$.

(7) *Each* $ij-\delta-$*open set* G *can be expressed as* $G=\cup\{F:F\in ji-\delta C(X), F\subset G\}$.

(8) *For each* $ij-\delta-$*closed set* $F, x\in X\backslash F$, *we have* $ji-Cl_\delta\{x\}\bigcap F=\phi$.

Proof. (1) ⇒ (2) By Definition 12 $ji-\ker_\delta\{x\}=\bigcap\{U\in ji-\delta O(X): x\in U\}$ and by Definition 14 each $ji-\delta-$open set G containing x contains $ij-Cl_\delta(\{x\})$. Hence $ij-Cl_\delta(\{x\})\subset ji-\ker_\delta(\{x\})$.

(2) ⇒ (3) If $y\in ij-\ker_\delta(\{x\})$ then $x\in ij-Cl_\delta(\{y\})$ and by (b), we have $x\in ji-\ker_\delta(\{y\})$.

(3) ⇒ (4) If $y\in ij-Cl_\delta(\{x\})$ then $x\in ij-\ker_\delta(\{y\})$ and by (c) $y\in ji-\ker_\delta(\{x\})$ and so $x\in ji-Cl_\delta(\{y\})$.

(4) ⇒ (5) Let F be $ij-\delta-$closed and $x\notin F$ then for any $y\in F$ $ij-Cl_\delta(\{y\})\subset F$ and so $x\notin ij-Cl_\delta(\{y\})$. By (4), $y\notin ji-Cl_\delta(\{x\})$i.e., there exists $ji-\delta-$open set G_y such that $y\in G_y$ and $x\notin G_y$. Let $G=\bigcup_{y\in F}\{G_y: G_y\in ji-\delta O(X),\ y\in G_y$ and $x\notin G_y\}$. Therefore, G is the required $ji-\delta-$open set such that $x\notin G$ and $F\subset G$.

(5) ⇒ (6) Let F be an $ij-\delta-$closed set and suppose that $H=\bigcap\{G: G\in ji-\delta O(X), F\subset G\}$. Clearly, $F\subset H$. To prove that $H\subset F$, let $x\notin F$. Hence by (5), there exists a $ji-\delta-$open set G such that $x\notin G$ and $F\subset G$, and hence $x\notin H$. Therefore $H\subset F$ and (f) follows.

(6) ⇒ (7) Obvious.

(7) $\Rightarrow$ (8) Let F be an $ij-\delta-$closed set and $x \notin F$. Then $X \backslash F$ is an $ij-\delta-$open set and $x \in X \backslash F$. By (g), there exists a $ji-\delta-$closed set H such that $x \in H \subset X \backslash F$ and so $ji-Cl_\delta(\{x\}) \subset X \backslash F$. Hence $ji-Cl_\delta(\{x\}) \bigcap F = \phi$.

(8) $\Rightarrow$ (1) Let G be an $ij-\delta-$open set and $x \in G$. Then $x \notin X \backslash G$ which $ij-\delta-$closed and by (h) $ji-Cl_\delta(\{x\}) \bigcap (X \backslash G) = \phi$ which implies that $ji-Cl_\delta(\{x\}) \subset G$. ■

Theorem 23. *Let (X, τ_1, τ_2) a bitopological space. If X is pairwise $\delta-R_0$ then for any x and y in X, we have either $p-Cl_\delta(\{x\}) = p-Cl_\delta(\{y\})$ or $p-Cl_\delta(\{x\}) \bigcap p-Cl_\delta(\{y\}) = \phi$.*

Proof. Let $x, y \in X$ and suppose $p-Cl_\delta(\{x\}) \bigcap p-Cl_\delta(\{y\}) \neq \phi$. Let $z \in 12-Cl_\delta(\{x\}) \bigcap 21-Cl_\delta(\{x\}) \cap 12-Cl_\delta(\{y\}) \bigcap 21-Cl_\delta(\{y\})$, then

$12-Cl_\delta(\{z\}) \subset 12-Cl_\delta(\{x\}) \bigcap 12-Cl_\delta(\{y\})$ and $21-Cl_\delta(\{z\}) \subset 21-Cl_\delta(\{x\}) \bigcap 21-Cl_\delta(\{y\})$. Also $z \in 12-Cl_\delta(\{x\})$ which implies that $21-Cl_\delta(\{x\}) \subset 21-Cl_\delta(\{y\})$, this because by (d) of Theorem 22 $z \in 12-Cl_\delta(\{x\})$. Then $x \in 21-Cl_\delta(\{z\})$ implies $21-Cl_\delta(\{x\}) \subset 21-Cl_\delta(\{z\}) \subset 21-Cl_\delta(\{y\})$. Similarly $z \in 21-Cl_\delta(\{x\})$ implies $12-Cl_\delta(\{x\}) \subset 12-Cl_\delta(\{y\})$ and $z \in 12-Cl_\delta(\{y\})$ implies $21-Cl_\delta(\{y\}) \subset 21-Cl_\delta(\{x\})$ and $z \in 21-Cl_\delta(\{y\})$ implies $12-Cl_\delta(\{y\}) \subset 12-Cl_\delta(\{x\})$. Therefore $12-Cl_\delta(\{x\}) \bigcap 21-Cl_\delta(\{x\}) \subset 12-Cl_\delta(\{y\}) \bigcap 21-Cl_\delta(\{y\})$ and $12-Cl_\delta(\{y\}) \bigcap 12-Cl_\delta(\{y\}) \subset 12-Cl_\delta(\{x\}) \bigcap 21-cl_\delta(\{x\})$. Hence $12-Cl_\delta(\{x\}) \bigcap 21-Cl_\delta(\{x\}) = 12-Cl_\delta(\{y\}) \bigcap 21-Cl_\delta(\{y\})$. This completes the proof. ■

Theorem 24. *Let (X, τ_1, τ_2) be a bitopological space. If X is pairwise $\delta-R_0$ then for any x and y in X, we have either $p-\ker_\delta(\{x\}) = p-\ker_\delta(\{y\})$ or $p-\ker_\delta(\{x\}) \bigcap p-\ker_\delta(\{y\}) = \phi$.*

Proof. In virtue of statement (c) of Theorem 22, the proof is similar to that of Theorem 24. ■

Theorem 25. *Let (X, τ_1, τ_2) a bitopological space. If X is pairwise $\delta-T_1$ then it is pairwise $\delta-R_0$.*

Proof. If X is pairwise $\delta-T_1$ then by Theorem 11, $12-Cl_\delta(\{x\}) = \{x\} = 21-Cl_\delta(\{x\})$. Thus X is pairwise $\delta-R_0$. ■

Theorem 26. *Let (X, τ_1, τ_2) be a bitopological space. If X is pairwise $\delta-R_0$ weakly pairwise $\delta-T_0$ then it is weakly pairwise $\delta-T_1$.*

Proof. Let $x, y \in X$ such that $x \neq y$. Since X is weakly pairwise $\delta-T_0$, there exists an $ij-\delta-$open or $ji-\delta-$open set G, containing one of the two points but not the other. Suppose $x \in G \in ij - \delta O(X)$. Since X is pairwise $\delta - R_0$, $x \in G$ implies $ji - Cl_\delta(\{x\}) \subset G$. Now $x \notin X \backslash ji - Cl_\delta(\{x\})$, thus G is an $ij - \delta-$open set containing x but not y and $X \backslash ji - Cl_\delta(\{x\})$ is a $ji - \delta-$open set containing y but not x. Thus X is weakly pairwise $\delta - T_1$. ■

Definition 15. *A bitopological space (X, τ_1, τ_2) is called pairwise $\delta - R_1$ if for every pair of distinct points x and y of X such that $ij - Cl_\delta(\{x\}) \neq ji - Cl_\delta(\{y\})$, there exist a $ji - \delta-$open set U and an $ij - \delta-$open set V that $x \in U$, $y \in V$ and $U \bigcap V = \phi$.*

Theorem 27. *Let (X, τ_1, τ_2) be a bitopological space. If X is pairwise $\delta - R_1$ then it is pairwise $\delta - R_0$.*

Proof. Let G be any $ij - \delta-$open set and $x \in G$. For each point $y \in X \backslash G$, $ji - Cl_\delta(\{x\}) \neq ij - Cl_\delta(\{y\})$ so there exist an $ij - \delta-$open set U_y and a $ji - \delta-$open set V_y such that $x \in U_y$, $y \in V_y$ and $U_y \bigcap V_y = \phi$. If $A = \bigcup\{V_y : y \in X \backslash G\}$ then $X \backslash G \subset A$ and $x \notin A$. Since A is a $ji - \delta-$open set, then $ji - Cl(\{x\}) \subset X \backslash A \subset G$. Hence X is pairwise $\delta - R_0$ ■

Theorem 28. *A bitopological space (X, τ_1, τ_2) is pairwise $\delta - R_1$ if and only if for every pair of points x and y of X such that $ij - Cl_\delta(\{x\}) \neq ji - Cl_\delta(\{y\})$, there exist an $ij - \delta-$open set U and a$ji - \delta-$open set V such that $ij - Cl_\delta(\{x\}) \subset V$, $ji - Cl_\delta(\{y\}) \subset U$ and $U \bigcap V = \phi$.*

Proof. Let (X, τ_1, τ_2) be pairwise $\delta - R_1$ and $x, y \in X$ such that $ij - Cl_\delta(\{x\}) \neq ji - Cl_\delta(\{y\})$. Then there exist an $ij - \delta - rmopen$ set U and a $ji - \delta-$open set V such that $x \subset V, y \subset U$ and $U \bigcap V = \phi$. Since a pairwise $\delta - R_1$ space is pairwise $\delta - R_0$ therefore $x \in V$ implies $ij - Cl_\delta(\{x\}) \subset V$ and $y \in U$ implies $ji - Cl_\delta(\{y\}) \subset U$. Hence the result.

The converse is obvious. ■

Theorem 29. *Let (X, τ_1, τ_2) be a bitopological space. If X is pairwise $\delta - R_1$ pairwise $\delta - T_1$, then it is strongly pairwise $\delta - T_2$.*

Proof. Let X be a pairwise $\delta - R_1$ pairwise $\delta - T_1$ space and $x, y \in X$ such that $x \neq y$. Since X is pairwise $\delta - T_1$, $ji - Cl_\delta(\{x\}) = \{x\}$ and $ij - Cl_\delta(\{y\}) = \{y\}$ and so $ji - Cl_\delta\{x\} \neq ij - Cl_\delta\{y\}$. Since X is pairwise $\delta - R_1$ there exist an $ij - \delta-$open set U and a $ji - \delta-$open set V such that $x \subset U$, $y \subset V$ and $U \bigcap V = \phi$. Therefore X is strongly pairwise $\delta - T_2$. ■

Theorem 30. *Let (X, τ_1, τ_2) be a pairwise $\delta - R_1$ bitopological space, $A \subset X$ and $x \in X$. If A is an $ij-$nearly compact subset of X such that $A \bigcap ji - Cl_\delta(\{x\}) = \phi$, then there exist an $ij - \delta-$open set U and a $ji - \delta-$open set V such that $ji - Cl_\delta(\{x\}) \subset U$, $A \subset V$ and $U \bigcap V = \phi$.*

Proof. For each $y \in A$, $ji - Cl_\delta(\{x\}) \neq ij - Cl_\delta(\{y\})$. So, there exist an $ij - \delta-$open set U_y and a $ji - \delta-$open set V_y such that $ij - Cl_\delta(\{y\}) \subset V_y$, $ji - Cl_\delta(\{x\}) \subset U_y$ and $U_y \bigcap V_y = \phi$. Thus $\{V_y : y \in A\}$ is an $ij - \delta-$open cover of A, which is $ij-$nearly compact and hence admits a finite subcover $\{V_{y_r} : r = 1, \ldots, n\}$. Let $U = \bigcap_{r=1}^{n} U_{y_r}$ and $V = \bigcup_{r=1}^{n} V_{y_r}$. Thus U is an $ij - \delta-$open set and V is a $ji - \delta-$open set. Also $ji - Cl_\delta(\{x\}) \subset U$, $A \subset V$ and $U \bigcap V = \phi$. ■

References

[1] ARYA S.P., NOUR T.M., Separation axioms for bitopological spaces, *Indian J. Pure. Apple. Math.,* 19(3)(1988) 42-50.

[2] BANERJEE G.K., On pairwise almost strongly θ-continuous mapping, *Bull. Calcutta Math. Soc.,* 79(1987), 314-320.

[3] KELLY J.C., Bitopological spaces, *Proc. London Math. Soc.,* 13(1963), 71- 89.

[4] KHEDR F.H, AL-SHIBANI A.M., NOIRI T., On δ-continuity in bitopological spaces, *J. Egypt. Math. Soc.,* 5(3)(1997), 57-63.

[5] MURDESHWAR G.M., NAIMPALLY S.A., *Quasi-Uniform Topological Spaces,* Monograph P. Noordhorff Ltd., 1966.

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